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Polynomial algorithm for the disjoint bilinear programming problem with an acute-angled polytope for a disjoint subset

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Abstract. We consider the disjoint bilinear programming problem in which one of the disjoint subsets has the structure of an acute-angled polytope. An optimality criterion for such a problem is formulated and proved, and based on this, a polynomial algorithm for its solving is proposed and grounded. We show that the proposed polynomial algorithm can be efficiently used for studying and solving the boolean programming problem and the piecewise linear concave programming problem.

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1 Introduction and problem formulation

The main results of the article are concerned with studying and solving the disjoint programming problem in the case when one of the disjoint subsets has the structure of an acute-angled polytope. We formulate and prove an optimality criterion for this problem and, based on this, propose a polynomial algorithm for its solving. Furthermore, we show that these results can be efficiently used for studying and solving the boolean programming problem and the piecewise linear concave programming problem.

The formulation of the disjoint bilinear programming problem is as follows [1–6]:

Minimize

$$z = x^T Cy + gx + ey \tag{1}$$

subject to

$$Ax \le a, \quad x \ge 0; \tag{2}$$

$$By \le b, \quad y \ge 0, \tag{3}$$

where

$$C = (c_{ij})_{n\times m}, \quad A = (a_{ij})_{q\times n}, \quad B = (b_{ij})_{l\times m},$$

$$a^T = (a_{10}, a_{20}, \dots, a_{q0}) \in R^q, \quad b^T = (b_{10}, b_{20}, \dots, b_{l0}) \in R^l,$$

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$$g = (g_1, g_2, \dots g_n) \in R^n, \quad e = (e_1, e_2, \dots, e_m) \in R^m,$$
$$x^T = (x_1, x_2, \dots, x_n) \in R^n, \quad y^T = (y_1, y_2, \dots, y_m) \in R^m.$$

Throughout this article we will assume that the solution sets X and Y of the corresponding systems (2) and (3) are nonempty and bounded. Our main investigations in the article are addressed to the case of problem (1)-(3) when solution set Y of system (3) has the structure of an acute-angled polytope.

In general disjoint bilinear programming problem (1)-(3) comprises a large class of integer and combinatorial optimization problems, including the well-known NP-complete problem of the existence of a boolean solution for a given system of linear inequalities

$$\begin{cases} \sum\limits_{i=1}^{n} a_{ij}x_j \le a_{i0}, & i = 1, 2, \dots, q, \\ \qquad x_j \ge 0, & j = 1, 2, \dots, n. \end{cases} \tag{4}$$

It is easy to show that this problem can be represented as the following disjoint bilinear programming problem:
Minimize

$$z = \sum_{j=1}^{n} (x_j + y_j - 2x_jy_j) \tag{5}$$

subject to

$$\begin{cases} \sum\limits_{j=1}^{n} a_{ij}x_j \le a_{i0}, & i = 1, 2, \dots, q, \\ \quad 0 \le x_j \le 1, & j = 1, 2, \dots, n; \end{cases} \tag{6}$$

$$0 \le y_j \le 1, \qquad j = 1, 2, \dots, n. \tag{7}$$

The relationship between this bilinear programming problem and the problem of determining a boolean solution of system (4) is the following one: system (4) has a boolean solution $x^* = (x_1^*, x_2^*, \dots, x_n^*)$ if and only if $x^* = (x_1^*, x_2^*, \dots, x_n^*)$ and $y^* = (y_1^*, y_2^*, \dots, y_n^*)$ with $y_j^* = x_j^*$, $j = 1, 2, \dots, n$, represents an optimal solution of the disjoint bilinear programming problem (5)-(7) where $z(x^*, y^*) = 0$. In [4, 5] for determining a boolean solution of system (4) have been considered the disjoin bilinear programming problem in which the minimizing objective function on the set of solutions of systems (6), (7) is $z' = \sum\limits_{j=1}^{n} (x_jy_j + (1 - x_j)(1 - y_j))$. In this case also the optimal value of the objective function of the problem is equal to zero if system (4) has a boolean solution and if $x^* = (x_1^*, x_2^*, \dots, x_n^*)$ is a boolean solution for system (4), then $x^* = (x_1^*, x_2^*, \dots, x_n^*)$ and $y^* = (y_1^*, y_2^*, \dots, y_n^*)$ with $y_i^* = 1 - x_i^*$, $i = 1, 2, \dots, n$, represents an optimal solution of the bilinear programming problem, i.e. $z'(x^*, z^*) = 0$.

To disjoint bilinear programming problem (1)-(3) also the following classical boolean linear programming problem can be reduced:

Minimize

$$z=\sum_{j=1}^{n} c_j x_j \tag{8}$$

subject to

$$\begin{cases} \sum\limits_{i=1}^{n} a_{ij}x_j \le a_{i0}, & i=1,2,\ldots,q,\\ \qquad x_j \in \{0,1\}, & j=1,2,\ldots,n. \end{cases} \tag{9}$$

This problem has an optimal boolean solution $x^*=(x_1^*,x_2^*,\ldots,x_n^*)$ if and only if system (9) has a feasible boolean solution. If coefficients a_{ij} and c_j of problem (8), (9) are integers, then $z^*=z(x^*)\in[-nH,nH]$, where $H=\max\{|c_i|, i=1,2,\ldots,n\}$. Therefore the optimal solution of boolean linear programming problem (8), (9) can be found by solving a sequence of disjoint bilinear programming problems:
Minimize

$$z=\sum_{j=1}^{n}(x_j+y_j-2x_jy_j)$$

subject to

$$\begin{cases} \sum\limits_{j=1}^{n} c_jx_j \le t_k, & \\ \sum\limits_{i=1}^{n} a_{ij}x_j \le a_{i0}, & i=1,2,\ldots,q,\\ 0\le x_j\le 1, & j=1,2,\ldots,n;\\ 0\le y_j\le 1, & j=1,2,\ldots,n, \end{cases}$$

with integer parameters t_k on $[-nH,nH]$ by applying the bisection method with a standard integer roundoff procedure for t_k.

Boolean programming problem (8), (9) can be also formulated as the following disjoint bilinear programming problem:
Minimize

$$\overline{z}=\sum_{j=1}^{n} c_jx_j + M\sum_{j=1}^{n}(x_j+y_j-2x_jy_j) \tag{10}$$

subject to

$$\begin{cases} \sum\limits_{j=1}^{n} a_{ij}x_j \le a_{i0}, & i=1,2,\ldots,q,\\ 0\le x_j\le 1, & j=1,2,\ldots,n; \end{cases} \tag{11}$$

$$0\le y_j\le 1, \qquad j=1,2,\ldots,n, \tag{12}$$

where M is a suitable large value. It is easy to show that if the coefficients in (8), (9) are integers, then M should satisfy the condition $M\ge n\cdot 2^{3L+1}$, where L is the length of the binary encoding of the coefficients of boolean problem (8), (11). In this case the relationship between boolean linear programming problem (8), (9) and disjoint bilinear programming problem (10)-(12) is the following: boolean linear programming problem (8), (9) has an optimal boolean solution $x^*=$

$(x_1^*, x_2^*, \dots, x_n*)$ if and only if $x^* = (x_1^*, x_2^*, \dots, x_n^*)$ and $y^* = (y_1^*, y_2^*, \dots, y_n^*)$ with $y_i^* = x_i^*, i = 1, 2, \dots, n$, represent a solution of disjoint bilinear programming problem (10)-(12) and $\overline{z}(x^*, y^*) = \sum_{j=1}^{n} c_j x^*$. So, the boolean programming problem can be reduced in polynomial time to disjoint programming problem (1)-(3) where the matrix B is identity one.

Another important problem which can be reduced to disjoint bilinear programming problem (1)-(3) is the following piecewise linear concave programming problem:
Minimize

$$z = \sum_{j=1}^{l} \min\{c^{jk}x + c_0^{jk}, \qquad k = 1, 2, \dots, m_j\} \tag{13}$$

subject to (2), where $x \in R^n,\ c^{jk} \in R^n,\ c_0^{jk} \in R^1$.

This problem arises as an auxiliary one when solving a class of resource allocation problems [7]. In [7] it is shown that this problem can be replaced by the following disjoint bilinear programming problem:
Minimize

$$z = \sum_{j=1}^{l} \sum_{k=1}^{m_j} (c^{jk}x + c_0^{jk}) y_{jk}$$

subject to

$$Ax \le a, \quad x \ge 0;$$

$$\begin{cases} \sum\limits_{k=1}^{m_j} y_{jk} = 1, & j = 1, 2, \dots, l; \\ y_{jk} \ge 0, & k = 1, 2, \dots, m_j, \quad j = 1, 2, \dots, l. \end{cases}$$

In this problem we can eliminate $y_{1,m_1}, y_{2,m_2}, \dots, y_{l,m_l}$, taking into account that

$$y_{jm_j} = 1 - \sum_{k=1}^{m_j - 1} y_{jk}, \qquad j = 1, 2, \dots l,$$

and we obtain the following disjoint bilinear programming problem:
Minimize

$$z = \sum_{j=1}^{l} \sum_{k=1}^{m_j - 1} (c^{jk} - c^{jm_j}) x y_{jk} + \sum_{j=1}^{l} c^{jm_j} x +$$

$$+ \sum_{j=1}^{l} \sum_{k=1}^{m_j - 1} (c_0^{jk} - c_0^{jm_j}) y_{jk} + \sum_{j=1}^{l} c_0^{jm_j} \tag{14}$$

subject to

$$Ax \le a, \quad x \ge 0; \tag{15}$$

$$\begin{cases} \sum\limits_{k=1}^{m_j - 1} y_{jk} \le 1, & j = 1, 2, \dots, l; \\ y_{jk} \ge 0, & j = 1, 2, \dots, l, \quad k = 1, 2, \dots, m_j - 1, \end{cases} \tag{16}$$

i.e. we obtain a special case of disjoint bilinear programming problem (1)-(3) where the corresponding matrix B is step-diagonal.

Disjoint bilinear programming problem (1)-(3) has been extensively studied in [1–4, 6–9, 15] and some general methods and algorithms have been developed. In this article we shall use a new optimization criterion that takes into account the structure of the disjoint subset Y.

It can be observed that in the presented above examples of disjoint programming problems the matrix B is either identity one or step diagonal. This means that the set of solutions Y has the structure of an acute-angled polyhedron [10, 11]. Acute-angled polyhedra are polyhedra in which all dihedral angles are acute or right. A detailed characterization of such polyhedra has been made by Coxeter [11] and Andreev [10, 12]. Moreover, in [10, 12] it has been proven that in an acute-angled polyhedron the hyperplanes of nonadjacent facets cannot intersect. Based on this property in the present article we show that for a disjoint bilinear programming problem with the structure of an acute-angled polytope for a disjoint subset an optimality criterion can be formulated that can be efficiently used for studying and solving the problem. In fact we show that the formulated optimality criterion is valid not only for the case when Y is an acute-angled polytope but it holds also for a more general class of polyhedra for which the hyperplanes of nonadjacent facets do not intersect. In this article such polyhedra are called *perfect polyhedra*; we call the corresponding disjoint bilinear programming problems with such a disjoint subset *disjoint bilinear programming problems with a perfect disjoint subset.* The aim of this paper is to show that for this class of problems an optimality criterion can be formulated that takes into account the mentioned structure of the disjoint subset Y and that can be efficiently used for solving the problems. Moreover, based on presented optimality criterion for such a problem we ground a polynomial algorithms for its solving. The optimality criterion for disjoint bilinear programming with an acute-angled polytope for a disjoint subset has been announced in [4, 5]. We present in this article the full proof of this optimality criterion and the results concerned with its application for elaboration of the polynomial algorithm for solving the disjoint bilinear programming problem with a perfect disjoint subset.

The article is organized as follows: In Section 2 we present the formulation of the disjoint bilinear programming problem with a prefect disjoint subset that generalizes the bilinear programming problem with a disjoint subset having the structure of an acute-angled polytope. In Sections 3 - 5 we present the basic properties of the optimal solutions of the disjoint bilinear programming problem (1) - (3) in general. An important result of Section 5 is Theorem 3 that gives an optimization criterion for problem (1) - (3) and allows to ground some general schemes for its solving. Section 7 provides some necessary auxiliary results related to redundant inequalities for consistent and inconsistent systems that in the next Section 8 are used for the proof of the optimality criterion for the problem with a perfect polytope. So, in Section 8 we formulate and prove the optimality criterion for the disjoint bilinear programming problem with a perfect disjoint subset (Theorem 8) that represents a refinement of Theorem 3 for the case when Y has a structure of perfect polytope (or acute-angled polytope). The main results of this section are Theorems 8, 9 on the basis of which in Sections 9 - 11 we present the main results concerned with elaboration of the polynomial algorithm for solving the disjoint bilinear programming problem with a perfect disjoint subset. In Section 12 it is shown how the proposed polynomial algorithm can be used for solving the boolean programming problem and the piecewise linear concave programming problem. In Section 13 the importance of the obtained in the article results for modern combinatorial problems is discussed.

2 Disjoint bilinear programming problem with a perfect disjoint subset

The formulation of the disjoint bilinear programming problem with a perfect disjoint subset is as follows:
Minimize

$$z = x^T Cy + gx + ey \tag{17}$$

subject to

$$Ax \leq a, \quad x \geq 0; \tag{18}$$

$$Dy \leq d. \tag{19}$$

This problem differs from problem (1)-(3) only by system (19) in which $D = (d_{ij})_{p \times m}$ and $d^T = (d_{10}, d_{20} \dots, d_{p0}) \in R^p$; in this problem A, C, a, g, e are the same as in problem (1)-(3). We call this problem *a disjoint bilinear programming problem with perfect disjoint subset* Y if system (19) has a full rank equal to m, where $m < p$, and this system possesses the following properties:

a) *system (19) does not contain redundant inequalities and its solution set* Y *is a bounded set with nonempty interior;*

b) *the set of solutions* Y' *of an arbitrary subsystem* $D'y \leq d'$ *of rank* m *with* m *inequalities of system (19) represents a convex cone with the origin(apex) at an extreme point* y' *of the set of solutions* Y *of system (19), i.e.* y' *is the solution of the system of equations* $D'y = d'$*;*

c) *at each extreme point* y' *of the set of solutions* Y *of system (19) exactly* m *hyperplanes of the facets of the polytope* Y *intersect*;

The main properties of perfect polyhedra as well as of the disjoint programming problem with a perfect disjoint subset are studied in Section 8. Note that if $Y = \{y \mid Dy \leq d\}$ is a nonempty set that has the structure of an acute-angled polytope, then it satisfies conditions a)-c) above.

3 Disjoint bilinear programming and the min-max linear problem with interdependent variables

Disjoint bilinear programming problem (1)-(3) is tightly connected with the following min-max linear programming problem with interdependent variables:

To find

$$z^* = \min_{y \in Y} \max_{u \in U(y)} (ey - a^T u) \tag{20}$$

and

$$y^* \in Y = \{y |\ By \leq b,\ y \geq 0\} \tag{21}$$

for which

$$z^* = \max_{u \in U(y^*)} (ey^* - a^T u), \tag{22}$$

where

$$U(y) = \{u \in R^q |\ -A^T u \leq Cy + g^T,\ \ u \geq 0\}. \tag{23}$$

The relationship between the solutions of problem (1)-(3) and min-max problem (20)-(23) can be obtained on the basis of the following theorems.

Theorem 1. *If (x^*, y^*) is an optimal solution of problem (1)-(3) and z^* is the optimal value of the objective function in this problem, then z^* and $y^* \in Y$ represent a solution of min-max problem (20)-(23) and vice versa: if z^* and $y^* \in Y$ represent a solution of min-max problem (20)-(23), then z^* is the optimal value of the objective function of problem (1)-(3) and y^* corresponds to an optimal point in this problem.*

Proof. The proof of the theorem is obtained from the following reduction procedure of bilinear programming problem (1)-(3) to min-max problem (20)-(23). We represent disjoint bilinear programming problem (1)-(3) as the problem of determining

$$\psi_1(y^*) = \min_{y \in Y} \psi_1(y), \tag{24}$$

where

$$\begin{cases} \psi_1(y) = \min\limits_{x \in X} (x^T C y + gx + ey), \\ \qquad\quad X = \{x \in R^n |\ Ax \leq a,\ x \geq 0\}. \end{cases} \tag{25}$$

If we replace linear programming problem (25) with respect to x by the dual problem

$$\psi_1(y) = \max_{u \in U(y)} (ey - a^T u)$$

and after that we introduce this expression in (24), then we obtain min-max problem (20)-(23). □

Similarly we can prove the following result.

Theorem 2. *If $(x^*, y^*) \in X \times Y$ is an optimal solution of disjoint bilinear programming problem (1)-(3) and z^* is the minimal value of the objective function of this problem, then z^* and x^* correspond to a solution of the following min-max linear problem:*
To find

$$z^* = \min_{x\in X} \max_{v\in V(x)} (gx - b^T v) \tag{26}$$

and

$$x^* \in X = \{x|\ Ax \leq a,\ x \geq 0\} \tag{27}$$

for which

$$z^* = \max_{v\in V(x^*)} (cx^* - b^T v), \tag{28}$$

where

$$V(x) = \{v \in R^l|\ -B^T v \leq C^T x + e^T, \quad v \geq 0\}. \tag{29}$$

Corollary 1. *If $(x^*, y^*) \in X \times Y$ is an optimal solution of the following disjoint bilinear programming problem:*
Minimize

$$z = x^T Cy + gx + ey$$

subject to

$$Ax \leq a, \quad x \geq 0, \quad Dy \leq d,$$

and z^* is the minimal value of the objective function of this problem, then z^* and x^* correspond to a solution of the following min-max linear problem:
To find

$$z^* = \min_{x\in X} \max_{v\in V(x)} (gx - d^T v)$$

and

$$x^* \in X = \{x|\ Ax \leq a,\ x \geq 0\}$$

for which

$$z^* = \max_{v\in V(x^*)} (gx^* - d^T v),$$

where

$$V(x) = \{v \in R^l|\ -D^T v = C^T x + e^T, \quad v \geq 0\}.$$

Problems (20)-(23) and (26)-(29) can be regarded as a couple of dual min-max linear problems with interdependent variables. It is easy to see that these min-max problems always have solutions if X and Y are nonempty and bounded sets.

4 Estimation of the optimal value of the objective function for problem (1)-(3)

Let L be the length of the input data of disjoint bilinear programming problem (1)-(3) with integer coefficients of matrices C, A, B and of vectors a, b, c, e [13,14], i.e.

$$L = L_1 + L_2 + (qm + q + m)(1 + \log(H + 1)),$$

where

$$L_1 = \sum_{i=0}^{q}\sum_{j=1}^{n} \log(|a_{ij}| + 1) + \log n(q + 1),$$

$$L_2 = \sum_{i=1}^{l}\sum_{j=0}^{m} \log(|b_{ij}| + 1) + \log l(m + 1),$$

$$H = \max\{|c_{ij}|, |g_i|, |e_j|,\ i = \overline{1,n},\ j = \overline{1,m}\}.$$

Then the following lemma holds.

Lemma 1. *If disjoint bilinear programming problem (1)-(3) with integer coefficients has optimal solutions, then the optimal value* z^* *of the objective function (1) is a rational number that can be expressed by an irreducible fraction* $\frac{M}{N}$ *with integer* M *and* N $(|N| \geq 1)$*, where* $|M|$ *and* $|N|$ *do not exceed* 2^L*, and* $-2^L \leq z^* \leq 2^L$*.*

Proof. If the optimal value of the objective function of problem (1)-(3) exists, then this value is attained at an extreme point (x', y') of the polyhedral set $X \times Y$ determined by (2), (3) where $x' \in X$ and $y' \in Y$ (see [1–3,9]). Then according to Lemma 1 from [20] (see also [21]) each component x'_i of x' is a rational value and it can be expressed by a fraction of form $x' = \frac{M_i^1}{N_0^1}$ with integer M_i^1 and N_0^1, where M_i^1 is a determinant of the extended matrix of system (2), N_0^1 is a nonzero determinant of matrix A and $|M_i^1|, |N_0^1| \leq \frac{2^{L_1}}{n(q+1)}$; similarly each component y'_j of y' is a rational value and it can be expressed by a fraction of form $y' = \frac{M_j^2}{N_0^2}$ with integer M_j^2 and N_0^2, where M_j^2 is a determinant of the extended matrix of system (3), N_0^2 is a nonzero determinant of matrix B and $|M_j^2|, |N_0^2| \leq \frac{2^{L_2}}{l(m+1)}$. Therefore

$$z^* = \frac{1}{N_0^1 N_0^2}\Big(\sum_{i=1}^{q}\sum_{j=1}^{m} c_{ij} M_i^1 M_j^2 + \sum_{i=1}^{q} g_i M_i^1 N_0^2 + \sum_{j=1}^{m} e_j M_j^2 N_0^1\Big)$$

where

$$|N_0^1 N_0^2| = |N_0^1||N_0^2| \leq \frac{2^{L_1+L_2}}{n(q+1)l(m+1)}$$

and

$$\Big|\sum_{i=1}^{q}\sum_{j=1}^{m} c_{ij}M_i^1M_j^2+\sum_{i=1}^{q} g_iM_i^1N_0^2+\sum_{j=1}^{m} e_jM_j^2N_0^1\Big| \le$$

$$H\Big|\sum_{i=1}^{q}\sum_{j=1}^{m} M_i^1M_j^2+\sum_{i=1}^{q} M_i^1N_0^2+\sum_{j=1}^{m} M_j^2N_0^1\Big| \le$$

$$2^{\log(H+1)}\Big(2^{L_1+L_2}+2^{L_1+L_2}+2^{L_1+L_2}\Big) \le 2^L.$$

So, the optimal value z^* of the objective function (1) is a rational number that can be represented by a fraction $\frac{M}{N}$ with integer M and N ($|N| \ge 1$), where $|M|$ and $|N|$ do not exceed 2^L, and $-2^L \le z^* \le 2^L$. □

5 An optimality criterion for disjoint bilinear programming problem (1)-(3)

Let us assume that the optimal value of the objective function of disjoint bilinear programming problem (1)-(3) is bounded. Then problem (1)-(3) can be solved by varying the parameter $h \in [-2^L, 2^L]$ in the problem of determining the consistency (compatibility) of the system

$$\begin{cases} Ax \le a, \\ x^TCy + gx + ey \le h, \\ By \le b, \\ x \ge 0,\ y \ge 0. \end{cases} \tag{30}$$

In order to study the consistency problem for system (30) we will reduce it to a consistency problem for a system of linear inequalities with a right-hand member depending on parameters using the following results.

Lemma 2. *Let solution sets X and Y of the corresponding systems (2) and (3) be nonempty and bounded. Then system (30) for a given $h \in R^1$ has no solutions if and only if the system of linear inequalities*

$$\begin{cases} -A^Tu \le Cy + g^T, \\ a^Tu < ey - h, \\ u \ge 0 \end{cases} \tag{31}$$

is consistent with respect to u for every y satisfying (3).

Proof. System (30) has no solutions if and only if for every $y \in Y$ the system of linear inequalities

$$\begin{cases} Ax \le a, \\ x^T(Cy + g^T) \le h - ey, \\ x \ge 0 \end{cases} \tag{32}$$

has no solutions with respect to x. If we apply the duality principle (Theorem 2.14 from [15]) for system (32) with respect to vector of variables x, then we obtain that it is inconsistent if and only if the system of linear inequalities

$$\begin{cases} A^T\lambda + (Cy + g^T)t \geq 0, \\ a^T\lambda + (h - ey)t < 0, \\ \lambda \geq 0, \ t \geq 0 \end{cases} \tag{33}$$

has solutions with respect to λ and t for every $y \in Y$. Note that for an arbitrary solution (λ^*, t^*) of system (33) the condition $t^* > 0$ holds. Indeed, if $t^* = 0$, then it means that the system

$$\begin{cases} A^T\lambda \geq 0, \\ a^T\lambda < 0, \\ \lambda \geq 0 \end{cases}$$

has solutions, which, according to Theorem 2.14 from [15], involves the inconsistency of system (2) that is contrary to the initial assumption. Consequently, $t^* > 0$. Since $t > 0$ in (33) for every $y \in Y$, then, dividing each inequality of this system by t and denoting $u = (1/t)\lambda$, we obtain system (31). So, system (30) is inconsistent if and only if system (31) is consistent with respect to u for every y satisfying (3). □

Corollary 2. *Let solution sets X and Y of the corresponding systems (2) and (3) be nonempty and bounded. Then for a given h system (30) has solutions if and only if there exists $y \in Y$ for which system (31) is inconsistent with respect to u. The minimal value h^* of parameter h for which such a property holds is equal to the optimal value of the objective function in disjoint bilinear programming problem (1)-(3), i.e. in this case for $h = h^*$ there exists $y = y^* \in Y$ such that system (31) is inconsistent with respect to u.*

Theorem 3. *Let solution sets X and Y of the corresponding systems (2) and (3) be nonempty and bounded. Then for a given $h \in R^1$ the system*

$$\begin{cases} Ax \leq a, \\ x^TCy + gx + ey < h, \\ By \leq b, \\ x \geq 0, \ y \geq 0 \end{cases} \tag{34}$$

is inconsistent if and only if the system of linear inequalities

$$\begin{cases} -A^Tu \leq Cy + g^T, \\ a^Tu \leq ey - h, \\ u \geq 0 \end{cases} \tag{35}$$

is consistent with respect to u for every y satisfying (3). The maximal value h^* of the parameter h for which system (35) is consistent with respect to u for every $y \in Y$ is equal to the minimal value of the objective function of disjoint bilinear programming problem (1)-(3). Moreover, for the considered systems the following properties hold:

1) If the system of linear inequalities

$$\left\{ \begin{array}{rcl} -A^Tu - Cy & \leq & g^T, \\ a^Tu - \;\; ey & < & -h^*, \\ u \geq 0 & & \end{array} \right. \tag{36}$$

is inconsistent with respect to u and y, then the system

$$\left\{ \begin{array}{c} Ax \leq a, \\ C^Tx = -e^T, \\ gx = h^*, \\ x \geq 0 \end{array} \right. \tag{37}$$

is consistent and an arbitrary solution to it x^* together with an arbitrary $y \in Y$ determine a solution (x^*, y) of disjoint bilinear programming problem (1)-(3).

2) If system (36) is consistent with respect to u and y, then there exists $y^* \in Y$ for which the system

$$\left\{ \begin{array}{r} -A^Tu \leq Cy^* + g^T, \\ a^Tu < ey^* - h^*, \\ u \geq 0 \end{array} \right. \tag{38}$$

has no solutions with respect to u. An arbitrary $y^* \in Y$ with such a property together with a solution x^* of the system of linear inequalities

$$\left\{ \begin{array}{l} Ax \leq a, \\ x^T(Cy^* + g^T) \leq h^* - ey^*, \\ x \geq 0 \end{array} \right. \tag{39}$$

with respect to x, represent an optimal solution (x^*, y^*) for disjoint bilinear programming problem (1)-(3). Moreover, h^* represents the maximal value of parameter h for which system (35) is consistent with respect to u for every $y \in Y$ and at the same time h^* represents the minimal value of parameter h in system (31) for which there exists $y = y^*$ such that system (31) is inconsistent with respect to u.

Proof. System (34) is inconsistent if and only if the system

$$\left\{ \begin{array}{l} Ax \leq a, \\ x^T(Cy + g^T) < h - ey, \\ x \geq 0 \end{array} \right. \tag{40}$$

is inconsistent with respect to x for every $y \in Y$. Taking into account that the set of solutions of system (2) is nonempty and bounded we can replace (40) by the following homogeneous system

$$\left\{ \begin{array}{l} Ax - at \leq 0, \\ x^T(Cy + g^T) - (ey - h)t < 0, \\ x \geq 0, t \geq 0 \end{array} \right. \tag{41}$$

preserving the inconsistency property with respect to x and t for every $y \in Y$. Therefore system (34) is inconsistent if and only if system (41) is inconsistent with respect to x and t for every $y \in Y$. Applying the duality principle for system (41) we obtain that it is inconsistent with respect to x and t for every $y \in Y$ if and only if system (35) is consistent with respect to u for every y satisfying (3). Based on this property we may conclude that the maximal value h^* of parameter h in system (35) for which it has solutions with respect to u for every $y \in Y$ is equal to the minimal value of the objective function in problem (1)-(3). At the same time h^*, according to Corollary 2, represents the minimal value of parameter h in system (31) for which there exists $y = y^*$ such that system (31) is inconsistent with respect to u.

Now let us prove property 1) from the theorem. Assume that system (36) is inconsistent with respect to u and y. Then the system

$$\begin{cases} -A^Tu - Cy \ - g^Tt \leq 0, \\ \quad a^Tu - ey + h^*t < 0, \\ \quad u \geq 0, \qquad\quad t \geq 0 \end{cases}$$

is inconsistent. This involves that system (37) has solutions. Then for a solution x^* of system (37) we have

$$x^{*T}Cy + gx^* + ey = y^T(C^Tx^* + e^T) + gx^* = gx^* = h^*,$$

where h^* is the minimal value of the objective function in problem (1)-(3). This means that x^* together with an arbitrary $y \in Y$ determine an optimal solution (x^*, y) of problem (1)-(3). Moreover, in this case the optimal value h^* of the objective function of the problem does not depend on the constraints (3) that define Y, i.e. Y may be an arbitrary subset from R^m.

The proof of property 2) of this theorem can be derived from Corollary 2 of Lemma 2 and presented above properties of solutions of system (35). Indeed, if h^* is the maximal value of parameter h for which system (35) has solutions with respect to u for every $y \in Y$, then according to Corollary 2, h^* can be treated as the minimal value of parameter h in (31) for which there exists $y = y^* \in Y$ such that system (38) has no solutions with respect to u. This means that system (38) is inconsistent and the corresponding homogeneous system

$$\begin{cases} -A^Tu - (Cy^* + g^T)t \ \leq 0, \\ \quad a^Tu - (ey^* - h^*)t < 0; \\ \quad u \geq 0, \qquad\quad t \geq 0 \end{cases}$$

with respect to u and t has no solutions. By applying the duality principle to this system we obtain that system (39) has solutions with respect to x. This means that $y^* \in Y$ together with a solution x^* of system (39) determine an optimal solution (x^*, y^*) of problem (1)-(3). $\square$

Corollary 3. *The linear programming problem:*
Maximize

$$z = h \tag{42}$$

subject to

$$\begin{cases} -A^T u - Cy \leq g^T, \\ \quad a^T u - ey \leq -h, \\ \quad u \geq 0 \end{cases} \tag{43}$$

has solutions if and only if the linear programming problem :
Minimize

$$z' = gx \tag{44}$$

subject to

$$\begin{cases} Ax \leq a, \\ C^T x = -e^T, \\ x \geq 0 \end{cases} \tag{45}$$

has solutions. If h^* is the optimal value of the objective function of linear programming problem (42),(43), then an optimal solution x^* of linear programming problem (44), (45) together with an arbitrary $y \in Y$ is an optimal solution of the disjoint bilinear programming problem with an arbitrary subset $Y \in R^m$.

Proof. If we dualize linear programming problem (42), (43), then we obtain the problem:
Minimize:

$$z' = gx$$

subject to

$$\begin{cases} -Ax + \ at \geq 0; \\ -C^T x - e^T t \geq 0; \\ \qquad\qquad t = 1; \\ \quad x \geq 0, \end{cases}$$

i.e. this problem is equivalent to problem (44), (45). So, problem (42), (43) has solutions if and only if problem (44), (45) has solutions. □

If in Lemma 2 we take into account that the set of solutions X of system (2) is nonempty and bounded, then we obtain the following result.

Corollary 4. *Linear programming problem (42), (43) has solutions if and only if system (45) is consistent. If system (45) is inconsistent then the objective function (42) is unbounded on the set of feasible solutions (43).*

Remark 1. Let $U_h(y)$ be the set of solutions of system (31) with respect to u for fixed $h \in R^1$ and $y \in R^m$. Additionally, let $\overline{U}_h(y)$ be the set of solutions of system (35) with respect to u for fixed $h \in R^1$ and fixed $y \in R^m$ and denote

$$Y_h = \{y \in R^m | U_h(y) \neq \emptyset\}; \quad \overline{Y}_h = \{y \in R^m | \overline{U}_h(y) \neq \emptyset\}.$$

In terms of these sets we can formulate the results above as follows:

1. For a given h system (30) has solutions if and only if $Y \not\subset Y_h$ and the minimal value h^* of h for which this property holds is equal to the optimal

value of the objective function of problem (1)-(3);

2. For a given h system (34) has no solutions if and only if $Y \subseteq \overline{Y}_h$ and the maximal value h^* of h for which this property holds is equal to the optimal value of the objective function of problem (1)-(3);

3. If system (36) is consistent, then $Y \subset \overline{Y}_h$ for $h < h^*$ and $Y \not\subset Y_h$ for $h \geq h^*$, i.e. $\overline{Y}_{h^*} \setminus Y_{h^*}$ represents the set of optimal points $y^* \in Y$ for problem (1)-(3);

4. If system (36) is inconsistent, then $Y_{h^*} = \emptyset$ and an arbitrary solution x^* of system (37) together with an arbitrary $y \in Y$ represent an optimal solution of problem (1)-(3), i.e. $Y_h = \emptyset$ for $h \geq h^*$ and $Y \subset \overline{Y}_h$ for $h < h^*$.

Corollary 5. *For a given* $h \in [-2^L, 2^L]$ *system (30) has solutions if and only if* $Y \not\subset Y_h$*; if* $Y \subset Y_h$*, then system (30) has no solutions.*

Thus, based on Theorem 3, we can replace disjoint bilinear programming problem (1)-(3) with the problem of determining the optimal value h^* of h and vector $y^* \in Y$ for which system (35) is consistent with respect to u. System (35) can be regarded as a parametric system with right-hand members that depend on the vector of parameters $y \in Y$ and $h \in R^1$. If in (30) we regard x as a vector of parameters, then we can prove a variant of Theorem 3 in which parametrical system (35) is replaced by the parametrical system

$$\begin{cases} -B^T v \leq C^T x + e^T, \\ \quad b^T v \leq gx - h, \\ \qquad v \geq 0, \end{cases} \tag{46}$$

where v is the column vector of variables in this system and x is the vector of parameters that satisfies (2). This means that for the considered parametric linear systems (35) and (46) we can formulate the following duality principle (see [16]):

Theorem 4. *The system of linear inequalities (35) is consistent with respect to* u *for every* y *satisfying (3) if and only if the system of linear inequalities (46) is consistent with respect to* v *for every* x *satisfying (2).*

All results formulated and proved in this section are also valid for the case when Y is determined by an arbitrary consistent system

$$Dy \leq d, \tag{47}$$

where $D = (d_{ij})$ is a $p \times m$ matrix and d is a column vector with p components. In the case when system (3) is replaced by system (47), the following duality principle holds.

Theorem 5. *The system of linear inequalities (35) is consistent with respect to* u *for every* y *satisfying (47) if and only if the system*

$$\begin{cases} -D^T v = C^T x + e^T, \\ d^T v \le gx - h, \\ v \ge 0 \end{cases} \tag{48}$$

is consistent with respect to v *for every* x *satisfying (2).*

6 The approach for solving the problem based on Theorem 3

The general scheme of the approach we shall use for solving the disjoint bilinear programming problem is based on Theorem 3 and is as follows:

We replace problem (1)-(3) by problem of determining the maximal value h^* of parameter h such that system (35) is consistent with respect to u for every y satisfying (3). Then we show how to determine the corresponding point x^* that satisfies the conditions of Theorem 2. To apply this approach it is necessary to develop algorithms for checking conditions 1) and 2) of Theorem 3, i.e. it is necessary to develop algorithms for checking if the condition $Y \not\subseteq Y_h$ holds for a given $h \in [-2^L, 2^L]$. In general case, the problem of checking such a condition is a difficult problem from computational point of view, however for some special cases suitable algorithms can be developed. In this article we show that if Y has a structure of an acute-angled polytope or perfect polytope, then the condition $Y \not\subseteq Y_h$ can be checked in polynomial time. Based on this we show how to determine the optimal solution of the disjoint bilinear programming problem when one of the disjoint subsets has the structure of an acute-angled polytope. To do this, in Section 8 we prove Theorem 8, that represents a refinement of Theorem 3 for the case when Y has a structure of the perfect polytope, and based on this, we show how to check if $Y \not\subseteq Y_h$.

In general, if Y is a bounded set then condition $Y \not\subseteq Y_h$ can be detected in finite time by checking the consistency (or inconsistency) of system (31) for all extreme points of Y. In analogous way condition $Y \subseteq \overline{Y}_h$ can be verified by checking the consistency of system (35) with respect to u for each extreme point of Y. It is evident that such an approach for checking condition $Y \subseteq Y_h$ (or $Y \not\subseteq Y_h$) is complicated from computational point of view. The algorithm we propose for the case when Y is a perfect polytope, avoids exhaustive search. Moreover, we show that in the case of problems (5)-(7) and (13)-(16) our approach allows to elaborate efficient algorithms for solving them.

7 Some auxiliary results related to redundant inequalities for linear systems

In this section we shall use some properties of redundant inequalities for linear systems in the cases of consistent and inconsistent systems. An inequality

$$\sum_{j=1}^{m} s_j y_j \le s_0 \tag{49}$$

is called *redundant* for a consistent system of linear inequalities

$$\sum_{j=1}^{m} d_{ij} y_j \le d_{i0}, \qquad i = 1, 2, \dots, p, \tag{50}$$

if (49) holds for an arbitrary solution of system (50). We call the redundant inequality (49) *non-degenerate* if $s_j \neq 0$ at least for an index $j \in \{1, 2, \dots, m\}$. If $s_j = 0$, $j = 1, 2, \dots, m$, and $s_0 \ge 0$, we say that the redundant inequality (49) is *degenerate*. We call the redundant inequality (49) for consistent system (50) *strongly redundant* if there exists $\epsilon > 0$ such that the corresponding inequality

$$\sum_{j=1}^{m} s_j y_j \le s_0 - \epsilon$$

is redundant for (50); if such an ϵ does not exist then we call inequality (49) *weakly redundant*. If an inequality

$$\sum_{j=1}^{m} d_{kj} y_j \le d_{k0} \tag{51}$$

of system (50) can be omitted without changing the set of its feasible solutions then we say that it is redundant in (50), i.e., inequality (51) is redundant in (50) if it is redundant for the system of the rest of its inequalities.

The redundancy property of linear inequality (49) for consistent system (50) can be checked based on following Farkas theorem [17] (see also [15, 18]:

Theorem 6. *Inequality (49) is redundant for consistent system (50) if and only if the system*

$$\begin{cases} s_j = \sum\limits_{i=1}^{p} d_{ij} v_i, \qquad j = 1, 2, \dots, m; \\ s_0 = \sum\limits_{i=1}^{p} d_{i0} v_i + v_0; \\ \qquad\qquad v_i \ge 0, \ \ i = 0, 1, 2, \dots, p, \end{cases} \tag{52}$$

has solutions with respect to $v_0, v_1, v_2, \dots, v_p$. Moreover, if inequality (49) is redundant for system (50), then system (52) has a basic feasible solution $v_0^*, v_1^*, v_2^*, \dots, v_p^*$ that satisfies the following conditions:

1) the set of column vectors

$$\left\{ D_i = \begin{pmatrix} d_{i1} \\ d_{i2} \\ \vdots \\ d_{im} \end{pmatrix} : \; v_i^* > 0, \; i \in \{1, 2, \dots, p\} \right\}$$

is linearly independent;

2) inequality (49) is redundant for the subsystem of system (49) induced by the inequalities that correspond to indices $i \in \{1, 2, \dots, p\}$ *with* $v_i^* > 0$*;*

3) if $v_0^* > 0$*, then inequality (49) is strongly redundant for system (50) and if* $v_0^* = 0$*, then inequality (49) is weakly redundant for system (50).*

The proof of this theorem can be found in [15,17,18].

Corollary 6. *Let redundant inequality (49) for consistent system (50) be given and consider the following linear programming problem:*
Minimize

$$z = \sum_{i=1}^{p} d_{i0} v_i \tag{53}$$

subject to

$$\begin{cases} s_j = \sum\limits_{i=1}^{p} d_{ij} v_i, \qquad j = 1, 2, \dots, m; \\ \qquad\qquad v_i \geq 0, \quad i = 1, 2, \dots, p. \end{cases} \tag{54}$$

Then this problem has an optimal solution $v_1^*, v_2^*, \dots, v_p^*$ *where* $s_0 \geq \sum\limits_{i=1}^{p} d_{i0} v_i^*$*. If* $s_0 > \sum\limits_{i=1}^{p} d_{i0} v_i^*$ *then inequality (49) is strongly redundant for system (50) and if* $s_0 = \sum\limits_{i=1}^{p} d_{i0} v_i^*$ *then inequality (49) is weakly redundant for system (50).*

Theorem 6 can be extended for the case when system (50) is inconsistent.

Definition 1. Assume that system (50) is inconsistent. Inequality (49) is called redundant for inconsistent system (50) if there exists a consistent subsystem

$$\sum_{j=1}^{m} d_{i_k j} y_j \leq d_{i_k 0}, \; k = 1, 2, \dots, p' \; (p' < p), \tag{55}$$

of system (50) such that inequality (49) is redundant for (55).

Theorem 7. *Inequality (49) is redundant for inconsistent system (50) if and only if system (52) has a basic feasible solution $v_0, v_1, v_2, \ldots, v_p$ for which the set of column vectors*

$$D^+ = \left\{ \begin{pmatrix} d_{i1} \\ d_{i2} \\ \vdots \\ d_{im} \end{pmatrix} : \; v_i > 0, \; i \in \{1, 2, \ldots, p\} \right\} \tag{56}$$

is linearly independent. Moreover, the subsystem of inconsistent system (50) induced by inequalities that correspond to indices with $v_i^0 > 0$ is a consistent subsystem of system (50).

Proof. $\Rightarrow$ Assume that inequality (49) is redundant for inconsistent system (50). Then, there exists a consistent subsystem (55) of system (50) such that (49) is redundant for (55). Then according to Theorem 6 there exists a basic feasible solution $v_0, v_{i_1}, v_{i_2}, \ldots, v_{i_{p'}}$ for the system

$$\begin{cases} s_j = \sum\limits_{k=1}^{p'} d_{i_k j} v_{i_k}, & j = 1, 2, \ldots, m; \\ s_0 = \sum\limits_{k=1}^{p'} d_{i_k 0} v_{i_k} + v_0; & \\ \qquad v_0 \geq 0, \; v_{i_k} \geq 0, & k = 1, 2, \ldots, p', \end{cases}$$

such that the set of column vectors

$$\left\{ d_{i_k} = \begin{pmatrix} d_{i_k 1} \\ d_{i_k 2} \\ \vdots \\ d_{i_k m} \end{pmatrix} : \; v_{i_k} > 0, \; k \in \{1, 2, \ldots, p'\} \right\}$$

is linearly independent.

$\Leftarrow$ Let (50) be an arbitrary inconsistent system and $v_0, v_1, v_2, \ldots, v_p$ be a solution of system (52) that contains $p' \geq 1$ nonzero components $v_{i_1}, v_{i_2}, \ldots, v_{i_{p'}}$ such that the corresponding system of column vectors $\{d_{i_k} : v_{i_k} > 0, k = 1, 2, \ldots, p'\}$ is linearly independent. Then $p' \leq \min\{m, p\}$ and the corresponding system

$$\sum_{j=1}^{m} d_{i_k j} y_j \leq d_{i_k 0}, \qquad k = 1, 2, \ldots, p',$$

has solutions. Based on Theorem 6 we obtain that inequality (49) is redundant for system (55). This means that inequality (49) is redundant for inconsistent system (50). $\square$

8 The optimality criterion for the disjoint bilinear programming problem with a perfect disjoint subset

In this section we present a refinement of Theorem 3 for the disjoint bilinear programming problem (17)-(19) with conditions $a) - c)$ for system (19). In fact, this refinement is related to the case when $Y = \{y|By \leq b,\ y \geq 0\}$ is replaced by $Y = \{y|Dy \leq d\}$ that satisfies conditions $a) - c)$. We show that in this case the optimality criterion for problem (17)-(19) with conditions $a) - c)$ can be formulated in terms of the existence of a basic solution with the given basic component for a system of linear equations with nonnegative conditions for the variables. We present the optimality criterion in new terms for problem (17)-(19) in the following extended form:
Minimize

$$z = \sum_{i=1}^{n}\sum_{j=1}^{m} c_{ij}x_i y_j + \sum_{i=1}^{n} g_i x_i + \sum_{j=1}^{m} e_j y_j \tag{57}$$

subject to

$$\begin{cases} \displaystyle\sum_{j=1}^{n} a_{ij}x_j \leq a_{i0}, & i = 1, 2, \ldots, q, \\ \qquad x_j \geq 0, & j = 1, 2, \ldots, n; \end{cases} \tag{58}$$

$$\sum_{j=1}^{m} d_{ij}y_j \leq d_{i0}, \quad i = 1, 2, \ldots, p \ \ (m < p), \tag{59}$$

So, we will assume that the set of solutions Y of system (59) in this problem satisfies the following conditions:

a) *system (59) does not contain redundant inequalities and the set of its solutions Y is a bounded set with nonempty interior;*

b) *the set of solutions of an arbitrary subsystem*

$$\sum_{j=1}^{m} d_{i_k j}y_j \leq d_{i_k 0}, \qquad k = 1, 2, \ldots, m,$$

of rank m represents a convex cone $Y^-(y^r)$ with the origin at an extreme point y^r from the set of extreme points $\{y^1, y^2, \ldots, y^N\}$ of the set of solutions Y of system (59);

c) *at each extreme point $y^r \in \{y^1, y^2, \ldots, y^N\}$ of the set of solutions Y of system (59) exactly m hyperplanes of the facets of polytope Y intersect.*

It is easy to see that if $D = \begin{pmatrix} B \\ -I \end{pmatrix}$, $d = \begin{pmatrix} b \\ 0 \end{pmatrix}$, I is the identity matrix and 0 is the column vector with zero components, then problem (57)-(59) becomes problem

(1)-(3). Additionally if matrix B is an identity one or step-diagonal, then we obtain a disjoint bilinear programming problems for which conditions $a) - c)$ hold.

The main results we describe in this section are concerned with elaboration of an algorithm that determines if the property $Y \not\subset Y_h$ holds.

8.1 The properties of the extreme points for the set of solutions of system (59)

Let $y^r = (y_1^r, y_2^r, \ldots, y_m^r), r = 1, 2, \ldots N$ be the extreme points of the set of solutions Y of system (59) that satisfies conditions $a) - c)$. Then for each $y^r, r \in \{1, 2, \ldots, N\}$, there exists a unique subsystem

$$\sum_{j=1}^{m} d_{i_k j} y_j \le d_{i_k 0}, \qquad k = 1, 2, \ldots, m, \tag{60}$$

of rank m of system (59) such that $y_1^r, y_2^r, \ldots, y_m^r$ is the solution of the system of linear equations

$$\sum_{j=1}^{m} d_{i_k j} y_j = d_{i_k 0}, \qquad k = 1, 2, \ldots, m. \tag{61}$$

Denote

$$I(y^r) = \{i \in \{1, 2, \ldots, p\} : \sum_{j=1}^{m} d_{ij} y_j^r = d_{i0}\}$$

and consider the *convex cone* $Y^-(y^r)$ *for system (59)* as the solution set of the following system

$$\sum_{j=1}^{m} d_{ij} y_j \le d_{i0}, \quad i \in I(y^r), \tag{62}$$

where $y^r = (y_1^r, y_2^r, \ldots, y_m^r)$ is the *origin(apex) of the cone* $Y^-(y^r)$ and $|I(y^r)| = m$. We call the solution set of the system

$$\sum_{j=1}^{m} d_{ij} y_j \ge d_{i0}, \quad i \in I(y^r)$$

the symmetrical cone for $Y^-(y^r)$ and denote it by $Y^+(y^r)$. Obviously, $Y^-(y^r)$, $Y^+(y^r)$ represent convex polyhedral sets with interior points such that $Y = \cap_{r=1}^{N} Y^-(y^r)$ and $Y^+(y^r) \cap Y^-(y^r) = y^r,\ r = 1, 2, \ldots, N$.

Based on mentioned above properties of perfect set Y, the following result can be proved.

Lemma 3. *If* $Y^+(y^1),\ Y^+(y^2),\ \ldots,\ Y^+(y^N)$ *represent the symmetrical cones for the corresponding cones* $Y^-(y^1),\ Y^-(y^2),\ \ldots,\ Y^-(y^N)$ *of system (59) with properties* $a) - c)$, *then* $Y^+(y^r) \bigcap Y^+(y^k) = \emptyset$ *for* $r \ne k$. *Additionally, if* $z^1,\ z^2,\ \ldots,\ z^N$ *represent arbitrary points of the corresponding cones* $Y^+(y^1),\ Y^+(y^2), \ldots, Y^+(y^N)$, *then the convex hull* $Conv\big(z^1, z^2, \ldots, x^N\big)$ *of points* $z^1,\ z^2,\ \ldots, z^N$ *contains* Y.

Proof. The property $Y^+(y^r) \cap Y^+(y^k) = \emptyset$ for $r \neq k$ can be proven by contradiction. Indeed, if we assume that $Y^+(y^r) \cap Y^+(y^k) \neq \emptyset$, then this polyhedral set contains an extreme point y^0, where $y^0 \notin Y$, because $Y^+(y^r) \cap Y^+(y^k)$ is determined by the system of inequalities consisting of inequalities that define $Y^+(y^r)$ and inequalities that define $Y^+(y^k)$. This means that $y^0 = (y_1^0, y_2^0, \dots, y_m^0)$ is the solution for a system of equations

$$\sum_{j=1}^{m} d_{i_k j} y_j = d_{i_k 0}, \qquad k \in \{1, 2, \dots, m\},$$

of rank m. Then according to properties $a) - c)$ we obtain that y^0 is a solution of system (59) which is in contradiction with the fact that $y^0 \notin Y$. So, $Y^+(y^r) \cap Y^+(y^k) = \emptyset$ for $r \neq k$.

Now let us show that if z^1, z^2, $\dots$, z^N represent arbitrary points of the corresponding sets $Y^+(y^1)$, $Y^+(y^2)$, $\dots$, $Y^+(y^N)$, then the convex hull $Conv(z^1, z^1, z^2, \dots, z^N)$ of points $z^1, z^1, z^2, \dots, z^N$ contains Y. Indeed, if we construct the convex hull $Y^1 = Conv(z^1, y^2, \dots, y^N)$ of points $z^1, y^2, \dots, y^N$, then $y^1 \in Y^1$ and $Y \subseteq Y^1$. If after that we construct the convex hull $Y^2 = Conv(z^1, z^2, y^3 \dots, y^N)$ of points $(z^1, z^2, y^3, \dots, y^N)$, then $y^2 \in Y^2$ and $Y^1 \subseteq Y^2$ and so on. Finally at step N we construct the convex hull $Y^N = Conv(z^1, z^2, \dots, z^N)$ of points $z^1, z^2, \dots, z^N$ where $y^N \in Y^N$ and $Y^{N-1} \subseteq Y^N$, i.e. $Y \subseteq Y^1 \subseteq Y^2 \subseteq \cdots \subseteq Y^N = Conv\big(z^1, z^2, \dots, z^N\big)$. □

From this lemma as a corollary we obtain the following result.

Corollary 7. *If system (59) satisfies conditions $a) - c)$, then the following system*

$$\sum_{j=1}^{m} d_{ij} y_j \geq d_{i0}, \qquad i = 1, 2, \dots, p,$$

is inconsistent and the inequalities of this system can be divided into N disjoint consistent subsystems

$$\sum_{j=1}^{m} d_{ij} y_j \geq d_{i0}, \qquad i \in I(y^r),\ r = 1, 2, \dots, N$$

such that $Y^+(y^l) \cap Y^+(y^k) = \emptyset$ for $l \neq k$.

Another important result for perfect set Y is the following lemma.

Lemma 4. *Assume that system (59) satisfies conditions $a) - c)$. If an inequality*

$$-\sum_{j=1}^{m} s_j y_j \leq -s_0 \tag{63}$$

is redundant for the inconsistent system

$$-\sum_{j=1}^{m} d_{ij} y_j \leq -d_{i0}, \qquad i = 1, 2, \ldots, p, \tag{64}$$

then such an inequality is redundant at least for a consistent subsystem

$$-\sum_{j=1}^{m} d_{ij} y_j \leq -d_{i0}, \qquad i \in I(y^r), \ r \in \{1, 2, \ldots, N\},$$

of inconsistent system (64).

Proof. Assume that inequality (63) is redundant for inconsistent system (64). Then, there exists a consistent subsystem of rank p'

$$-\sum_{j=1}^{m} d_{i_k j} y_j \leq -d_{i0}, \qquad k = 1, 2, \ldots p',$$

for inconsistent system (64) such that $p' \leq m$ and inequality (63) is redundant for this system. If $p' = m$, then the set of solution of this system will represent a convex cone $Y^+(y^r), r \in \{1, 2, \ldots, N\}$, i.e in this case lemma holds. If $p' < m$, then the set of solutions of this system contains at least a convex cone $Y^+(y^r), r \in \{1, 2, \ldots, N\}$, so lemma holds. □

8.2 A criterion for checking if $Y \not\subset Y_h$ based on properties of redundant inequalities for inconsistent systems

In this subsection we present a criterion for checking the condition $Y \not\subset Y_h$ that we shall use in the next subsection for the proof of the optimality criterion for problem (57)-(59). We formulate and prove such a criterion by using the results from Section 7 related to redundant inequalities for an inconsistent system of linear inequalities.

If for problem (57)-(59) we consider system (35) (see Theorem 3)

$$\begin{cases} -\sum_{j=1}^{q} a_{ji} u_j - \sum_{j=1}^{m} c_{ij} y_j \leq g_i, \qquad i = 1, 2, \ldots, n, \\ \sum_{j=1}^{q} a_{j0} u_j - \sum_{j=1}^{m} e_j y_j \leq -h, \\ \qquad u_j \geq 0, \qquad j = 1, 2, \ldots, q, \end{cases} \tag{65}$$

then either this system has solutions with respect to u_1,$u_2, \ldots$,$u_q, y_1, y_2, \ldots$,y_m for an arbitrary $h \in R^1$ or there exists a minimal value h^* of h for which this system has a solution. According to Corollary 3 if for system (65) there exists a minimal value h^* for which it is consistent, then the optimal solution of problem (57)-(59) can be found by solving linear programming problem (44), (45). Therefore in what follows we will analyze the case when system (65) has solutions for every $h \in R^1$, i. e. the case when system (45) has no solutions.

Lemma 5. *Let disjoint bilinear programming problem (57)-(59) be such that system (59) satisfies conditions* $a) - c)$ *and the set of solutions* X *of system (58) is nonempty and bounded. If system (45) has no solutions, then for a given* h *the property* $Y \not\subset Y_h$ *holds if and only if for consistent system (65) there exists a non-degenerate redundant inequality*

$$\sum_{j=1}^{m} s_j y_j \le s_0 \tag{66}$$

such that the corresponding symmetrical inequality (63) is redundant for the inconsistent system (64).

Proof. $\Rightarrow$Assume that system (45) has no solutions. Then, according to Corollary 4 of Theorem 3, we have $Y_h \neq \emptyset$. Therefore, if $Y \not\subset Y_h$, then among the extreme points $y^1, y^2, \dots, y^N$ of Y there exists at least one that does not belong to Y_h. Denote by $y^1, y^2, \dots, y^{N'}$ the extreme points of Y that do not belong to Y_h and by $y^{N'+1}, y^{N'+2}, \dots, y^N$ the extreme points of Y that belong to Y_h. Each extreme point $y^l = (y_1^l, y_2^l, \dots, y_m^l)$, $l \in \{1, 2, \dots, N\}$, of Y represents the apex of the cone $Y^-(y^l)$ that is determined by the solution set of system (62) for $r = l$. At the same time each extreme point y^l of Y is the apex of the symmetrical cone $Y^+(y^l)$ that is determined by the solution set of the subsystem of linear inequalities

$$-\sum_{j=1}^{m} d_{ij} y_j \le -d_{i0}, \qquad i \in I(y^l), \tag{67}$$

of inconsistent system (64). According to Lemma 3 and Corollary 7 we have $Y^+(y^l) \cap Y^+(y^k) = \emptyset$ for $k \neq l$. Let us show that among the extreme points $y^1, y^2, \dots, y^{N'}$ there exists a point y^{j_0} for which the corresponding cone $Y^+(y^{j_0})$ has no common points with Y_h, i.e. $Y^+(y^{j_0}) \bigcap Y_h = \emptyset$. This fact can be proved using the rule of contraries. If we assume that $Y^+(y^l) \bigcap Y_h \neq \emptyset$, $l = 1, 2, \dots, N'$, then we can select from each set $Y^+(y^l) \bigcap Y_h$ an element z^l and construct the convex hull $Conv(z^1,\ z^2,\ \dots,\ z^{N'}, y^{N'+1},\ y^{N'+2},\ \dots,\ y^N)$ for the set of the points $\{z^1,\ z^2,\ \dots, z^{N'},\ y^{N'+1},\ y^{N'+2},\ \dots,\ y^N\}$. Taking into account that $z^r \in Y_h$, $r = 1,\ 2,\ \dots,\ N'$ and $y^{N'+l} \in Y_h$, $r = 1, 2, \dots, N - N'$ we have $Conv(z^1, z^2, \dots, z^{N'}, y^{N'+1}, y^{N'+2}, \dots, y^N) \subseteq Y_h$. However according to Lemma 3 we have $Y \subseteq Conv(z^1, z^2, \dots, z^{N'}, y^{N'+1}, y^{N'+2}, \dots, y^N)$, i.e. we obtain $Y \subseteq Y_h$. This is in contradiction with the condition $y^r \notin Y_h$ for $r = 1, 2, \dots, N'$.

Thus, among $Y^+(y^1), Y^+(y^2), \dots, Y^+(y^{N'})$ there exists a cone $Y^+(y^{j_0})$ with apex y^{j_0} for which $Y^+(y^{j_0}) \bigcap Y_h = \emptyset$. Therefore for convex sets $Y^+(y^{j_0})$ and Y_h there exists a separating hyperplane [18, 19]

$$\sum_{j=1}^{m} s_j y_j = s_0$$

such that

$$\sum_{j=1}^{n} s_j y_j < s_0, \quad \forall (y_1, y_2, \dots, y_n) \in Y_h,$$

and

$$-\sum_{j=1}^{m} s_j y_j \leq -s_0, \quad \forall (y_1, y_2, \dots, y_n) \in \ Y^+(y^{j_0}).$$

So, the inequality $\sum_{j=1}^{n} s_j y_j \leq s_0$ is redundant for system (65) and the inequality $-\sum_{j=1}^{n} s_j y_j \leq -s_0$ is redundant for consistent subsystem (67) of inconsistent system (64). Moreover, according to Lemma 4, the inequality $-\sum_{j=1}^{n} s_j y_j \leq -s_0$ is redundant at least for a consistent subsystem (67) of inconsistent system (64).

$\Leftarrow$ Assume that for system (65) there exists a non-degenerate redundant inequality (66) such that the corresponding inequality (63) is redundant for inconsistent system (64). Then there exists a consistent subsystem (67) of system (64) for which the conical subset $Y^-(y^{i_0})$ has no common points with Y_h where $Y_h \neq \emptyset$, i.e. $y^{j_0} \notin Y_h$. Taking into account that $y^{j_0} \in Y$ we obtain $Y \not\subset Y_h$. □

Corollary 8. *Assume that the conditions of Lemma 5 are satisfied. Then the minimal value z^* of the objective function of problem (57)-(59) is equal to the minimal value h^* of parameter h in system (65) for which there exists a non-degenerate redundant inequality*

$$\sum_{j=1}^{m} s_j^* y_j \leq s_0^* \tag{68}$$

such that the corresponding symmetrical inequality

$$-\sum_{j=1}^{m} s_j^* y_j \leq -s_0^* \tag{69}$$

is redundant for inconsistent system (64). An optimal point y^* for problem (57)-(59) can be found by solving the following system

$$\begin{cases} \sum_{j=1}^{m} d_{ij} y_j \leq d_{i0}, \qquad i = 1, 2, \dots, p \ , \\ \sum_{j=1}^{m} s_j^* y_j = s_0^*. \end{cases} \tag{70}$$

Proof. Assume that for system (65) with given h there exists a non-degenerate redundant inequality (66) such that symmetrical inequality (63) is redundant for system (64). Then according to Corollary 2 of Lemma 2 and Theorem 3, for problem (57)-(59) there exists a feasible solution such that the corresponding value of the objective function is not greater than h. Therefore the minimal value z^* of the objective function of problem (57)-(59) is equal to minimal value h^* of parameter h in system (65) for which there exists a non-degenerate redundant inequality (68) such that the corresponding symmetrical inequality (69) is redundant for inconsistent system (64). In this case $y^{j_0} = y^* \in Y \cap bd(\overline{Y}_{h^*})$ and the optimal point $y^* \in Y$ can be found by solving system (70). □

8.3 The optimality criterion for the disjoint bilinear programming problem (57)-(59)

Based on results from previous subsection we can prove the following optimality criterion for problem (57)-(59).

Theorem 8. *Let disjoint bilinear programming problem (57)-(59) be such that system (59) satisfies conditions a) – c) and the set of solutions X of system (58) is nonempty and bounded. If system (45) has no solutions, then for a given h the property* $Y \not\subset Y_h$ *holds if and only if the following system*

$$\begin{cases} \sum_{j=1}^{n} a_{ij}x_j + x_{n+i} = a_{i0}, & i = 1,2,\dots,q, \\ \sum_{i=1}^{n} c_{ij}x_i + \sum_{k=1}^{p} d_{kj}v_k = -e_j, & j = 1,2,\dots,m, \\ \sum_{i=1}^{n} g_i x_i - \sum_{k=1}^{p} d_{k0}v_k + v_{p+1} = h, & \\ \quad x_i \ge 0,\ i = 0,1,2,\dots,n+q;\ v_k \ge 0,\ k = 1,2,\dots,p+1, & \end{cases} \tag{71}$$

has a basic feasible solution $x_1^0, x_2^0, \dots, x_{n+q}^0, v_1^0, v_2^0, \dots, v_{p+1}^0$ *for which the set of vectors*

$$\left\{ D_k = \begin{pmatrix} d_{k1} \\ d_{k2} \\ \vdots \\ d_{km} \end{pmatrix} : \ v_k^0 > 0, \quad k \in \{1,2,\dots,p\} \right\} \tag{72}$$

is linearly independent. The minimal value h^* *of parameter* h *with the property* $Y \not\subset Y_h$ *is equal to the optimal value of the objective function of problem (57)-(59). Moreover, for* $h = h^*$ *system (71) has a basic solution* $x_1^*, x_2^*, \dots, x_{n+q}^*, v_1^*, v_2^*, \dots, v_{p+1}^*$ *with the degenerate basic component* $v_{p+1}^* = 0$.

Proof. According to Lemma 5 the condition $Y \not\subset Y_h$ holds if and only if there exist $s_0, s_1, s_2, \dots, s_n$ such that (66) is a non-degenerate redundant inequality for system

(65) and (63) is a redundant inequality for inconsistent system (64). If for (66) and 65) we apply the Minkowski-Farkas theorem (Theorem 6), then we obtain that (66) is redundant for (65) if and only if there exist $x_0, x_1, x_2, \dots, x_{n+q}, t$ such that

$$\begin{cases} 0 = -\sum\limits_{j=1}^{n} a_{ij}x_j - x_{n+i} + a_{i0}t, & i = 1, 2, \dots, q, \\ s_j = -\sum\limits_{i=1}^{n} c_{ij}x_i - e_jt, & j = 1, 2, \dots, m, \\ s_0 = \sum\limits_{i=1}^{n} g_ix_i - ht + x_0, & \\ \qquad\qquad x_j \ge 0, \ i = 0, 1, 2, \dots, n+q; \ t \ge 0. & \end{cases} \tag{73}$$

Now, let us apply Theorem 7 for inequality (63) and inconsistent system (64). According to this theorem, inequality (63) is redundant for system (64) if and only if there exist $v_0, v_1, v_2, \dots, v_m$ such that

$$\begin{cases} -s_j = -\sum\limits_{k=1}^{p} d_{kj}v_k, & j = 1, 2, \dots, m, \\ -s_0 = -\sum\limits_{k=1}^{n} d_{k0}v_k + v_0, & \\ \qquad\qquad v_k \ge 0, \quad k = 0, 1, 2, \dots, p \ , & \end{cases} \tag{74}$$

where $\sum_{k=1}^{p} d_{kj}v_i \neq 0$ at least for an index $j \in \{1, 2, \dots, m\}$ and the nonempty set of column vectors (72) is linearly independent.

So, Lemma 5 can be formulated in terms of solutions of systems (73), (74) as follows: the condition $Y \not\subset Y_h$ holds if and only if there exist s_0, s_1, $s_2, \dots, s_m$, $x_0, x_1, x_2, \dots, x_{n+q}$, $v_0, v_1, v_2, \dots, v_{p+1}, t$ that satisfy (73), (74), where $\sum_{k=1}^{p} d_{kj}v_k \neq 0$ at least for an index $j \in \{1, 2, \dots, m\}$ and the set of column vectors (72) is linearly independent. If we eliminate $s_1, s_2, \dots, s_m$ from (73) by introducing (74) in (73) and after that denote $v_{p+1} = v_0 + x_0$, then we obtain the following system

$$\begin{cases} \sum\limits_{j=1}^{n} a_{ij}x_j + x_{n+i} = a_{i0}t, & i = 1, 2, \dots, q, \\ \sum\limits_{i=1}^{n} c_{ij}x_i + \sum\limits_{k=1}^{p} d_{kj}v_k = -e_jt, & j = 1, 2, \dots, m, \\ \sum\limits_{i=1}^{n} g_ix_i - \sum\limits_{k=1}^{p} d_{k0}v_k + v_{p+1} = ht; & \\ \quad x_j \ge 0, \ j = 1, 2, \dots, n+q; \ v_k \ge 0, k = 1, 2, \dots, p+1, \ t \ge 0. & \end{cases} \tag{75}$$

This means that Lemma 5 in terms of solutions of system (75) can be formulated as follows: the property $Y \not\subset Y_h$ holds if and only if system (75) has a solution

$x_1,\ x_2,\ \ldots, x_{n+q},\ v_1,\ v_2\ ,\ldots,\ v_p,\ v_{p+1},\ t$ where $\sum_{k=1}^{p} d_{kj}v_i \neq 0$ at least for an index $j \in \{1,2,\ldots,m\}$ and the nonempty set of column vectors (72) is linearly independent. In (75) the subsystem

$$\begin{cases} \displaystyle\sum_{j=1}^{n} a_{ij}x_j + x_{n+i} = a_{i0}t, \qquad i = 1,2,\ldots,q, \\ x_j \geq 0,\ j = 1,2,\ldots,n+q, t \geq 0, \end{cases} \tag{76}$$

has a nonzero solution $x_1, x_2,\ \ldots,\ x_{n+q}, t$ if and only if $t > 0$, because the set of solutions of system (58) is nonempty and bounded. Based on this and taking into account that we are seeking for a solution of system (75) such that $\sum_{k=1}^{p} d_{kj}v_k^0 \neq 0$ at least for an index $j \in \{1,2,\ldots,m\}$, we can set $t = 1$ in (75) and therefore finally obtain that $Y \not\subset Y_h$ if and only if system (71) has a solution $x_1^0,\ x_2^0,\ \ldots,\ x_{n+q}^0,\ v_1^0,\ v_2^0,\ \ldots,\ v_{p+1}^0$ for which the set of column vectors (72) is linearly independent and $\sum_{k=1}^{p} d_{kj}v_k^0 \neq 0$ at least for an index $j \in \{1,2,\ldots,m\}$.

If h^* is the minimal value of parameter h for which system (71) has a basic feasible solution where the system of vectors (72) is linearly independent, then according to Corollary 8 of Lemma 5, h^* is equal to the optimal value of the objective function of problem (57)-(59). Obviously, if $h = h^*$, then system (71) has a basic solution $x_1^*, x_2^*, \ldots, x_{n+q}^*, v_1^*, v_2^*, \ldots, v_{p+1}^*$ with the degenerate basic component $v_{p+1}^* = 0$. □

Corollary 9. *Let disjoint bilinear programming problem (57)-(59) be such that system (59) satisfies conditions a) − c) and the set of solutions X of system (58) is nonempty and bounded. If system (45) has no solutions, then for a given h the property $Y \not\subset Y_h$ holds if and only if system (71) has a basic feasible solution $x_1^0, x_2^0, \ldots, x_{n+q}^0, v_1^0, v_2^0, \ldots, v_{p+1}^0$, where v_{p+1}^0 is a basic component. The minimal value h^* of parameter h for which system (71) has a basic solution $x_1^*, x_2^*, \ldots, x_{n+q}^*, v_1^*, v_2^*, \ldots, v_{p+1}^*$ with the degenerate basic component $v_{p+1}^* = 0$ represents the optimal value of objective function of problem (57)-(59).*

Remark 2. Theorem 8 holds also when for a given h system (71) has a basic feasible solution $x_1^0, x_2^0, \ldots, x_{n+q}^0, v_1^0, v_2^0, \ldots, v_{p+1}^0$ for which the set of vectors (72) is an empty set because, based on Corollary 3, we obtain that $x^* = (x_1^0, x_2^0, \ldots, x_n^0)^T$ is an optimal point for the disjoint bilinear programming problem (57)-(59). In this case x^* together with an arbitrary $y \in Y$ is an optimal solution of problem (57)-(59).

Remark 3. From Theorem 8 we can derived the conditions for the existence of a boolean solution for system (4) if in (71) we take

$h^* = 0,\quad D = \begin{pmatrix} I_n \\ -I_n \end{pmatrix},\ d = \begin{pmatrix} \mathbf{i_n} \\ 0 \end{pmatrix}$, where I_n is an identity matrix, $\mathbf{i_n}$ is a column vector with units elements, 0 is a column vector with zero components and

$C = (c_{ij})_{n\times n},\ \ g = (g_1, g_2, \ldots, g_n),\ \ e = (e_1, e_2, \ldots, e_n)$ are defined as

$$c_{ij} = \begin{cases} -2 & \text{if } i = j, \\ 0 & \text{if } i \neq j, \end{cases} \quad g_i = 1,\ e_j = 1,\quad i = 1,2,\ldots n; j = 1,2,\ldots,n.$$

Remark 4. Theorem 8 is valid for problem (57)-(59) and it is not valid for problem (1)-(3) in general case. Additionally, if for a given $h \in [2^{-L}, 2^L]$ system (71) has a basic solution $x_1^0, x_2^0, \ldots, x_{n+q}^0, v_1^0, v_2^0, \ldots, v_{p+1}^0$ with the properties mentioned in Theorem 8 and Corollary 9, then such a basic solution may be not unique.

Taking into account the presented above results we can prove the following theorem.

Theorem 9. *Let disjoint bilinear programming problem (57)-(59) be such that system (59) satisfies conditions* $a) - c)$ *and the set of solutions* X *of system (2) is nonempty and bounded. Then the minimal value* z^* *of the objective function of problem (57)-(59) is equal to the minimal value* h^* *of parameter* h *in system (71) for which this system has a basic feasible solution* $x_1^*,\ x_2^*,\ \ldots,\ x_{n+q}^*, v_1^*,\ v_2^*,\ \ldots, v_{p+1}^*$ *where either the system of column vectors*

$$\left\{ D_k = \begin{pmatrix} d_{k1} \\ d_{k2} \\ \vdots \\ d_{km} \end{pmatrix} :\ v_k^* > 0, \quad k \in \{= 1, 2, \ldots, p\} \right\} \tag{77}$$

is linearly independent or this system of vectors is an empty set. If such a solution for system (71) with $h = h^*$ *is known, then* $x_1^*, x_2^*, \ldots, x_n^*$ *together with an arbitrary solution* $y_1^*, y_2^*, \ldots, y_m^*$ *of the system*

$$\left\{ \begin{array}{ll} \displaystyle\sum_{j=1}^{m} d_{kj} y_j \le d_{k0}, & k = 1, 2, \ldots, p, \\ \displaystyle\sum_{j=1}^{m} (\sum_{k=1}^{p} d_{kj} v_k^*) y_j = \sum_{k=1}^{p} d_{k0} v_k^* & \end{array} \right. \tag{78}$$

represent an optimal solution $x_1^*, x_2^*, \ldots, x_{n+q}^*, y_1^*, y_2^*, \ldots, y_m^*$ *of disjoint programming problem (57)-(59). If the system of column vectors (77) is an empty set, then* $v_1^* = 0, v_2^* = 0, \ldots, v_{p+1}^* = 0$ *and* $\sum_{k=1}^{p} d_{kj} v_k^* = 0, \quad j = 1, 2, \ldots, m$*; in this case* $x_1^*, x_2^*, \ldots, x_q^*$ *together with an arbitrary solution* $y_1, y_2, \ldots, y_m$ *of system (59) represent a solution of problem (57)-(59).*

Proof. Let h^* be the minimal value of parameter h for which system (71) has a basic feasible solution $x_1^*,\ x_2^*,\ \ldots,\ x_{n+q}^*, v_1^*,\ v_2^*,\ \ldots,\ v_{p+1}^*$ for which either the system of vectors (77) is linearly independent or this system of vectors is an empty set. Then, according to Theorem 8 and Corollary 3 of Theorem 3, the optimal value of the objective function of problem (57)-(59) with properties $a) - c)$ is equal to h^* and $v_{p+1}^* = 0$.

Now, let us prove the second part of the theorem. According to Corollary 1 of Theorem 2, for disjoint bilinear programming problem (57)-(59) we can consider the following min-max problem:

Find

$$h^* = \min_{(x_1,x_2,\dots,x_n)\in X} \quad \max_{(v_1,v_2,\dots,v_p)\ \in V(x_1,x_2,\dots,x_n)} \left(\sum_{j=1}^{n} g_j x_j \ - \sum_{k=1}^{p} d_{k0} v_k\right)$$

and $(x_1^*,\ x_2^*,\ \dots,\ x_n^*) \in X$,

$$X = \left\{(x_1, x_2, \dots, x_n) \middle| \sum_{i=1}^{n} a_{ij} x_j \le a_{i0},\ i = 1, 2, \dots, q;\ \ x_i \ge 0,\ i = 1, 2, \dots, q\right\},$$

such that

$$h^* = \max_{(v_1,v_2\dots,v_p)\ \in V(x_1^*,x_2^*,\dots,x_n^*)} \left(\sum_{i=1}^{n} g_j x_j^* \ - \sum_{k=1}^{p} d_{k0} v_k\right),$$

where

$$V(x_1, x_2, \dots, x_n) = \left\{(v_1, v_2, \dots, v_p) \middle| - \sum_{k=1}^{p} d_{kj} v_k = \sum_{i=1}^{n} c_{ij} x_i + e_j, j = 1, 2, \dots m\right\}.$$

We can observe that $x_1^*,\ x_2^*,\ \dots,\ x_{n+q}^*, v_1^*,\ v_2^*,\ \dots,\ v_{p+1}^*$ is a solution of this min-max problem and $x^* = (x_1^*,\ x_2^*,\ \dots,\ x_{n+q}^*)$ is an optimal point for problem (57)-(59) with properties $a) - c)$. Taking into account that $\sum\limits_{k=1}^{p} d_{kj} v_k^* = - \sum\limits_{i=1}^{n} c_{ij} x_i^* - e_j,\ j = 1, 2, \dots, m$, and $\sum\limits_{k=1}^{p} d_{k0} v_k^* = -h^* + \sum\limits_{i=1}^{p} g_i x_i^*$ we obtain that system (78) coincides with system (70), because $s_j^* = \sum\limits_{k=1}^{p} d_{kj} v_k^*$ and $s_0^* = \sum\limits_{k=1}^{p} d_{k0} v_k^*$. So, for the optimal point $x^* = (x_1^*,\ x_2^*,\ \dots,\ x_{n+q}^*)$ the corresponding optimal point $y^* = (y_1^*, y_2^*, \dots, y_m^*)$ for problem (57)-(59) can be found by solving system (78). If the system of column vectors (77) is an empty set, then it is evident that $v_1^* = 0, v_2^* = 0, \dots, v_{p+1}^* = 0$ and $\sum_{k=1}^{p} d_{kj} v_k^* = 0,\ \ j = 1, 2, \dots, m$; in this case $x_1^*, x_2^*, \dots, x_q^*$ together with an arbitrary solution $y_1, y_2, \dots, y_m$ of system (59) represents a solution of problem (57)-(59), i.e. we have the case of Corollary 3. $\square$

9 The main results concerned with checking the conditions of Theorem 8

The main results of this section are concerned with studying and solving the following problem: to determine if for a given h, system (71) has a basic feasible solution $x_1^0, x_2^0, \dots, x_{n+q}^0, v_1^0, v_2^0, \dots, v_{p+1}^0$, where v_{p+1}^0 is a basic component. We show that for the considered problem there exists a polynomial algorithm that determines if such a basic solution for system (71) exists or doesn't exist. According to Corollary 9 of Theorem 8, a basic solution for system (71) that has v_{p+1}^0 as a basic component exists if and only if $Y \not\subset Y_h$. We present the results that allow to check if the condition $Y \not\subset Y_h$ holds and show how to determine a basic solution

$x_1^0, x_2^0, \dots, x_{n+q}^0, v_1^0, v_2^0, \dots, v_{p+1}^0$ for system (71) where v_{p+1}^0 is a basic component when the condition $Y \not\subset Y_h$ holds. Note that all results presented in this section are related to the case when system (45) has no solutions, because in the case when this system is consistent the solution of the problem can be easy found.

9.1 Some preliminary results for the problem of checking if $Y \not\subset Y_h$

Let us consider system (71) in the following matrix form

$$\begin{cases} W\alpha = W_0, \\ \quad \alpha \geq 0, \end{cases} \tag{79}$$

where α is the column vector of variables of system (71), i.e.

$$\alpha^T = (x_1, x_2, \dots x_n, x_{n+1} \dots, x_{n+q}, v_1, v_2, \dots, v_{p+1}),$$

$W = (w_{i,j})$ is the $(q+m+1) \times (n+q+p+1)$ matrix of coefficients of system (71), and W_0 is the column vector of right hand sides of this system, i.e.

$$W = \begin{pmatrix} A & I & 0 & 0 \\ C^T & 0 & D^T & 0 \\ g & 0 & -d^T & 1 \end{pmatrix}, \tag{80}$$

and

$$W_0^T = (w_{1,0}, w_{2,0}, \dots, w_{q+m+1,0}) = (a_{1,0}, a_{2,0}, \dots, a_{q,0}, -e_1, -e_2, \dots, -e_m, h).$$

In what follows we will denote the components of column vector α by α_j, i.e.

$$\alpha^T = (\alpha_1, \alpha_2, \dots, \alpha_{n+q}, \alpha_{n+q+1}, \dots, \alpha_{n+q+p+1}),$$

and will take into account that $\alpha_1, \alpha_2, \dots, \alpha_{n+q+p+1}$ represent the corresponding variables $x_1, x_2, \dots, x_{n+q}, v_1, v_2, \dots, v_{p+1}$ for system (71). We denote the column vectors of matrix W by $W_1, W_2, \dots, W_{n+q+p+1}$, where

$$W_j^T = (w_{1,j}, w_{2,j}, \dots, w_{q+m+1,j}), \quad j = 1, 2, \dots, n+q+p+1 \ (m \leq p)$$

and each W_j corresponds to the variable α_j.

Recall that

$$C = (c_{ij})_{n\times m}, \quad A = (a_{ij})_{q\times n}, \quad D = (d_{ij})_{p\times m}, \ a^T = (a_{10}, a_{20}, \dots a_{q0}),$$

$$d^T = (d_{10}, d_{20}, \dots, d_{m0}), \ g = (g_1, g_2, \dots, g_n), \ e = (e_1, e_2, \dots, e_m),$$

$$rank(A) = q, \qquad rank(D) = m \qquad \text{and} \qquad rank(W) = q + m + 1.$$

Using the vector notations $x^T = (x_1, x_2, \dots, x_n)$, $\overline{x}^T = (x_{n+1}, x_{n+2}, \dots, x_{n+q})$, $v^T = (v_1,\ v_2,\ \dots,\ v_p)$, system (79) can be written as follows

$$\left\{\begin{array}{llllcl} Ax & + \ I\overline{x} & & & = & a^T, \\ C^Tx & & -\ D^Tv & & = & -e^T, \\ gx & & -\ d^Tv & + \ v_{p+1} & = & h, \\ \multicolumn{4}{r}{x \geq 0,\ \overline{x} \geq 0,\ v \geq 0,\ v_{p+1}} & \geq & 0. \end{array}\right. \tag{81}$$

Let us assume that for a given $h \in [2^L, 2^L]$ system (79) has a basic feasible solution α^{0^T}. Without loss of generality, we may assume that the basic components of α^{0^T} are $\alpha^0_1, \alpha^0_2, \dots, \alpha^0_{q+m+1}$ which correspond to the linearly independent column vectors $W_1, W_2, \dots, W_{q+m+1}$ that form a basis for the set of column vectors of matrix W (if it is not so, then we can relabel variables such that the first $q+m+1$ components of α^{0^T} are the basic ones). We denote by W_B the matrix consisting of column vectors $W_1, W_2, \dots, W_{q+m+1}$, and by W_N the matrix consisting of column vectors $W_{q+m+2},\ W_{q+m+3},\ \dots,\ W_{n+q+p+1}$. Then system (79) can be represented as

$$\left\{\begin{array}{l} W_B \cdot \alpha_B \ + \ W_N \cdot \alpha_N = W_0, \\ \qquad \alpha_B \geq 0, \qquad \alpha_N \geq 0, \end{array}\right.$$

where $\alpha_B^T = (\alpha_1, \alpha_2, \dots, \alpha_{q+m+1})$, $\alpha_N^T = (\alpha_{q+m+2}, \alpha_{q+m+3}, \dots, \alpha_{n+q+p+1})$. For W_B there exists W_B^{-1}, therefore this system is equivalent to the system

$$\left\{\begin{array}{l} I \cdot \alpha_B + W_B^{-1} \cdot W_N \cdot \alpha_N = W_B^{-1} \cdot W_0, \\ \quad \alpha_B \geq 0, \qquad\qquad \alpha_N \geq 0. \end{array}\right.$$

By introducing the notations $\mathbf{W} = W_B^{-1} \cdot W_N$ and $\mathbf{W}_0 = W_B^{-1} \cdot W_0$, we obtain the system

$$\left\{\begin{array}{l} \alpha_B + \mathbf{W} \cdot \alpha_N = \mathbf{W}_0, \\ \alpha_B \geq 0, \quad \alpha_N \geq 0, \end{array}\right. \tag{82}$$

where $\mathbf{W} = (\mathbf{w}_{i,j})$ is a $(q+m+1) \times (n+p-m)$ matrix and $\mathbf{W}_0$ is a column vector with nonnegative components $\mathbf{w}_{1,0}, \mathbf{w}_{2,0}, \dots, \mathbf{w}_{q+m+1,0}$, i.e. $\alpha^0_B = \mathbf{W}_0$.

Now let us consider system (83) in the following unfolded form:

$$\left\{\begin{array}{l} \alpha_1 \qquad\qquad + \sum\limits_{j=r+1}^{r+s-1} \mathbf{w}_{1,j}\alpha_j + \mathbf{w}_{1,r+s}\alpha_{r+s} = \mathbf{w}_{1,0}, \\ \quad \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ \qquad\quad \alpha_k \quad + \sum\limits_{j=r+1}^{r+s-1} \mathbf{w}_{k,j}\alpha_j + \mathbf{w}_{k,r+s}\alpha_{r+s} = \mathbf{w}_{k,0}, \\ \quad \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ \qquad\qquad\qquad \alpha_r + \sum\limits_{j=r+1}^{r+s-1} \mathbf{w}_{r,j}\alpha_j + \mathbf{w}_{r,r+s}\alpha_{r+s} = \mathbf{w}_{r,0}, \\ \qquad\qquad\qquad\qquad \alpha_j \geq 0, \quad j = 1,\ 2,\ \dots,\ r+s, \end{array}\right. \tag{83}$$

where $r = q + m + 1$, $s = n + p - m$, and $\mathbf{w}_{i,0} \geq 0$, $i = 1, 2, \ldots, r$. So, ${\alpha^0}^T = (\alpha_1^0, \alpha_2^0, \ldots, \alpha_r^0, 0, 0, \ldots, 0)$ is a basic feasible solution of system (83), where $\alpha_i^0 = \mathbf{w}_{i,0}$, $i = 1, 2, \ldots, r$. Obviously, if in (83) one of the basic component α_i, $i \in \{1, 2, \ldots, r\}$ corresponds to v_{p+1} then $v_{p+1}^0 = \alpha_i^0$ is the basic component for ${\alpha^0}^T$ and according to Theorem 9 we have $Y \not\subset Y_h$. If in (83) among $\alpha_1, \alpha_2, \ldots, \alpha_r$ there is no component α_i, $i \in \{1, 2, \ldots, r\}$ that corresponds to v_{p+1}, then without loss of generality we may assume that v_{p+1} in this system is represented by the nonbasic variable α_{r+s} with the corresponding column vector $\mathbf{W}_{r+s}$, $\mathbf{W}_{r+s}^T = (\mathbf{w}_{1,r+s}, \mathbf{w}_{2,r+s}, \ldots, \mathbf{w}_{r,r+s})$, where $\alpha_{r+s} = v_{p+1}$ and at least a component $\mathbf{w}_{i,r+s}$ of $\mathbf{W}_{r+s}$ is different from zero. If it is not so, we always can relabel variables such that α_{r+s} will correspond to v_{p+1}. Then in this case the following lemma holds.

Lemma 6. *Let* ${\alpha^0}^T = (\alpha_1^0, \alpha_2^0, \ldots, \alpha_r^0, 0, 0, \ldots, 0)$ *be a basic feasible solution of system (83) where the nonbasic variable* α_{r+s} *in (83) corresponds to* v_{p+1}. *If the column vector* $\mathbf{W}_{r+s}$ *contains positive components, then* $Y \not\subset Y_h$. *If the column vector* $\mathbf{W}_{r+s}$ *contains a nonzero component* $\mathbf{w}_{i_0,r+s}$ *(positive or negative) and* $\mathbf{w}_{i_0,0} = 0$, *then* $Y \not\subset Y_h$.

Proof. Assume that column vector $\mathbf{W}_{r+s}$ contains positive components. Then we can find $i_0 \in \{1, 2, \ldots, r\}$ such that

$$\frac{\mathbf{w}_{i_0,0}}{\mathbf{w}_{i_0,r+s}} = \min_{\mathbf{w}_{i,r+s}>0} \frac{\mathbf{w}_{i,0}}{\mathbf{w}_{i,r+s}}$$

and determine a new basic feasible solution α' by applying for system (83) the standard pivoting procedure with pivot element $\mathbf{w}_{i_0,r+s}$ in the same way as in the simplex algorithm for linear programming. After such a pivoting procedure we determine a new basic feasible solution α' the components of which are calculated as follows:

$$\alpha'_i = \begin{cases} \mathbf{w}_{i,0} - \dfrac{\mathbf{w}_{i,r+s}\mathbf{w}_{i_0,0}}{\mathbf{w}_{i_0,r+s}} & \text{for} \quad i = 1, 2, \ldots, r \ (i \neq i_0, r+s), \\ 0 & \text{for} \quad i = i_0, r+1, r+2, \ldots, r+s-1, \\ \dfrac{\mathbf{w}_{i_0,0}}{\mathbf{w}_{i_0,r+s}} & \text{for} \quad i = r+s. \end{cases}$$

In such a way we determine the basic feasible solution

$${\alpha'}^T = (\alpha'_1,\ \alpha'_2,\ \ldots,\ \alpha'_{i_0-1},\ \alpha'_{r+s},\ \alpha'_{i_0+1},\ \ldots,\ \alpha'_r,\ 0,\ 0,\ \ldots,\ 0,\ 0)$$

that in initial system (79) corresponds to the basis

$$\{W_1, W_2, \ldots, W_{i_0-1}, W_{r+s}, W_{i_0+1}, \ldots, W_r\}$$

obtained from

$$\{W_1, W_2, \ldots, W_{i_0-1}, W_{i_0}, W_{i_0+1}, \ldots, W_r\}$$

by replacing the column vector W_{i_0} with the column vector W_{r+s}. We can observe that the basic component α'_{r+s} of this basic feasible solution corresponds to v_{p+1} and therefore according to Theorem 9 we have $Y \not\subset Y_h$.

In the case $\mathbf{w}_{i_0,0} = 0$, $\mathbf{w}_{i_0,r+s} \neq 0$ where $\mathbf{w}_{i_0,r+s}$ may be positive or negative, we also can apply for system (83) the same pivoting procedure with the pivot element $\mathbf{w}_{i_0,r+s}$ in spite of the fact that the sign of $\mathbf{w}_{i_0,r+s}$ may be negative. After such a procedure, we determine the basic feasible solution

$$\alpha'^T = (\alpha'_1, \alpha'_2, \dots, \alpha'_{i_0-1}, \alpha'_{r+s}, \alpha'_{i_0+1}, \dots, \alpha_r, 0, 0, \dots, 0),$$

for which $\alpha'_{r+s} = \alpha^0_{i_0} = 0$ and $\alpha^0 = \alpha'$, i.e. α' is the basic solution that corresponds to the basis $W_1, W_2, \dots, W_{i_0-1}, W_{r+s}, W_{i_0+1}, \dots, W_r$, obtained from $W_1, W_2, \dots, W_{i_0-1}, W_{i_0}, W_{i_0+1}, \dots, W_r$, by replacing W_{i_0} with W_{r+s}. So, α' is a basic feasible solution for which the component α'_{r+s} corresponds to v_{p+1}, and therefore according to Theorem 9 we have $Y \not\subset Y_h$. □

Corollary 10. *If $Y \subset Y_h$, then for an arbitrary basic feasible solution α^0 of system (83) the column vectors $\mathbf{W}_{r+s}$ possess the following properties:*

1) all components of $\mathbf{W}_{r+s}$ are non-positive;

2) if $\mathbf{w}_{i,r+s} < 0$ for some $i \in \{1, 2, \dots, r\}$, then $\mathbf{w}_{i,0} > 0$.

9.2 The main results for checking if $Y \not\subset Y_h$

In the previous subsection we showed that the problem of the existence of a basic solution for system (71) that has v_{p+1} as a basic component can be reduced to the problem of the existence of a basic solution for system (83) that has α_{r+s} as a basic component, where α_{r+s} corresponds to v_{p+1}. In this subsection, we show how to determine if such a solution for system (83) exists in the case when the column vectors $\mathbf{W}_{r+s}$ and $\mathbf{W}_0$ satisfy the following condition

$$\begin{cases} \mathbf{w}_{i,r+s} \leq 0, \;\; i = 1, 2, \dots, r; \\ if \;\; \mathbf{w}_{i,r+s} < 0 \; for \; some \; i \in \{1, 2, \dots, r\}, \;\; then \; \mathbf{w}_{i,0} > 0. \end{cases} \tag{84}$$

If this condition is not satisfied, then according to Lemma 6 we have $Y \not\subset Y_h$. If condition (84) for system (83) is satisfied, then by relabeling its equations and variables we can represent this system in the following form

$$\left\{\begin{array}{llll}
\alpha_1 & + \sum\limits_{j=r+1}^{r+s-1} \mathbf{w}_{1,j}\alpha_j & & = \mathbf{w}_{1,0}, \\
\cdots & \cdots\cdots\cdots & \cdots\cdots & \cdots \\
\quad \alpha_{k-1} & + \sum\limits_{j=r+1}^{r+s-1} \mathbf{w}_{k-1,j}\alpha_j & & = \mathbf{w}_{k-1,0}, \\
\qquad \alpha_k & + \sum\limits_{j=r+1}^{r+s-1} \mathbf{w}_{k,j}\alpha_j & + \ \mathbf{w}_{k,r+s}\alpha_{r+s} & = \mathbf{w}_{k,0}, \\
\cdots & \cdots\cdots\cdots & \cdots\cdots & \cdots \\
\qquad\qquad \alpha_r & + \sum\limits_{j=r+1}^{r+s-1} \mathbf{w}_{r,j}\alpha_j & + \ \mathbf{w}_{r,r+s}\alpha_{r+s} & = \mathbf{w}_{r,0}, \\
 & \alpha_j \geq 0, \quad j = 1,\ 2,\ \ldots,\ r+s, & &
\end{array}\right. \tag{85}$$

where $\mathbf{w}_{i,0} \geq 0,\ i = 1, 2, \ldots, k-1;\ \mathbf{w}_{i,0} > 0,\ i = k,\ k+1,\ \ldots,\ r$, and $\mathbf{w}_{i,r+s} < 0$, $i = k, k{+}1, \ldots, r$. In this system, we will assume that the variable α_{r+s} corresponds to the variable v_{p+1} of system (71). Note that in (85) the variables $\alpha_k, \alpha_{k+1}, \ldots, \alpha_r$ correspond to some of the variables $v_1, v_2, \ldots, v_p$ of system (71), and if a variable α_i corresponds to one of the bounded variables $x_1, x_2, \ldots, x_{n+q}$ of system (71), then $1 \leq i \leq k$, where $q + n + 1 \leq k \leq r$.

It can be easily observed that if for an arbitrary feasible solution of system (85) the following condition holds

$$\sum_{j=r+1}^{r+s-1} \mathbf{w}_{r,j}\alpha_j < \mathbf{w}_{r,0}, \tag{86}$$

then the problem of the existence of a basic feasible solution for system (85) that has α_{r+s} as a basic component can be reduced to a similar problem for a new system with fewer equations and fewer variables. Such a reduction can be made on the base of the following lemma.

Lemma 7. *Let the consistent system (85) with $k < r$ be given. If for an arbitrary feasible solution of this system condition (86) holds, then system (85) has a basic feasible solution $\alpha^{0^T} = (\alpha_1^0, \alpha_2^0, \ldots, \alpha_{r+s}^0)$ such that α_{r+s}^0 is a basic component if and* *only if the following system*

$$\begin{cases} \alpha_1 + \sum\limits_{j=r+1}^{r+s-1} \mathbf{w}_{1,j}\alpha_j = \mathbf{w}_{1,0}, \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots \\ \alpha_{k-1} + \sum\limits_{j=r+1}^{r+s-1} \mathbf{w}_{k-1,j}\alpha_j = \mathbf{w}_{k-1,0}, \\ \alpha_k + \sum\limits_{j=r+1}^{r+s-1} \mathbf{w}_{k,j}\alpha_j + \mathbf{w}_{k,r+s}\alpha_{r+s} = \mathbf{w}_{k,0}, \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots \\ \alpha_{r-1} + \sum\limits_{j=r+1}^{r+s-1} \mathbf{w}_{r-1,j}\alpha_j + \mathbf{w}_{r-1,r+s}\alpha_{r+s} = \mathbf{w}_{r-1,0}, \\ \alpha_j \geq 0, \ \forall j \in \{1, 2, \dots, r+s\} \setminus \{r\}, \end{cases} \tag{87}$$

has a basic feasible solution $\overline{\alpha}^{0^T} = (\alpha_1^0, \alpha_2^0, \dots, \alpha_{r-1}^0, \alpha_{r+1}^0 \dots \alpha_{r+s}^0)$ *such that* α_{r+s}^0 *is a basic component. If* $k = r$ *and for an arbitrary feasible solution of system (85) condition (86) holds, then system (85) has no basic feasible solution* $\alpha^{0^T} = (\alpha_1^0, \alpha_2^0, \dots, \alpha_{r+s}^0)$ *such that* α_{r+s}^0 *is a basic component.*

Proof. Indeed, if $k < r$ and for an arbitrary solution of system (85) condition (86) holds, then $\alpha_r > 0$ for an arbitrary solution of system (85), i.e. α_r is a basic component for an arbitrary basic solution of this system. Therefore, we can eliminate the last equation from (85) and conclude that system (85) has a basic solution $\alpha^{0^T} = (\alpha_1^0, \alpha_2^0, \dots, \alpha_{r+s}^0)$ such that α_{r+s}^0 is a basic component if and only if system (87) has a basic solution $\overline{\alpha}^{0^T} = (\alpha_1^0, \alpha_2^0, \dots, \alpha_{r-1}^0, \alpha_{r+1}^0 \dots \alpha_{r+s}^0)$ such that α_{r+s}^0 is a basic component. In the case $k = r$ it is evident that system (85) couldn't contain a basic feasible solution $\alpha^{0^T} = (\alpha_1^0, \alpha_2^0, \dots, \alpha_{r+s}^0)$ that has α_{r+s}^0 as a basic component. □

Corollary 11. *If for an arbitrary feasible solution of system (85) the following condition holds*

$$\sum_{j=r+1}^{r+s-1} \mathbf{w}_{i,j}\alpha_j < \mathbf{w}_{i,0}, \ i = k, k+1, \dots, r,$$

then system (85) has no basic feasible solution $\alpha^{0^T} = (\alpha_1^0, \alpha_2^0, \dots, \alpha_{r+s}^0)$ *that has* α_{r+s}^0 *as a basic component.*

In the case when for an arbitrary feasible solution of system (85) the following condition holds

$$\sum_{j=r+1}^{r+s-1} \mathbf{w}_{r,j}\alpha_j \leq \mathbf{w}_{r,0} \tag{88}$$

Lemma 7 can be specified as follows.

Lemma 8. *Assume that for an arbitrary feasible solution of consistent system (85) with* $k \leq r$ *condition (88) holds, and at least for a feasible solution*

$\alpha^T = (\alpha_1, \alpha_2, \ldots, \alpha_{r+s})$ of system (85) condition (88) is satisfied as equality, i.e. $\sum\limits_{j=r+1}^{r+s-1} \mathbf{w}_{r,j}\alpha_j = \mathbf{w}_{r,0}$. Then system (85) has a basic feasible solution $\alpha^{0^T} = (\alpha_1^0, \alpha_2^0, \ldots, \alpha_{r+s}^0)$ that has $\alpha_{r+s}^0 = 0$ as a degenerate basic component. Such a basic solution for system (85) can be found by using an optimal basic solution $\overline{\alpha}^{0^T} = (\overline{\alpha}_1^0, \overline{\alpha}_2^0, \ldots, \overline{\alpha}_{k-1}^0, \overline{\alpha}_{r+1}^0, \overline{\alpha}_{r+2}^0, \ldots, \overline{\alpha}_{r+s-1}^0)$ of the following linear programming problem:
Maximize

$$z_r = \sum_{j=r+1}^{r+s-1} \mathbf{w}_{r,j}\alpha_j - \mathbf{w}_{r,0} \tag{89}$$

subject to

$$\begin{cases} \alpha_1 + \sum\limits_{j=r+1}^{r+s-1} \mathbf{w}_{1,j}\alpha_j = \mathbf{w}_{1,0}, \\ \quad \alpha_2 + \sum\limits_{j=r+1}^{r+s-1} \mathbf{w}_{2,j}\alpha_j = \mathbf{w}_{2,0}, \\ \quad \cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ \qquad \alpha_{k-1} + \sum\limits_{j=r+1}^{r+s-1} \mathbf{w}_{k-1,j}\alpha_j = \mathbf{w}_{k-1,0}, \\ \alpha_j \geq 0, \ j = 1, 2, \ldots, k-1, r+1, r+2, \ldots, r+s-1. \end{cases} \tag{90}$$

If $\overline{\alpha}^{0^T}$ is known, then a basic feasible solution $\alpha^{0^T} = (\alpha_1^0, \alpha_2^0, \ldots, \alpha_{r+s}^0)$ of system (85) that has $\alpha_{r+s}^0 = 0$ as a degenerate basic component can be found as follows:

$$\alpha_i^0 = \begin{cases} \overline{\alpha}_i^0 & \textit{if } \ i = 1, 2, \ldots, k-1, r+1, \ldots, r+s-1, \\ -\sum\limits_{j=r+1}^{r+s-1} \mathbf{w}_{i,j}\overline{\alpha}_j^0 + \mathbf{w}_{i,0} & \textit{if } \ i = k, k+1, \ldots, r-1, \\ 0 & \textit{if } \ i = r, \ r+s, \end{cases} \tag{91}$$

where $\alpha_r^0 = 0$ is a nonbasic component of α^{0^T}. In this case inequality (88) is weakly redundant for the following system

$$\begin{cases} \sum\limits_{j=r+1}^{r+s-1} \mathbf{w}_{i,j}\alpha_j \leq \mathbf{w}_{i,0}, & i = 1, 2, \ldots, k-1, \\ \qquad \alpha_j \geq 0, & j = r+1, r+2, \ldots, r+s-1, \end{cases} \tag{92}$$

and an arbitrary inequality

$$\sum_{j=r+1}^{r+s-1} \mathbf{w}_{i,j}\alpha_j \leq \mathbf{w}_{i,0}, \ i \in \{k, k+1, \ldots, r\} \tag{93}$$

is redundant for system (92).

Proof. Assume that the conditions of the lemma are satisfied and consider the following linear programming problem: *Maximize (89) subject to (85).* Then this problem

has solutions and the optimal value of the objective function (89) is equal to zero. If we dualize the linear programming problem (85), (89), then we obtain the following problem:
Minimize

$$z' = \sum_{i=1}^{k-1} \mathbf{w}_{i,0}\beta_i \;+\; \sum_{i=k}^{r} \mathbf{w}_{i,0}\beta_i - \mathbf{w}_{r,0} \tag{94}$$

subject to

$$\begin{cases} \sum\limits_{i=1}^{k-1} \mathbf{w}_{i,j}\beta_i \;+\; \sum\limits_{i=k}^{r} \mathbf{w}_{i,j}\beta_i \geq \mathbf{w}_{r,j}, \quad j = \overline{r+1, r+s-1}, \\ \sum\limits_{i=k}^{r} \mathbf{w}_{i,r+s}\beta_i \geq 0, \\ \beta_i \geq 0, \quad i = 1, 2, \dots, r. \end{cases} \tag{95}$$

This problem has solutions and the optimal value of the objective function is equal to zero. Taking into account that $\mathbf{w}_{i,r+s} < 0,\ i = k,\ k+1,\ \dots,\ r$ we obtain $\beta_i = 0,\ i = k, k+1, \dots, r$. Therefore, this dual problem can be written as follows:
Minimize

$$z' = \sum_{i=1}^{k-1} \mathbf{w}_{i,0}\beta_i - \mathbf{w}_{r,0} \tag{96}$$

subject to

$$\begin{cases} \sum\limits_{i=1}^{k-1} \mathbf{w}_{i,j}\beta_i \geq \mathbf{w}_{r,j}, \quad j = \overline{r+1, r+s-1}, \\ \beta_i \geq 0, \quad j = 1, 2, \dots, r+s-1. \end{cases} \tag{97}$$

Problem (96), (97) can be regarded as the dual problem for linear programming problem (89), (90) and for linear programming problem (89), (92), i.e. for these problems there exist optimal solutions and the corresponding optimal values of the objective functions are equal to zero. So, the linear programming problem (89), 92) has an optimal solution that is attained at an extreme point $(\overline{\alpha}^0_{r+1}, \overline{\alpha}^0_{r+2}, \dots, \overline{\alpha}^0_{r+s-1})$ of the set of solutions of system (92) and the optimal value of the objective function (89) is equal to zero. This means that $\sum\limits_{j=r+1}^{r+s-1} \mathbf{w}_{r,j}\overline{\alpha}^0_j - \mathbf{w}_{r,0} = 0$, i.e. inequality (88) is weakly redundant for system (92). Moreover, inequality (88) is weakly redundant for the following system

$$\begin{cases} \sum\limits_{j=r+1}^{r+s-1} \mathbf{w}_{i,j}\alpha_j \leq \mathbf{w}_{i,0}, \quad i = 1, 2, \dots, k-1, \\ \sum\limits_{j=r+1}^{r+s-1} \mathbf{w}_{i,j}\alpha_j + \mathbf{w}_{i,r+s}\alpha_{r+s} \leq \mathbf{w}_{i,0}, \quad i = k, k+1, \dots, r, \\ \alpha_j \geq 0, j = r+1, r+2, \dots, r+s, \end{cases} \tag{98}$$

where $(\overline{\alpha}^0_{r+1}, \overline{\alpha}^0_{r+2}, \dots, \overline{\alpha}^0_{r+s-1}, \overline{\alpha}^0_{r+s})$ with $\overline{\alpha}^0_{r+s} = 0$ represents an extreme point of the set of solutions of system (98) at which the maximal value of the objective function (89) on the set of solutions of this system is attained, i.e. $z^0_r = \sum\limits_{j=r+1}^{r+s-1} \mathbf{w}_{r,j}\overline{\alpha}^0_j - \mathbf{w}_{r,0} = 0$. Therefore $(\overline{\alpha}^0_1, \overline{\alpha}^0_2, \dots, \overline{\alpha}^0_{r+s-1}, \overline{\alpha}^0_{r+s})$ with $\overline{\alpha}^0_i = -\sum\limits_{j=r+1}^{r+s-1} \mathbf{w}_{i,j}\overline{\alpha}^0_j + \mathbf{w}_{r,0} \geq 0$, $i = 1, 2, \dots, r$; $\overline{\alpha}^0_{r+s} = 0$, represents an optimal basic solution of linear programming problem: *Maximize the objective function (89)* subject to (85). Thus, we may conclude that ${\alpha^0}^T = (\alpha^0_1, \alpha^0_2, \dots, \alpha^0_{r+s})$ with the components determined according to (91) represents a basic feasible solution for system (85) that has $\alpha^0_{r+s} = 0$ as a degenerate basic component. Indeed, from (91) we can see that ${\alpha^0}^T$ contains no more than $k-1$ basic components of ${\overline{\alpha}^0}^T$, no more than $r-k$ components $\alpha^0_i = -\sum\limits_{j=r+1}^{r+s-1} \mathbf{w}_{i,j}\overline{\alpha}^0_j + \mathbf{w}_{r,0} \geq 0$, $i = k, k+1, \dots, r-1$, and a degenerate basic component $\alpha^0_{r+s} = 0$, i.e. in the whole we obtain that ${\alpha^0}^T$ contains no more than r basic components, including a degenerate basic component $\alpha^0_{r+s} = 0$. In general, if the conditions of Lemma 8 are satisfied, then formula (91) is valid for an arbitrary basic feasible solution ${\alpha^0}^T = (\alpha^0_1, \alpha^0_2, \dots, \alpha^0_{r+s})$ of system (85) with $\alpha^0_{r+s} = 0$, because $\alpha^0_i = -\sum\limits_{j=r+1}^{r+s-1} \mathbf{w}_{i,j}\alpha^0_j + \mathbf{w}_{r,0} \geq 0$, $i = k, k+1, \dots, r-1$. Therefore, an arbitrary inequality (93) is redundant for system (92). □

Corollary 12. *The consistent system (85) with $k = r$ has a basic feasible solution ${\alpha^0}^T = (\alpha^0_1, \alpha^0_2, \dots, \alpha^0_{r+s})$ that has $\alpha^0_{r+s} = 0$ as a degenerate basic component if and only if inequality (88) is weakly redundant for system of inequalities (92).*

Remark 5. Lemma 8 has been proven assuming that for an arbitrary feasible solution of system (85) inequality (88) holds, and at least for a feasible solution of this system inequality (88) is satisfied as equality. In general, such a condition in this lemma can be considered with respect to an arbitrary inequality

$$\sum_{j=r+1}^{r+s-1} \mathbf{w}_{i_0,j}\alpha_j \leq \mathbf{w}_{i_0,0}, \quad i_0 \in \{k, k+1, \dots, r\},$$

and based on this similarly we can prove the existence of a basic solution ${\alpha^0}^T = (\alpha^0_1, \alpha^0_2, \dots, \alpha^0_{r+s})$ for system (85) that has $\alpha^0_{r+s} = 0$ as a degenerate basic component.

Taking into account Remark 5 to Lemma 8 we obtain the following result.

Theorem 10. *The consistent system (85) with $k \leq r$ has a basic feasible solution ${\alpha^0}^T = (\alpha^0_1,\ \alpha^0_2,\ \dots,\ \alpha^0_{r+s})$ that has $\alpha^0_{r+s} = 0$ as a basic component if and only if* *there exists* $i_0 \in \{k, k+1, \dots, r\}$ *for which the inequality*

$$\sum_{j=r+1}^{r+s-1} \mathbf{w}_{i_0,j}\alpha_j \le \mathbf{w}_{i_0,0}$$

is weakly redundant for system (92) and an arbitrary inequality (93) is redundant for system (92).

Based on the results above we can prove the following result.

Theorem 11. *Let the consistent system (85) with* $k \le r$ *be given and consider the linear programming problem: Maximize the objective function (89) on the set of solutions of system (85). If this linear programming problem has solutions and the maximal value of the objective function is* $z_r^* = t_r^*$ *where* $0 \le t_r^* < \infty$, *then system (85) has a basic feasible solution* $\alpha^{*T} = (\alpha_1^*, \alpha_2^*, \dots, \alpha_{r+s}^*)$ *that has* $\alpha_{r+s}^* = -\dfrac{1}{\mathbf{w}_{r,r+s}} t_r^*$ *as a basic component and* $a_r^* = 0$. *Such a basic solution for system (85) can be found by using an optimal basic solution* $\overline{\alpha}^{*T} = (\overline{\alpha}_1^*, \overline{\alpha}_2^*, \dots, \overline{\alpha}_{k-1}^*, \overline{\alpha}_{r+1}^*, \overline{\alpha}_{r+2}^*, \dots, \overline{\alpha}_{r+s-1}^*)$ *of the linear programming problem: Maximize the objective function (89) on the set of solutions of system (90). If* $\overline{\alpha}^{*T}$ *is known, then a basic feasible solution* $\alpha^{*T} = (\alpha_1^*, \alpha_2^*, \dots, \alpha_{r+s}^*)$ *of system (85) that has* $\alpha_{r+s}^* = -\dfrac{1}{\mathbf{w}_{r,r+s}} t_r^*$ *as a basic component can be found as follows:*

$$\alpha_i^* = \begin{cases} \overline{\alpha}_i^*, & i = 1, 2, \dots, k-1, r+1, \dots, r+s\text{–}1, \\ -\sum\limits_{j=r+1}^{r+s-1} \mathbf{w}_{i,j}\overline{\alpha}_j^* + \dfrac{\mathbf{w}_{i,r+s}}{\mathbf{w}_{r,r+s}} t_r^* + \mathbf{w}_{i,0}, & i = k, k+1, \dots, r-1, \\ 0, & i = r, \\ -\dfrac{1}{\mathbf{w}_{r,r+s}} t_r^*, & i = r+s, \end{cases} \tag{99}$$

where $\alpha_r^* = 0$ *is a nonbasic component of* α^{*T}. *In this case the inequality*

$$\sum_{j=r}^{r+s-1} \mathbf{w}_{r,j}\alpha_j \le \mathbf{w}_{r,0} + t_r^* \tag{100}$$

is weakly redundant for system (92) and an arbitrary inequality

$$\sum_{j=r+1}^{r+s-1} \mathbf{w}_{i,j}\alpha_j \le \mathbf{w}_{i,0} + \frac{\mathbf{w}_{i,r+s}}{\mathbf{w}_{r,r+s}} t_r^*, \quad i \in \{k, k+1, \dots, r\}, \tag{101}$$

is redundant for system (92).

If $k < r$ *and the maximal value of the objective function* $z_r^* = t_r^* < 0$, *then system (85) has a basic feasible solution* $\alpha^{*T} = (\alpha_1^*, \alpha_2^*, \dots, \alpha_{r+s}^*)$ *with a basic component* α_{r+s}^* *if and only if system (87) has a basic feasible solution* $\alpha^{*T} = (\alpha_1^*, \alpha_2^*, \dots, \alpha_{r+s}^*)$

such that α^*_{r+s} *is a basic component. If* $k = r$ *and the maximal value of the objective function* $z^*_r = t^*_r < 0$*, then system (85) has no basic solution* α^{*T} *that has* α^*_{r+s} *as a basic component.*

Proof. The proof of the first part of this theorem is similar to the proof of Lemma 8. We consider the linear programming problem: *Maximize (89) subject to (85).* This problem has solutions and the optimal value of the objective function (89) is equal to t^*_r. If we dualize this problem, then we obtain the linear programming problem (94), (95) for which the optimal value of objective function (94) is equal to t^*_r. As we have shown, in this problem $\beta_i = 0$, $i = k, k+1, \dots, r$, because $\mathbf{w}_{i,r+s} < 0$, $i = k,\ k+1,\ \dots,\ r$. Therefore, problem (94), (95) can be written as the linear programming problem (96), (97) that can be regarded as the dual problem for the linear programming problem (89), (90) and for the linear programming problem (89), (92). So, for these problems there exist optimal solutions and the corresponding optimal values of the objective functions are equal to t^*_r. Linear programming problem (89), (92) has an optimal solution that is attained at an extreme point $(\overline{\alpha}^*_{r+1}, \overline{\alpha}^*_{r+2}, \dots, \overline{\alpha}^*_{r+s-1})$ of the set of solutions of system (92) and the optimal value of the objective function (89) is equal to t^*_r, i. e. $z^*_r = \sum\limits_{j=r+1}^{r+s-1} \mathbf{w}_{r,j}\overline{\alpha}^*_j - \mathbf{w}_{r,0} = t^*_r$. This means that inequality (100) is weakly redundant for system (92) and $\overline{\alpha}^{*T} = (\overline{\alpha}^*_1, \overline{\alpha}^*_2, \dots, \overline{\alpha}^*_{k-1}, \overline{\alpha}^*_r, \dots, \overline{\alpha}^*_{r+s-1})$ with $\overline{\alpha}^*_i = -\sum\limits_{j=r+1}^{r+s-1} \mathbf{w}_{i,j}\overline{\alpha}^*_j + \mathbf{w}_{r,0}$, $i = 1, 2, \dots, k-1$, represents an optimal basic solution of linear programming problem (89), (90). At the same time, the inequality (100) is weakly redundant for system (98) where $(\overline{\alpha}^*_{r+1}, \overline{\alpha}^*_{r+2}, \dots, \overline{\alpha}^*_{r+s-1}, \overline{\alpha}^*_{r+s})$ with $\overline{\alpha}^*_{r+s} = -\dfrac{1}{\mathbf{w}_{r,r+s}}t^*_r$ represents an extreme point of the set of solutions of system (98) at which the maximal value of the objective function (89) is attained, i.e. $z^*_r = \sum\limits_{j=r+1}^{r+s-1} \mathbf{w}_{r,j}\overline{\alpha}^*_j - \mathbf{w}_{r,0} = t^*$. Thus, if $\overline{\alpha}^{*T} = (\overline{\alpha}^*_{r+1}, \overline{\alpha}^*_{r+2}, \dots, \overline{\alpha}^*_{r+s-1})$ is an optimal basic solution of linear programming problem (89), (90), then $(\overline{\alpha}^*_1,\ \overline{\alpha}^*_2,\ \dots\ \overline{\alpha}^*_{r+s-1},\ \overline{\alpha}^*_{r+s})$, with $\alpha^*_{r+s} = -\dfrac{1}{\mathbf{w}_{r,r+s}}t^*_r$ and $\overline{\alpha}^*_i = -\sum\limits_{j=r+1}^{r+s-1} \mathbf{w}_{i,j}\overline{\alpha}^*_j + \dfrac{\mathbf{w}_{i,r+s}}{\mathbf{w}_{r,r+s}}t^*_r + \mathbf{w}_{i,0} \geq 0$, $i = k, k+1, \dots, r$, represents an optimal basic solution of linear programming problem (89), (85), i.e. $\alpha^{*T} = (\alpha^*_1, \alpha^*_2, \dots, \alpha^*_{r+s})$ with components determined according to (99) represents a basic feasible solution for system (85) that has $\alpha^*_{r+s} = -\dfrac{1}{\mathbf{w}_{r,r+s}}t^*_r$ as a basic component. Indeed, α^{*T} contains no more than $k-1$ basic components of $\overline{\alpha}^{*T}$, no more than $r-k$ components $\alpha^*_i = -\sum\limits_{j=r+1}^{r+s-1} \mathbf{w}_{i,j}\overline{\alpha}^*_j + \dfrac{\mathbf{w}_{i,r+s}}{\mathbf{w}_{r,r+s}}t^*_r + \mathbf{w}_{i,0}$, $i = k, k+1, \dots, r-1$, and the basic component $\alpha^*_{r+s} = \dfrac{1}{\mathbf{w}_{r,r+s}}t^*_r$, i.e., in the whole we obtain no more than r basic components. Therefore, the inequality (100) is weakly redundant for system

(92), and an arbitrary inequality (101) is redundant for system (92). The proof of the second part of the theorem for $z_r^* = t_r^* < 0$ follows from Lemma 7. $\square$

Remark 6. Theorem 11 has been proven assuming that the upper bound of function (89) on the set of solutions of system (85) is nonnegative finite value. In general, this theorem can be proven using such a bounded condition with respect to an arbitrary function $z_i = \sum_{j=r}^{r+s-1} \mathbf{w}_{i,j}\alpha_j - \mathbf{w}_{i,0}$, $i \in \{k, k+1, \dots, r\}$. In the case when all these upper bounds $z_i^* = t_i^*$, $i = k, k+1, \dots, r$, are negative, system (85) has no basic solution $\alpha^{*T} = (\alpha_1^*, \alpha_2^*, \dots, \alpha_{r+s}^*)$ that has α_{r+s}^* as a basic component. Additionally, if all upper bounds t_i^*, $i \in \{k, k+1, \dots, r\}$ of the corresponding functions z_i, $i \in \{k, k+1, \dots, r\}$ on the set of solutions of system (85) are nonnegative finite values, then system (85) may have different basic feasible solutions $\alpha^{*T}(i) = (\alpha_1^*(i), \alpha_2^*(i), \dots, \alpha_{r+s}^*(i))$ with different basic components $\alpha_{r+s}^*(i) = -\dfrac{1}{\mathbf{w}_{i,r+s}} t_i^*$, $i \in \{k, k+1, \dots, r\}$. So, if consistent system (85) with $k \le r$ has a basic feasible solution $\alpha^{*T} = (\alpha_1^*, \alpha_2^*, \dots, \alpha_{r+s}^*)$ suchis that α_{r+s}^* is a basic component, then such a basic solution with such a basic component may be not unique.

In the case when function (89) is upper unbounded on the set of solutions of system (85), the following theorem holds.

Theorem 12. *Assume that system (85) with $k < r$ has solutions and consider the linear programming problem: Maximize the objective function (89) on the set of solutions of system (85). If the objective function(89) on the set of solutions of system (85) is upper unbounded, then system (85) has no basic feasible solution $\alpha^{*T} = (\alpha_1^*, \alpha_2^*, \dots, \alpha_{r+s}^*)$ that has α_{r+s}^* as a basic component.*

Proof. We prove the theorem by contradiction, assuming that the function (89) on the set of solutions of system (85) is upper unbounded and system (85) has a basic feasible solution $\alpha^{*T} = (\alpha_1^*, \alpha_2^*, \dots, \alpha_{r+s}^*)$ that has α_{r+s}^* as a basic component, where $\alpha_{r+s}^* < \infty$. If it is so, then it easy to check that the following system

$$\begin{cases} \alpha_1 + \sum\limits_{j=r+1}^{r+s-1} \mathbf{w}_{1,j}\alpha_j = \mathbf{w}_{1,0}, \\ \dots\dots\dots\dots\dots\dots\dots \\ \alpha_{k-1} + \sum\limits_{j=r+1}^{r+s-1} \mathbf{w}_{k-1,j}\alpha_j = \mathbf{w}_{k-1,0}, \\ \alpha_k + \sum\limits_{j=r+1}^{r+s-1} \mathbf{w}_{k,j}\alpha_j + \mathbf{w}_{k,r+s}\alpha_{r+s} = \mathbf{w}_{k,0} - \mathbf{w}_{k,r+s}\alpha_{r+s}^*, \\ \alpha_{k+1} + \sum\limits_{j=r+1}^{r+s-1} \mathbf{w}_{i+1,j}\alpha_j + \mathbf{w}_{k+1,r+s}\alpha_{r+s} = \mathbf{w}_{k+1,0} - \mathbf{w}_{k+1,r+s}\alpha_{r+s}^*, \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots \\ \alpha_r + \sum\limits_{j=r+1}^{r+s-1} \mathbf{w}_{r,j}\alpha_j + \mathbf{w}_{r,r+s}\alpha_{r+s} = \mathbf{w}_{r,0} - \mathbf{w}_{r,r+s}\alpha_{r+s}^*, \\ \alpha_j \ge 0, \quad j = 1,\ 2,\ \dots,\ r+s, \end{cases} \tag{102}$$

has a basic feasible solution $\alpha^{0^T} = (\alpha_1^0, \alpha_2^0, \dots, \alpha_{r+s-1}^0, \alpha_{r+s}^0)$ that has $\alpha_{r+s}^0 = 0$ as a degenerate basic component and $\alpha_i^0 = \alpha_i^*$, $i = 1, 2, \dots, r+s-1$. The converse also holds, i.e. if for a given α_{r+s}^* system (102) has a basic feasible solution $\alpha^{0^T} = (\alpha_1^0, \alpha_2^0, \dots, \alpha_{r+s-1}^0, \alpha_{r+s}^0)$ that has $\alpha_{r+s}^0 = 0$ as a degenerate basic component, then system (85) has a basic feasible solution $\alpha^{*T} = (\alpha_1^*, \alpha_2^*, \dots, \alpha_{r+s}^*)$ that has α_{r+s}^* as a basic component and $\alpha_i^* = \alpha_i^0$, $i = 1, 2, \dots, r+s-1$.

So, $\alpha^{0^T} = (\alpha_1^0, \alpha_2^0, \dots, \alpha_{r+s}^0)$ with $\alpha_i^0 = \alpha_i^*$, $i = 1, 2, \dots, r+s-1$, and $\alpha_{r+s}^0 = 0$ represents a basic feasible solution with degenerate basic component $\alpha_{r+s}^0 = 0$ for system (102). Then according to Theorem 10 there exists $i_0 \in \{k, k+1, \dots, r\}$ for which the inequality

$$\sum_{j=r+1}^{r+s-1} \mathbf{w}_{i_0,j}\alpha_j \le \mathbf{w}_{i_0,0} - \mathbf{w}_{i_0,r+s}\alpha_{r+s}^* \tag{103}$$

is weakly redundant for system (92) and an arbitrary inequality

$$\sum_{j=r+1}^{r+s-1} \mathbf{w}_{i,j}\alpha_j \le \mathbf{w}_{i,0} - \mathbf{w}_{i,r+s}\alpha_{r+s}^*, \;\; i \in \{k, k+1 \dots, r\}, \tag{104}$$

is redundant for system (92). This means that an arbitrary function

$$z_i = \sum_{j=r+1}^{r+s-1} \mathbf{w}_{i,j}\alpha_j - \mathbf{w}_{i,0}, \;\; i \in \{k, k+1 \dots, r\},$$

is upper bounded on the set of solutions of system (92) and, based on Theorem 11, these functions are upper bounded on the set of solutions of system (85), i. e., we obtain a contradiction. □

Remark 7. Theorem 12 has been proven for the case when the function $z_r = \sum\limits_{j=r+1}^{r+s-1} \mathbf{w}_{rj}\alpha_j - \mathbf{w}_{r,0}$ is unbounded on the set of solutions of system (85). Obviously, this theorem is valid also for the case when an arbitrary function $z_{i_0} = \sum\limits_{j=r+1}^{r+s-1} \mathbf{w}_{i_0,j}\alpha_j - \mathbf{w}_{i_0,0}$, $i_0 \in \{k, k+1 \dots, r\}$, is unbounded on the set of solutions of system (85).

The presented above results prove the following theorem.

Theorem 13. *The consistent system (85) with* $k \le r$ *has a basic feasible solution* $\alpha^{*T} = (\alpha_1^*, \alpha_2^*, \dots, \alpha_{r+s}^*)$ *that has* α_{r+s}^* *as a basic component if and only if there exist* $i_0 \in \{k, k+1, \dots, r\}$ *and* $t_{i_0}^*$ $(0 \le t_{i_0}^* < \infty)$, *such that* $t_{i_0}^*$ *is the upper bound of the function*

$$z_{i_0} = \sum_{j=r+1}^{r+s-1} \mathbf{w}_{i_0,j}\alpha_j - \mathbf{w}_{i_0,0} \tag{105}$$

on the set of solutions of system (85). In this case system (85) with $k \le r$ *has a basic feasible solution* $\alpha^{*T} = (\alpha_1^*, \alpha_2^*, \dots, \alpha_{r+s}^*)$, *where* $\alpha_{r+s}^* = -\frac{1}{\mathbf{w}_{i_0,r+s}} t_{i_0}^*$ *is a basic component. Such a basic feasible solution for system system (85) can be found by using an optimal basic solution* $\overline{\alpha}^{*T} = (\overline{\alpha}_1^*, \overline{\alpha}_2^*, \dots, \overline{\alpha}_{i_0-1}^*, \overline{\alpha}_{i_0+1}^*, \dots, \overline{\alpha}_{r+s-1}^*)$ *of the following linear programming problem: Maximize the objective function (105) on the set of solution of system (90). If such an optimal basic solutions for this problem is known, then the components of the basic solution* α^{*T} *for system (85) that has* $\alpha_{r+s}^* = -\frac{1}{\mathbf{w}_{i_0,r+s}} t_{i_0}^*$ *as a basic component can be found as follows:*

$$\alpha_i^* = \begin{cases} \overline{\alpha}_i^*, & i = 1, 2, \dots, k-1, r+1, \dots, r+s-1, \\ -\sum\limits_{j=r+1}^{r+s-1} \mathbf{w}_{i,j} \overline{\alpha}_j^* + \frac{\mathbf{w}_{i,r+s}}{\mathbf{w}_{i_0,r+s}} t_{i_0}^* + \mathbf{w}_{i,0}, & i = k, k+1 \dots, r-1 \ (i \ne i_0), \\ 0, & i = i_0, \\ -\frac{1}{\mathbf{w}_{i_0,r+s}} t_{i_0}^*, & i = r+s, \end{cases} \tag{106}$$

where $\alpha_{i_0}^* = 0$ *is a nonbasic component of* α^{*T}. *Additionally, the inequality*

$$\sum_{j=r}^{r+s-1} \mathbf{w}_{i_0,j} \alpha_j \le \mathbf{w}_{i_0,0} + t_{i_0}^*,$$

is weakly redundant for system (92) and an arbitrary inequality

$$\sum_{j=r+1}^{r+s-1} \mathbf{w}_{i,j} \alpha_j \le \mathbf{w}_{i,0} + \frac{\mathbf{w}_{i,r+s}}{\mathbf{w}_{i_0,r+s}} t_{i_0}^*, \quad i \in \{k, k+1, \dots, r\},$$

is redundant for system (92).

System (85) has no basic solution $\alpha^{*T} = (\alpha_1^*, \alpha_2^*, \dots, \alpha_{r+s}^*)$ *with the basic component* α_{r+s}^* *if and only if for an arbitrary* $i_0 \in \{k.k+1, \dots, r\}$ *either the function* $z_{i_0} = \sum\limits_{j=r+1}^{r+s-1} \mathbf{w}_{i_0,j} \alpha_j - \mathbf{w}_{i_0,0}$ *is upper unbounded on the set of solutions of system (85) or for an arbitrary feasible solution of system (85)* $z_{i_0} = \sum\limits_{j=r+1}^{r+s-1} \mathbf{w}_{i_0,j} \alpha_j - \mathbf{w}_{i_0,0} < 0$ *holds.*

In fact, the first part of this theorem represents Theorem 11 for the case $i_0 = r$. The second part of this theorem follows from Lemma 7, Corollary 11 and Theorem 12.

To check if a function $z_i = \sum\limits_{j=r+1}^{r+s-1} \mathbf{w}_{i,j} \alpha_j - \mathbf{w}_{i,0}$, $i \in \{k, k+1, \dots, r\}$, on the set of solutions of system (85) is upper bounded or it is upper unbounded on the set of solutions of this system the following lemma can be used.

Lemma 9. *Let the consistent system (85) with $k \le r$ be given. Then a function*

$$z_{i_0} = \sum_{j=r+1}^{r+s-1} \mathbf{w}_{i_0,j}\alpha_j - \mathbf{w}_{i_0,0},\ \ i_0 \in \{k, k+1, \dots, r\}, \tag{107}$$

is upper bounded on the set of solutions of system (85) if an only if the following system

$$\begin{cases} \sum\limits_{j=r+1}^{r+s-1} \mathbf{w}_{i,j}\alpha_j \le 0, & i = 1, 2, \dots, k-1, \\ \sum\limits_{j=r+1}^{r+s-1} \mathbf{w}_{i_0,j}\alpha_j = 1, & \\ \qquad \alpha_j \ge 0, & j = r+1, r+2, \dots, r+s-1. \end{cases} \tag{108}$$

has no solutions; if system (108) has solutions then the function (107) is upper unbounded on the set of solutions of system (85).

Proof. $\Rightarrow$ Assume that the function (107) for a given $i_0 \in \{k, k+1, \dots, r\}$ is upper bounded on the set of solutions of system (85). Then based on Lemma 8 and Theorems 10, 11 we may conclude that the linear function

$$z'_{i_0} = \sum_{j=r+1}^{r+s-1} \mathbf{w}_{i_0,j}\alpha_j$$

is upper bounded on the set of solutions of system (92). Therefore, according to Lemma 1.10 from [15] (page 96), for an arbitrary solution of the homogeneous system

$$\begin{cases} \sum\limits_{j=r+1}^{r+s-1} \mathbf{w}_{i,j}\alpha_j \le 0, & i = 1, 2, \dots, k-1, \\ \qquad \alpha_j \ge 0, & j = r+1, r+2, \dots, r+s-1. \end{cases} \tag{109}$$

$\sum\limits_{j=r+1}^{r+s-1} \mathbf{w}_{i_0,j}\alpha_j \le 0$ holds, i.e. system (108) has no solutions.

$\Leftarrow$ Assume that the function (107) for a given $i_0 \in \{k, k+1, \dots, r\}$ is upper unbounded on the set of solutions of system (85). Then, it is easy to observe that this function is upper unbounded also on the set of solutions of system (92). So, for an arbitrary positive value N, system (92) has solutions such that $\sum\limits_{j=r+1}^{r+s-1} \mathbf{w}_{i_0,j}\alpha_j - \mathbf{w}_{i_0,0} \ge N$ and this involves that the set of solutions of the system

$$\begin{cases} \sum\limits_{j=r+1}^{r+s-1} \mathbf{w}_{i,j}\alpha_j \qquad\qquad \le \ \mathbf{w}_{i,0}, & i = 1, 2, \dots, k-1, \\ -\sum\limits_{j=r+1}^{r+s-1} \mathbf{w}_{i_0,j}\alpha_j + \alpha_{r+s+1} \le -\mathbf{w}_{i_0,0}, & \\ \qquad \alpha_j \ge 0 \quad j = r+1, r+2, \dots, r+s-1; \ \alpha_{r+s+1} \ge 0 & \end{cases} \tag{110}$$

is unbounded with respect to the positive direction of coordinate α_{r+s+1}. Therefore, according to Theorem 1.13 and Remark 2 from [15] (pages 93,96), the set of solutions of system (110) is unbounded with respect to the positive direction of coordinate α_{r+s+1} if and only if the homogeneous system

$$\begin{cases} \sum\limits_{j=r+1}^{r+s-1} \mathbf{w}_{i,j}\alpha_j \qquad\qquad \leq 0, \quad i = 1, 2, \dots, k-1, \\ -\sum\limits_{j=r+1}^{r+s-1} \mathbf{w}_{i_0,j}\alpha_j + \alpha_{r+s+1} \leq 0, \\ \qquad\qquad \alpha_j \geq 0, \ j = r+1, r+2, \dots, r+s-1; \ \alpha_{r+s+1} \geq 0 \end{cases} \tag{111}$$

is unbounded with respect to the positive direction of coordinate α_{r+s+1} of the set of solutions of system (111). So, system (108) has solutions. □

Remark 8. Lemma 9 represents a refinement of the corresponding Lemma 9 from [5].

Based on results obtained in this section, in the next section we propose a polynomial algorithm for checking if $Y \not\subset Y_h$.

10 Polynomial algorithm to check if the condition $Y \not\subset Y_h$ holds for a given $h \in [-2^L, 2^L]$

Let us assume that the coefficients of disjoint bilinear programming problem (57)-(59) with a perfect disjoint subset Y determined by the set of solutions of system (59) are integer. In this section, based on the results of the previous section, we propose a polynomial algorithm for checking if system (71) has a basic solution $x_1^*, x_2^*, \dots, x_{n+q}^*, v_1^*, v_2^*, \dots, v_{p+1}^*$ that has v_{p+1}^* as basic component for a fixed $h \in [-2^L, 2^L]$. Note that in system (71), parameter h may be not integer, however it is a rational value represented by an irreducible rational fraction $h = \dfrac{M}{N}$, where $|M|$ and $|N|$ do not exceed 2^L. Therefore, by multiplying the coefficients of equation $r = q+m+1$ of system (71) by a suitable integer value of the form 2^L, all coefficients of this equation become integer.

According to Theorem 8 and Corollary 9, if system (45) has no solutions, then for a given $h \in [-2^L, 2^L]$, the property $Y \not\subset Y_h$ holds if and only if system (71) has a basic feasible solution that has v_{p+1} as a basic component. In the case when system (45) has solutions, the problem of checking the condition $Y \not\subset Y_h$ can be easily solved on the basis of Corollary 4 and Theorem 3. Moreover, in this case, the optimal value h^* of the disjoint bilinear programming problem (57)-(59) and an optimal point $x^* \in X$ for this problem can be found by solving the linear programming problem (44)-(45). Therefore here we propose a polynomial algorithm for checking the existence of a basic solution for system (71) that has v_{p+1} as a basic component when system (45) has no solutions. In this case system (71) is consistent for an arbitrary $h \in [-2^L, 2^L]$, and the proposed algorithm can be applied for checking if

$Y \not\subset Y_h$. In this algorithm we assume that system (71) is represented in matrix form by system (79) consisting of r linear equations with nonnegative conditions for the vector of variables $\alpha^T = (\alpha_1, \alpha_2, \dots, \alpha_{r+s})$, where $r = q + m + 1$, $s = n + p - m$ and the variable α_{r+s} corresponds to the variable v_{p+1} in system (71). Thus, the proposed algorithm determines if either the property $Y \not\subset Y_h$ holds or fails to holds. In the case when for a given $h \in [-2^L, 2^L]$ the condition $Y \not\subset Y_h$ holds the algorithm determines a basic feasible solution $\alpha^{*T} = (\alpha_1^*, \alpha_2^*, \dots, \alpha_{r+s}^*)$ that has α_{r+s}^* as a basic component and consequently algorithm determines a basic solution $x_1^*,\ x_2^*,\ \dots,\ x_{n+q}^*, v_1^*,\ v_2^*,\ \dots, v_{p+1}^*$ for system (71) that has v_{p+1}^* as a basic component.

Algorithm 1:

Step 1 (Preliminary step):
Fix $h \in [-2^L,\ 2^L]$ and consider system (71) with the given h. In matrix form this system is represented by system (79) with respect to the vector of variables $\alpha^T = (\alpha_1, \alpha_2, \dots, \alpha_{r+s})$, where the component α_{r+s} in (79) corresponds to component v_{p+1} in (71), i.e. $\alpha_{r+s} = v_{p+1}$. If the conditions of Theorem 8 are satisfied, then for a given $h \in [2^{-L}, 2^L]$, system (79) has solutions.

Step 2:
Find a basic feasible solution $\alpha^{0^T} = (\alpha_1^0, \alpha_2^0, \dots, \alpha_{r+s}^0)$ of system (79). If α_{r+s}^0 is a basic component of α^{0^T}, then, according to Theorem 8, system (71) has a basic feasible solution that has $v_{p+1}^0 = \alpha_{r+s}^0$ as a basic component, i. e. $Y \not\subset Y_h$ *holds and Stop.* If α_{r+s}^0 is a nonbasic component of α^{0^T}, then all basic components are among $\alpha_1^0, \alpha_2^0, \dots, \alpha_{r+s-1}^0$. Without loss of generality, we may assume that the basic components of α^{0^T} represent the first its r components $\alpha_1^0, \alpha_2^0, \dots, \alpha_r^0$, which correspond to the linearly independent column vectors $W_1, W_2, \dots, W_r$ that form a basis for the set of column vectors of matrix W in (79); if this is not so, then we can relabel variables in (79) so that the first r variables in (79) will correspond to the basic variables. In this case, system (79) in extended form has the structure of system (83) with the first r variables corresponding to the basic ones, where basic variables are expressed explicitly via the nonbasic variables and $\alpha_{r+s} = v_{p+1}$. Let $\mathbf{W}_{r+s}$ be the column vector of coefficients of system (83) that corresponds to α_{r+s}, i. e. $\mathbf{W}_{r+s}^T = (\mathbf{w}_{1,r+s}, \mathbf{w}_{2,r+s}, \dots, \mathbf{w}_{r,r+s})$, and $\mathbf{W}_0$ be the column vector of right-hand side of coefficients of system (83), i.e. $\mathbf{W}_0^T = (\mathbf{w}_{1,0}, \mathbf{w}_{2,0}, \dots, \mathbf{w}_{r,0})$. If the components of these vectors satisfy conditions of Lemma 6, i.e. if either $\mathbf{W}_{r+s}$ contains a positive component $\mathbf{w}_{i_0,r+s}$ ($\mathbf{w}_{i_0,r+s} > 0$) or $\mathbf{W}_{r+s}$ contains a nonzero component $\mathbf{w}_{i_0,r+s} \neq 0$ (positive or negative) such that the corresponding component $\mathbf{w}_{i_0,0}$ of vector $\mathbf{W}_0^T$ is equal to zero, then according to Lemma 6, system (83) has a basic feasible solution $\alpha^{*T} = (\alpha_1^*, \alpha_2^*, \dots, \alpha_{r+s}^*)$ that has α_{r+s}^* as a basic component; such a basic solution can be obtained by using one step of the standard pivoting calculating procedure from Lemma 6. So, in this case, system (71) has a basic feasible solution $x_1^*,\ x_2^*,\ \dots,\ x_{n+q}^*, v_1^*,\ v_2^*,\ \dots, v_{p+1}^*$ that has v_{p+1}^* as a basic component, i. e. $Y \not\subset Y_h$ *holds and Stop.* If the components of the vectors $\mathbf{W}_{r+s}$

and $\mathbf{W}_0$ of system (83) do not satisfy the conditions of Lemma 6, then go to next *Step 3.*

Step 3:
We relabel the equations and variables $\alpha_1, \alpha_2, \dots, \alpha_r$ of system (83) so that it has the form of system (85), where

$$\mathbf{w}_{i,r+s} = 0, \quad i = 1, 2, \dots, k-1; \qquad \mathbf{w}_{i,r+s} < 0, \quad i = k, k+1, \dots, r,$$

and

$$\mathbf{w}_{i,0} \ \geq \ 0, \quad i = 1, 2, \dots, k-1; \qquad \mathbf{w}_{i,0} \ > 0, \quad i = k, k+1, \dots, r.$$

Then go to next *Step 4.*

Step 4:
The problem of the existence of the basic solution $\alpha^{*T} = (\alpha_1^*, \alpha_2^*, \dots, \alpha_{r+s}^*)$ for system (85) that has α_{r+s}^* as a basic component can be solved on the basis of Theorems 11-13 and Lemma 9 as follows. For an arbitrary $i_0 \in \{k, k+1, \dots, r\}$ check if system (108) is consistent. If at least for an $i_0 \in \{k, k+1, \dots, r\}$ system (108) has solutions, then based on Lemma 9 and Theorem 12, system (85) has no basic solution with basic component α_{r+s}^* and consequently system (71) has no basic solution that has v_{p+1}^* as a basic component, i.e. the *condition* $Y \not\subset Y_h$ *fails to hold and Stop*; if for an $i_0 \in \{k, k+1, \dots, r\}$ system (108) has no solutions, then we consider for each $i_0 \in \{k, k+1, \dots, r\}$ the linear programming problem: Maximize the objective function (107) subject to (85). By solving such linear programming problems for $i_0 = k, k+1, \dots, r$, we determine the corresponding optimal values $z_k^*, z_{k+1}^*, \dots, z_r^*$ of the objective functions for these linear programming problems. If there exists $z_{i_0}^* \in \{z_k^*, z_{k+1}^*, \dots, z_r^*\}$ such that $z_{i_0}^* \geq 0$, then, according to Theorem 11, system (85) has a basic solution $\alpha^{*T} = (\alpha_1^*, \alpha_2^*, \dots, \alpha_{r+s}^*)$ such that α_{r+s}^* is a basic component and consequently system (71) has a basic solution that has v_{p+1}^* as a basic component, i.e. $Y \not\subset Y_h$ *holds and Stop*; if $z_i^* < 0$, $i = k, k+1, \dots, r$, then according to Corollary 11 system (85) has no basic solution $\alpha^{*T} = (\alpha_1^*, \alpha_2^*, \dots, \alpha_{r+s}^*)$ that has α_{r+s}^* as a basic component, and consequently system (71) has no basic solution that has v_{p+1}^* as a basic component, i.e. *condition* $Y \not\subset Y_h$ *fails to hold and Stop.*

Remark 9. Algorithm 1 determines if the condition $Y \not\subset Y_h$ holds for a given $h \in [-2^L, 2^L]$. In the case when this condition holds, the algorithm determines also a basic feasible solution $\alpha^{*T} = (\alpha_1^*, \alpha_2^*, \dots, \alpha_{r+s}^*)$ of system (85) that has α_{r+s}^* as a basic component for a given h, and consequently determines a basic feasible solution $x_1^*,\ x_2^*,\ \dots,\ x_{n+q}^*, v_1^*,\ v_2^*,\ \dots, v_{p+1}^*$ of system (71) that has v_{p+1}^* as a basic component. Step 4 of the above Algorithm 1 differs from Step 4 of Algorithm 1 from [5] only by the refinement that the consistency of system (108) is checked for any $i_0 \in \{k, k+1, \dots, r\}$.

In the worst case Algorithm 1 solves not more that $2r$ linear programming problems. Therefore the computational complexity of this algorithm depends on the computational complexity of linear programming algorithms used in Algorithm 1. Taking into account that for solving a linear programming problem there exist polynomial algorithms (see [20–25]), we can formulate the following result.

Theorem 14. *Algorithm 1 for checking the condition* $Y \not\subset Y_h$ *and determining a basic feasible solution* $x_1^*,\ x_2^*,\ \ldots,\ x_{n+q}^*, v_1^*,\ v_2^*,\ \ldots, v_{p+1}^*$ *of system (71) that has* v_{p+1}^* *as a basic component for fixed* $h \in [-2^L, 2^L]$ *is polynomial.*

11 Polynomial algorithm for solving the disjoint bilinear programming problem (57)-(59)

We present a polynomial algorithm for solving the disjoint bilinear programming problem (57)-(59) in which the set of solutions Y of system (59) satisfies conditions a)-c), i.e. when Y has the structure of a perfect polytope.

Algorithm 2:

If for disjoint bilinear programming problem (57) - (59) the corresponding system (45) has solutions, then an optimal point $x^* \in X$ for this problem can be found by solving the following linear programming problem: Minimize the objective function (44) subject to (45). Then x^* together with an arbitrary $y \in Y$ represent an optimal solution of disjoint bilinear programming problem (57)-(59). In this case, the optimal value of the disjoint bilinear programming problem is $h^* = gx^*$. If system (45) is inconsistent, then we use the following iterative calculation procedure for determining the optimal solution of disjoint bilinear programming problem (57) - (59):

Step 0:(Preliminary step :
Fix $h_1^1 = -2^L,\ \ h_1^2 = 2^L,\ \ \varepsilon = 2^{-2L-2}$.

General step (step k, $k \geq 1$*):*
Check if $h_k^2 - h_k^1 \leq \varepsilon$. If $h_k^2 - h_k^1 \leq \varepsilon$, then go to *Final step*; If $h_k^2 - h_k^1 > \varepsilon$, then find $h_k^0 = \dfrac{h_k^1 + h_k^2}{2}$, and consider system (79) with $h = h_k^0$, and apply Algorithm 1 to determine if this system has a basic feasible solution $\alpha^{*T} = (\alpha_1^*, \alpha_2^*, \ldots, \alpha_{r+s}^*)$ that has α_{r+s}^* as a basic component. If system (79) has such a solution, then $Y \not\subset Y_h$ and based on the relationship of this system with system (71), we obtain a basic feasible solution $x_1^*, x_2^*, \ldots, x_{n+q}^*, v_1^*, v_2^*, \ldots, v_{p+1}^*$ for system (71) with $h = h_k^0$ that has v_{p+1}^* as a basic component. In this case, we set $h_{k+1}^1 = h_k^1$, $h_{k+1}^2 = h_k^0$ and go to step $k+1$. If system (79) has no basic feasible solution $\alpha^{*T} = (\alpha_1^*, \alpha_2^*, \ldots, \alpha_{r+s}^*)$ that has α_{r+s}^* as a basic component, then the condition $Y \not\subset Y_h$ fails to hold, and we put $h_{k+1}^1 = h_k^0$, $h_{k+1}^2 = h_k^2$ and go to step $k+1$.

Final Step:
After $3L+2$ iterations of the general step of the algorithm, we determine the value $\widetilde{h} = h_{3L+2}$ with no more than L digits before the decimal point and no more than $3L+2$ digits after the decimal point. The value $\widetilde{h}$ approximates the exact optimal

value h^* that is a rational value representing an irreducible fraction $h^* = \dfrac{M}{N}$ such that $|M|, |N| \le 2^L$ and $|\widetilde{h} - \dfrac{M}{N}| \le 2^{-2L-2}$. Therefore, here, in a similar way as it has been done in [20, 21] for the linear programming problem, we find the exact value h^* by representing $\widetilde{h}$ as an infinite fraction expansion. After that, fix $h = h^*$ in (71) and find an optimal basic solution $x_1^*,\ x_2^*,\ \dots,\ x_{n+q}^*, v_1^*,\ v_2^*,\ \dots, v_{p+1}^*$ for system (71) that has v_{p+1}^* as a degenerate basic component. According to Theorem 9, $x_1^*,\ x_2^*,\ \dots,\ x_n^*$ represent the components of an optimal point x^* for the disjoint linear programming problem (57)-(59). If x^* is known, then by finding a solution of system (78) we determine the components $y_1^*, y_2^*, \dots, y_m^*$ of the optimal point y^*, where (x^*, y^*) is an optimal solution of disjoint bilinear programming (57)-(59) for which $z^* = x^*Cy^* + gx^* + ey^*$ and *Stop.*

Note that the computational complexity of Algorithm 2 for solving problem (57) - (59) depends in the most part on the computational complexity of the linear programming algorithms used at the general and final steps of the algorithm. Taking into account that for the linear programming problem there exist polynomial algorithms (see [20–25]) and that the number of iterations at the general step of the algorithm is polynomial, we can formulate the following theorem.

Theorem 15. *Algorithm 2 finds an optimal solution of the disjoint bilinear programming problem (57)-(59) in polynomial time.*

12 Application of Algorithm 2 for solving the problems from Section 1

In this section, we show how Algorithm 2 can be applied for solving the boolean linear programming problem (8), (9) and the piecewise linear concave programming problem (2), (13).

12.1 Application of Algorithm 2 for finding a boolean solution of system (4) and solving problem (8), (9)

The problem of finding a boolean solution for system (4) can be solved by using Algorithm 2, because this problem can be represented as the disjoint bilinear programming problem (5)-(7) where the set of solutions Y has the structure of an acute-angled polytope. Indeed, in this case the matrix $D = (d_{ij})_{p \times n}$ and the vector

$d_0 = (d_{10}, d_{20}, \dots, d_{m0})^T$ have the following structure

$$D = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -1 & 0 & 0 & \cdots & 0 \\ 0 & -1 & 0 & \cdots & 0 \\ 0 & 0 & -1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & -1 \end{pmatrix}, \quad d_0 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ \vdots \\ 1 \\ 0 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix},$$

where $p = 2n,\ m = n$ and $C = (c_{ij})_{n\times n},\ g = (g_1, g_2, \dots, g_n),\ e = (e_1, e_2, \dots, e_n)$ are defined as follows:

$$c_{ij} = \begin{cases} -2 & \text{if } i = j, \\ 0 & \text{if } i \neq j, \end{cases} \quad g_i = 1,\ e_j = 1, \quad i = 1, 2, \dots n; j = 1, 2, \dots, n,$$

In this problem the solution set X is determined by (6) and therefore the corresponding matrix and column vector that determine X should be specified according to (6). By applying Algorithm 2 for this problem with such matrices and column vectors, we determine a boolean solution for the system of linear inequalities (4) if the optimal value z^* of the object function of disjoint bilinear programming problem (5)-(7) is equal to zero; otherwise system (4) has no boolean solution. If system (4) has a boolean solution, then for finding such a solution it is sufficient to find for the corresponding system (71) with $h = 0$ a basic solution $x_1^*, x_2^*, \dots, x_{n+q}^*, v_1^*, v_2^*, \dots, v_{p+1}^*$ with degenerate basic component $v_{p+1}^* = 0$, i.e. in this case it is sufficient to apply only Algorithm 1.

If it is necessary to solve the classical boolean linear programming problem (8), (9), then we can use the disjoint bilinear programming problem (10)-(12) with the corresponding M chosen as mentioned in Section 1. In this case the matrix D and column vector d_0 are the same as above, and $C = (c_{ij})_{n\times n},\ g = (g_1, g_2, \dots, g_n)$, $e = (e_1, e_2, \dots, e_m)$ are defined as follows:

$$c_{ij} = \begin{cases} -2M & \text{if } i = j, \\ 0 & \text{if } i \neq j, \end{cases} \quad g_i = c_i + M,\ e_j = M, \quad i = 1, 2, \dots n; j = 1, 2, \dots, n.$$

12.2 Polynomial time algorithm for solving the piecewise linear concave programming problem (2), (13)

The piecewise linear concave programming problem consists of minimizing the piecewise linear concave function (13) on the set of solutions of system (2). In Section 1 it has been shown that this problem can be represented as the bilinear programming problem (14)-(16). It can be observed that in this problem the set of solutions Y of system (16) has the structure of an acute-angled polytope. So, we obtain a disjoint bilinear programming problem (57)-(59) in which the matrix $D = (d_{ij})_{p\times m}$ and vector $d_0 = (d_{10}, d_{20}, \dots, d_{p0})^T$ are defined as follows:

$$D = \begin{pmatrix} 1 & 1 & \dots & 1 & 0 & 0 & \dots & 0 & \dots & 0 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 & 1 & 1 & \dots & 1 & \dots & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 & \dots & 1 & 1 & \dots & 1 \\ -1 & 0 & \dots & 0 & 0 & 0 & \dots & 0 & \dots & 0 & 0 & \dots & 0 \\ 0 & -1 & \dots & 0 & 0 & 0 & \dots & 0 & \dots & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 & \dots & 0 & 0 & \dots & -1 \end{pmatrix}, \; d_0 = \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix},$$

where $m = \sum_{i=1}^{l} m_i - l$; $p = l + m$. The matrix C in this problem is determined by the column vectors $\overline{c}^{j,k} = (c^{jk} - c^{jm_j})^T, j = 1, 2, \dots, l; k = 1, 2, \dots, m_j - 1$, i.e.

$$C = [\overline{c}^{11}, \overline{c}^{12}, \dots, \overline{c}^{1r_1-1}, \overline{c}^{21}, \overline{c}^{22}, \dots, \overline{c}^{2m_2-1}, \dots\dots, \overline{c}^{l1}, \overline{c}^{l2}, \dots, \overline{c}^{lm_l-1}];$$

$g = \sum\limits_{j=1}^{l} c^{jm_j}$ and e is the vector determined by the following components

$$\overline{c}_0^{jk} = c_0^{jk} - c_0^{jm_j}, \; j = 1, 2, \dots, l; k = 1, 2, \dots, m_j - 1,$$

i. e.

$$e = (\overline{c}_0^{11}, \overline{c}_0^{12}, \dots, \overline{c}_0^{1m_1-1}, \overline{c}_0^{21}, \overline{c}_0^{22}, \dots, \overline{c}_0^{2m_2-1}, \dots\dots, \overline{c}_0^{l1}, \overline{c}_0^{l2}, \dots, \overline{c}_0^{lm_l-1}).$$

So, by applying Algorithm 2 for the disjoint bilinear programming problem with given matrix D and column vector d_0, we obtain the optimal solution of the piecewise linear concave programming problem (2), (13).

13 Conclusion

A polynomial algorithm for solving the disjoint bilinear programming problem with a perfect disjoint subset has been proposed and grounded. This means that a polynomial algorithm exists for the disjoint bilinear programming problem in which one of the disjoint subsets has the structure of an acute-angled polytope. Taking into account that the boolean linear programming problem and the piecewise linear concave programming problem can be represented as disjoint bilinear problems with an acute-angled polytope for one of the disjoint subsets, these problems can be solved in polynomial time using Algorithm 2. Based on this an important conclusion concerned with the coincidence of classes P and NP (see [13,14,23,26]) can be made.

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