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Pure and Mixed Stationary Equilibria for Dynamic Positional Games on Graphs

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Abstract. A class of m-player dynamic positional games on graphs that extends the two-player zero-sum mean payoff games on graphs is formulated and studied. We consider dynamic positional games with average and discounted payoffs criteria for the players. We show that for an arbitrary game with average payoffs there exists Nash equilibrium in mixed stationary strategies and for an arbitrary two-player zero-sum average positional game there exists Nash equilibrium in pure stationary strategies. Additionally we show that for an arbitrary dynamic positional game with discounted payoffs there exists a Nash equilibrium in pure stationary strategies. Some approaches for determining the optimal stationary strategies of the players in such games are proposed.

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1 Introduction

In this article we formulate and study a class of m-player dynamic positional games on graphs that extends and generalizes the two-player zero-sum positional games on graphs from [1, 2, 4, 11, 12]. The formulation of this class of dynamic positional games is the following.

Let $G = (X, E)$ be a finite directed graph in which an arbitrary vertex $x \in X$ has at least one outgoing directed edge $e = (u, v) \in E$. The vertex set X of G is divided into m disjoint subsets $X_1, X_2, \dots, X_m$ ($X = X_1 \cup X_2 \cup \dots \cup X_m$; $X_i \cap Xj = \emptyset, i \neq j$) which are regarded as position sets of m players. On edge set m functions $r^i : F \to R,\ i = 1, 2, \dots, m$ are defined that assign to each directed edge $e = (x, y) \in E$ the values $r^1_e, r^2_e, \dots, r^m_e$ that are regarded as the rewards for the corresponding players $1, 2, \dots, m$ when in G is made a move through a directed edge $e = (x, y) \in E$ from x to y. On G the following m-person dynamic game is considered.

The game starts at a given position $x_0 \in X$ at the moment of time $t = 0$, where the player $i \in \{1, 2, \dots, m\}$ who is the owner of the starting position x_0 makes a move from x_0 to a neighbor position $x_1 \in V$ through a directed edge $e_0 = (x_0, x_1) \in E$. After that players $1, 2, \dots, m$ receive the corresponding rewards $r^1_{e_0}, r^2_{e_0}, \dots, c^m_{r_0}$. Then at the moment of time $t = 1$ the player $k \in \{1, 2, \dots, m\}$ who is the owner of

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position x_1 makes a move from x_1 to a position $x_2 \in X$ through the directed edge $e_1 = (x_1, x_2) \in E$. After that players $1, 2, \ldots, m$ receive the corresponding rewards $r^1_{e_1}, r^2_{e_1}, \ldots, r^m_{e_1}$, and so on, indefinitely. Such a play of the game on G produces the sequence of positions $x_0, x_1, x_2, \ldots, x_t \ldots$ induced by strategies of moves of the corresponding players $1, 2, \ldots, m$. If in this game each player $i \in \{1, 2, \ldots, m\}$ makes moves from his positions set trough outgoing directed edges in order to maximize his average reward per transition

$$\omega^i_{x_o} = \lim_{t\to\infty} \inf \frac{1}{t} \sum_{\tau=0}^{t-1} r^i_{e_\tau} \quad i = 1, 2, \ldots, m,$$

then such a game on G we call *average positional game.* If in this game each player $i \in \{1, 2, \ldots, m\}$ makes moves from his positions set trough outgoing directed edges in order to maximize the discounted sum of rewards

$$\psi^i_{x_o} = \sum_{\tau=0}^{\infty} \lambda^i r^i_{e_\tau}, \quad i = 1, 2, \ldots, m,$$

with given discount factor λ, $0 < \lambda < 1$, then such a game on G we call *discounted positional game.*

The considered games in the case $m = 2$ and $r^1_e = -r^2_e = r_e$, $\forall e \in E$ become the two-player zero-sum positional games with average and discounted payoffs criteria for the players, respectively. These cases of antagonistic positional games where studied in [2,4,5,11]. In [2,4] has been shown that for a zero-sum average positional game of two players there exists the value $v(x_0)$ such that the first player has a strategy of moves that insures $\lim_{t\to\infty} \inf \frac{1}{t} \sum_{\tau=0}^{t-1} r(e_\tau) \geq v(x_0)$ and the second player has a strategy of moves that insure $\lim_{t\to\infty} \sup \frac{1}{t} \sum_{\tau=0}^{t-1} r(e_\tau) \leq v(x_0)$, and players in such a game can achieve the values $v(x_0)$ by applying the strategies of moves that do not depend on t but depend only on the vertex (position) from which the player is able to move. Therefore such strategies in [2,12] where called positional strategies; in [4,11] such strategies where called stationary strategies. In fact, such strategies in a tw0-player zero-sum game can be specified as *pure stationary strategies* because each move through a directed edge at each position of the game is chosen by the corresponding player from the set of feasible strategies with the probability equal to 1 and in each position such a strategy of the move does not change in time.

In general case, for a non-zero-sum average positional game on a graph a Nash equilibrium in pure stationary strategies may not exist. This fact has been shown in [4], where an example of two-player non-zero-sum average positional game that has no Nash equilibria in pure strategies is constructed.

In this article we extend the notion of pure stationary strategies to mixed stationary strategies for the players in average positional games, assuming that players in their positions at different moment of time may make moves randomly according to a probability distribution over the set of possible moves. Such an extinction of the average positional game leads to a stochastic positional game studied in [6, 11]. Using the results from [6, 11] we show that for an arbitrary m-player average positional game on a given graph there exists a Nash equilibrium in mixed stationary strategies and for an arbitrary two-player zero-sum average positional game there exists a Nash equilibrium in pure stationary strategies. Additionally we show that for an arbitrary dynamic discounted positional game on a given graph graph there exists a Nash equilibrium in pure stationary strategies.

2 Stationary Nash equilibria for stochastic positional games

In this section we presents some results from [6–8] concerned with the existence and determining Nash equilibria for stochastic positional games and based on these results in the next section we show how to determine the existence of Nash equilibria for dynamic positional games with average and discounted payoffs on graphs. Stochastic positional games represents a special class of stochastic game in which the set of states of the Markov decision process is divided into m disjoint subsets and in each subset of the states the Marlov processes is controlled only by one player. The detailed formulation of such game we present in the following subsection.

2.1 Formulation of stochastic positional games with average and discounted payoffs for the players

An m-player stochastic positional game consists of the following elements:

- a state space X (which we assume to be finite);
- a partition $X = X_1 \cup X_2 \cup \cdots \cup X_m$ where X_i represents the position set of player $i \in \{1, 2, \dots, m\}$;
- a finite set $A(x)$ of actions in each state $x \in X$;
- a step reward $r^i_{x,a}$ with respect to each player $i \in \{1, 2, \dots, m\}$ in each state $x \in X$ and for an arbitrary action $a \in A(x)$;
- a transition probability function $p : X \times \prod_{x \in X} A(x) \times X \to [0, 1]$ that gives the probability transitions $p^a_{x,y}$ from an arbitrary $x \in X$ to an arbitrary $y \in X$ for a fixed action $a \in A(x)$, where $\sum_{y \in X} p^a_{x,y} = 1, \ \ \forall x \in X, \ a \in A(x)$;
- a starting state $x_0 \in X$.

The stochastic positional game starts at the moment of time $t = 0$ in a given state x_0 where the player $i \in \{1, 2, \dots, m\}$ who is the owner this state position x_0 ($x_0 \in X_i$) chooses an action $a_0 \in A(x_0)$ and determines the rewards $r^1_{x_0,a_0}, r^2_{x_0,a_0}, \dots, r^m_{x_0,a_0}$ for the corresponding players $1, 2, \dots, m$. After that the game passes to a state $y = x_1 \in X$ according to a certain probability distribution $\{p^{a_0}_{x_0,y}\}$. At the moment of time $t = 1$ the player $k \in \{1, 2, \dots, m\}$ who is the owner of the state position x_1 ($x_1 \in X_k$) chooses an action $a_1 \in A(x_1)$ and players $1, 2, \dots, m$ receive the corresponding rewards $r^1_{x_1,a_1}, r^2_{x_1,a_1}, \dots, r^m_{x_1,a_1}$. Then the game passes to a state $y = x_2 \in X$ according to a probability distribution $\{p^{a_1}_{x_1,y}\}$ and so on indefinitely. Such a play of the game produces a sequence of states and actions $x_0, a_0, x_1, a_1, \dots, x_t, a_t, \dots$ that defines a stream of stage rewards $r^1_{x_t,a_t}, r^2_{x_t,a_t}, \dots, r^m_{x_t,a_t}$, $t = 0, 1, 2, \dots$. For this stochastic process controlled by m players in [6, 7, 9] the following two stochastic positional games where formulated and studied: *the average stochastic positional game* and *the discounted stochastic positional game. The average stochastic positional game* is the game with payoffs of the players

$$\omega^i_{x_0} = \lim_{t\to\infty} \inf \ \mathsf{E}\Big(\frac{1}{t}\sum_{\tau=0}^{t-1} r^i_{x_\tau,a_\tau}\Big), \quad i = 1, 2, \dots, m,$$

where E is the expectation operator with respect to the probability measure in the Markov process induced by actions chosen by players in their position sets and given starting state x_0. *The discounted stochastic positional game* is the game with payoffs of the players

$$\sigma^i_{x_0} = \mathsf{E}\Big(\frac{1}{t}\sum_{\tau=0}^{t-1} \lambda^i r^i_{x_\tau,a_\tau}\Big), \quad i = 1, 2, \dots, m,$$

where λ is a given discount factor such that $0 < \lambda < 1$. These stochastic games extend and generalize deterministic positional games.

2.2 Pure and mixed stationary strategies for stochastic positional games

A *strategy* of player $i \in \{1, 2, \dots, m\}$ in a stochastic positional game is a mapping s^i that provides for every state $x_t \in X_i$ a probability distribution over the set of actions $A(x_t)$. If these probabilities take only values 0 and 1, then s^i is called a *pure strategy*, otherwise s^i is called *mixed strategy.* If these probabilities depend only on the state $x_t = x \in X_i$ (i. e. s^i do not depend on t), then s^i is called a *stationary strategy*, otherwise s^i is called a *non-stationary strategy.* Thus, a pure stationary strategy of player i is a mapping

$$s^i : x \to a \in A(x) \ \text{ for } \ x \in X_i$$

that determines an action $a \in A(x)$ for each state $x \in X_i$, i.e. $s^i(x) = a$ for $x \in X_i$. This means that a pure stationary strategy s^i of player i we can identify with the set of boolean variables $s^i_{x,a} \in \{0, 1\}$, where for a given $x \in X_i$ $s^i_{x,a} = 1$ if and only if player i fixes the action $a \in A(x)$. So, we can represent the set of pure stationary strategies $\mathbb{S}^i$ of player i as the set of solutions of the following system:

$$\begin{cases} \sum\limits_{a\in A(x)} s^i_{x,a} = 1, & \forall x \in X_i; \\ s^i_{x,a} \in \{0, 1\}, & \forall x \in X_i, \ \ \forall a \in A(x). \end{cases} \tag{1}$$

Obviously the sets of pure strategies $\mathbb{S}^1, \mathbb{S}^2, \dots, \mathbb{S}^m$ of players are finite sets. If in system (1) we change the restrictions $s^i_{x,a} \in \{0, 1\}$ for $x \in X_i$, $a \in A(x)$ by the conditions $0 \le s^i_{x,a} \le 1$ then we obtain the set of mixed stationary strategies where $s^i_{x,a}$ is treated as the probability of the choices of the action a by player i every time when the state x is reached by any route in the dynamic stochastic game. Thus, we can identify the set of mixed stationary strategies S^i of player i as the set of solutions of the system

$$\begin{cases} \sum\limits_{a\in A(x)} s^i_{x,a} = 1, & \forall x \in X_i; \\ \qquad s^i_{x,a} \ge 0, & \forall x \in X_i, \quad \forall a \in A(x) \end{cases} \tag{2}$$

and for a given profile **s**$= (s^1, s^2, \dots, s^m) \in S = S^1 \times S^2 \times \dots \times S^m$ of mixed strategies $s^1, s^2, \dots, s^m$ of the players the probability transition matrix $P^s = (p^s_{x,y})$ induced by s can be calculated as follows

$$p^s_{x,y} = \sum_{a\in A(x)} s^i_{x,a} p^a_{x,y} \quad \text{for} \quad x \in X_i, \quad i = 1, 2, \dots, m. \tag{3}$$

In the sequel we will distinguish stochastic games in pure and mixed stationary strategies

An average stochastic positional game in pure and mixed stationary strategies can be defined as follows. Let $s = (s^1, s^2, \dots, s^m)$ be a profile of stationary strategies (pure or mixed strategies) of the players. Then the elements of probability transition matrix $P^s = (p^s_{x,y})$ in the Markov process induced by s can be calculated according to (3). Therefore if $Q^s = (q^s_{x,y})$ is the limiting probability matrix of P^s, then the average payoffs per transition $\omega^1_{x_0}(s),\ \omega^2_{x_0}(s), \dots, \omega^m_{x_0}(s)$ for the players are determined as follows

$$\omega^i_{x_0}(s) = \sum_{k=1}^{m} \sum_{y\in X_k} q^s_{x_0,y} \cdot r^i_{y,s^k}, \quad i = 1, 2, \dots, m, \tag{4}$$

where

$$r^i_{y,s^k} = \sum_{a\in A(y)} s^k_{y,a} \cdot r^i_{y,a}, \quad \text{for} \quad y \in X_k, \ k \in \{1, 2, \dots, m\} \tag{5}$$

expresses the average reward (step reward) of player i in the state $y \in X_k$ when player k uses the strategy s^k.

The functions $\omega^1_{x_0}(s), \omega^2_{x_0}(s), \dots, \omega^m_{x_0}(s)$ on $S = S^1 \times S^2 \times \dots \times S^m$ defined according to (4),(5) determine a game in normal form that we denote $\langle \{S^i\}_{i=\overline{1,m}}, \{\omega^i_{x_0}(s)\}_{i=\overline{1,m}} \rangle$. This game corresponds to the *average stochastic positional game in mixed stationary strategies* that in extended form is determined by the tuple $(\{X_i\}_{i=\overline{1,m}}, \{A(x)\}_{x\in X}, \{r^i\}_{i=\overline{1,m}}, p)$. The functions $\omega^1_{x_0}(s), \omega^2_{x_0}(s), \dots, \omega^m_{x_0}(s)$ on $\mathbb{S} = \mathbb{S}^1 \times \mathbb{S}^2 \times \dots \times \mathbb{S}^m$ determine the static game in normal form $\langle \{\mathbb{S}^i\}_{i=\overline{1,m}}, \{\omega^i_{x_0}(s)\}_{i=\overline{1,m}} \rangle$ that corresponds to the *discounted stochastic positional game in pure strategies* and in extended form it also is determined by the tuple $(\{X_i\}_{i=\overline{1,m}}, \{A(x)\}_{x\in X}, \{r^i\}_{i=\overline{1,m}}, p)$.

A discounted stochastic positional game in pure and mixed stationary strategies can be defined as follows. Let $s = (s^1, s^2, \dots, s^m)$ be a profile of stationary strategies (pure or mixed strategies) of the players in a stochastic positional game. Then the elements of probability transition matrix $P^s = (p^s_{x,y})$ induced by s can be calculated according to (3) and we can find the matrix $W^s = (w^s_{x,y})$, where $W^s = (I - \gamma P^s)$. After that we can find the payoff for the players as follows

$$\sigma^i_{x_0}(s) = \sum_{y \in X} w^s_{x_0,y} \cdot r^i_{y,s}, \quad i = 1, 2, \ldots, m,$$

where $r^i_{y,s}$ is determined according to (5).

The functions $\sigma^1_{x_0}(s),\ \sigma^2_{x_0}(s),\ \ldots,\ \sigma^m_{x_0}(s)$ on $S = S^1 \times S^2 \times \cdots \times S^m$, defined according to (4),(5), determine a static game in normal form that we denote $\langle \{S^i\}_{i=\overline{1,m}}, \{\sigma^i_{x_0}(s)\}_{i=\overline{1,m}} \rangle$. This game corresponds to the *discounted stochastic positional game in mixed stationary strategies* that in extended form is determined by the tuple $(\{X_i\}_{i=\overline{1,m}},\ \{A(x)\}_{x\in X},\ \{r^i\}_{i=\overline{1,m}},\ p, \lambda)$. The functions $\sigma^1_{x_0}(s),\ \sigma^2_{x_0}(s), \ldots, \sigma^m_{x_0}(s)$ on $\mathbb{S} = \mathbb{S}^1 \times \mathbb{S}^2 \times \cdots \times \mathbb{S}^m$, determine the static game in normal form $\langle \{\mathbb{S}^i\}_{i=\overline{1,m}}, \{{}_{x_0}{}^i(s)\}_{i=\overline{1,m}} \rangle$ that corresponds to the *average stochastic positional game in pure strategies*. In the extended form this game also is determined by the tuple $(\{X_i\}_{i=\overline{1,m}},\ \{A(x)\}_{x\in X},\ \{r^i\}_{i=\overline{1,m}},\ p, \lambda)$.

2.3 Stochastic positional games with a random starting state:

For a stochastic positional games with average payoffs (or with discounted payoffs) we can consider the games in which the starting state is chosen randomly according to a given distribution $\{\theta_x\}$ on X. In the case of an average stochastic positional game with given distribution $\{\theta_x\}$ on X we can define the game with the payoff functions

$$\psi^i_\theta(s) = \sum_{x \in X} \theta_x \cdot \omega^i_x(s), \quad i = 1, 2, \ldots, m$$

on S and we obtain a static game in normal form $\langle \{S^i\}_{i=\overline{1,m}},\ \{\psi^i_\theta(s)\}_{i=\overline{1,m}} \rangle$ that in extended form is determined by $(\{X_i\}_{i=\overline{1,m}}, \{A(x)\}_{x\in X}, \{r^i\}_{i=\overline{1,m}}, p, \{\theta_x\}_{x\in X})$.

Similarly we can define the static game in normal form $\langle \{S^i\}_{i=\overline{1,m}}, \{\phi^i_\theta(s)\}_{i=\overline{1,m}} \rangle$ for a discounted stochastic positional in which the starting state is chosen randomly according to a given distribution $\{\theta_x\}$ on X and we obtain the game with payoffs

$$\phi^i_\theta(s^1, s^2, \ldots, s^m) = \sum_{x \in X} \theta_x \cdot \sigma^i_x(s^1, s^2, \ldots, s^m), \quad i = 1, 2, \ldots, m,$$

that is determined by $(\{X_i\}_{i=\overline{1,m}}, \{A(x)\}_{x\in X}, \{r^i\}_{i=\overline{1,m}}, p, \{\theta_x\}_{x\in X}, \lambda)$.

2.4 Stationary Nash equilibria for an average stochastic positional game

In this subsection we show that an arbitrary average stochastic positional game possesses a Nash equilibrium in mixed stationary strategies. To prove this we shall use the normal form of this game in stationary strategies $\langle \{S^i\}_{i=\overline{1,m}},\ \{\psi^i_\theta(s)\}_{i=\overline{1,m}} \rangle$, where each S^i, $i \in \{1, 2, \dots, m\}$ is the set of solutions of system (2) representing the set of stationary strategies of player i. Each S^i is a convex compact set and an arbitrary extreme point corresponds to a basic solution s^i of system (2), where $s^i_{x,a} \in \{0, 1\}$, $\forall x \in X_i$, $a \in A(x)$, i.e. each basic solution of this system corresponds to a pure stationary strategy $s^i \in \mathbb{S}^i$ of player i.

On the set $S = S^1 \times S^2 \times \cdots \times S^m$ we define m payoff functions

$$\psi^i_\theta(s^1, s^2, \dots, s^m) = \sum_{k=1}^{m} \sum_{x \in X_k} \sum_{a \in A(x)} s^k_{x,a} \cdot r^i_{x,a} \cdot q_x, \qquad i = 1, 2, \dots, m, \tag{6}$$

where q_x for $x \in X$ are determined uniquely from the following system of linear equations

$$\begin{cases} q_y - \sum\limits_{k=1}^{m} \sum\limits_{x \in X_k} \sum\limits_{a \in A(x)} s^k_{x,a} \cdot p^a_{x,y} \cdot q_x = 0, & \forall y \in X; \\ q_y + w_y - \sum\limits_{k=1}^{m} \sum\limits_{x \in X_k} \sum\limits_{a \in A(x)} s^k_{x,a} \cdot p^a_{x,y} \cdot w_x = \theta_y, & \forall y \in X \end{cases} \tag{7}$$

for a fixed profile $s = (s^1, s^2, \dots, s^m) \in S$. The functions $\psi^i_\theta(s^1, s^2, \dots, s^m)$, $i = 1, 2, \dots, m$, represent the payoff functions for the average stochastic game in normal form $\langle \{S^i\}_{i=\overline{1,m}}, \ \{\psi^i_\theta(s)\}_{i=\overline{1,m}} \rangle$. This game is determined by the tuple $(\{X_i\}_{i=\overline{1,m}}, \ \{A(x)\}_{x \in X}, \ \{r^i\}_{i=\overline{1,m}}, \ p, \ \{\theta_y\}_{y \in X})$ where θ_y for $y \in X$ are given nonnegative values such that $\sum_{y \in X} \theta_y = 1$.

If $\theta_y = 0$, $\forall y \in X \setminus \{x_0\}$ and $\theta_{x_0} = 1$, then we obtain an average stochastic game in normal form $\langle \{S^i\}_{i=\overline{1,m}}, \ \{\omega^i_{x_0}(s)\}_{i=\overline{1,m}} \rangle$ when the starting state x_0 is fixed, i.e. $\psi^i_\theta(s^1, s^2, \dots, s^m) = \omega^i_{x_0}(s^1, s^2, \dots, s^m)$, $i = 1, 2, \dots, m$. So, in this case the game is determined by $(X, \ \{A^i(x)\}_{i=\overline{1,m}}, \ \{r^i\}_{i=\overline{1,m}}, \ p, \ x_0)$.

If $\theta_y > 0$, $\forall y \in X$ and $\sum_{y \in X} \theta_y = 1$, then we obtain an average stochastic game when the play starts in the states $y \in X$ with probabilities θ_y. In this case for the payoffs of the players in the game in normal form we have

$$\psi^i_\theta(s^1, s^2, \dots, s^m) = \sum_{y \in X} \theta_y \cdot \omega^i_y(s^1, s^2, \dots, s^m), \quad i = 1, 2, \dots, m.$$

Let $\langle \{S^i\}_{i=\overline{1,m}}, \{\psi^i_\theta(s)\}_{i=\overline{1,m}} \rangle$ be the non-cooperative game in normal form that corresponds to the average stochastic positional game in stationary strategies determined by $(\{X_i\}_{i=\overline{1},m}, \ \{A(x)\}_{x \in X}, \ \{r^i\}_{i=\overline{1,m}}, p, \ \{\theta_y\}_{y \in X})$. So, $S^i, i = 1, 2, \dots, m$, and $\psi^i_\theta(s)$, $i = 1, 2, \dots, m$, are defined according to (2) and (6),(7).

In [6, 8] is proven the following theorem.

Theorem 1. *The game $\langle \{S^i\}_{i=\overline{1,m}}, \ \{\psi^i_\theta(s)\}_{i=\overline{1,m}} \rangle$ possesses a Nash equilibrium $\mathbf{s}^* = (s^{1*}, s^{2*}, \dots, s^{m*}) \in S$ which is a Nash equilibrium in mixed stationary strategies for the average stochastic positional game determined by $(\{X_i\}_{i=\overline{1},n}, \ \{A(x)\}_{x \in X}, \{r^i\}_{i=\overline{1,m}}, \ p, \ \{\theta_y\}_{y \in X})$. If $\theta_y > 0$, $\forall y \in X$, then $s^* = (s^{1*}, s^{2*}, \dots, s^{m*})$ is a Nash equilibrium in mixed stationary strategies for the average stochastic positional game $\langle \{S^i\}_{i=\overline{1,m}}, \{\omega^i_y(s)\}_{i=\overline{1,m}} \rangle$ with an arbitrary starting state $y \in X$.*

2.5 Pure stationary equilibria for a two-player zero-sum average positional game

For a two-player zero-sum stochastic positional game the existence of equilibria in pure stationary strategies can be derived from the following theorem.

Theorem 2. *Let a two-player zero-sum average stochastic positional game game that is determined by the tuple* $(X = X_1 \cup X_2, \{A(x)\}_{x\in X}, \{r_{x,a}\}_{x\in X}, p)$ *be given. Then the system of equations*

$$\begin{cases} \varepsilon_x + \omega_x = \max\limits_{a\in A(x)} \left\{ r_{x,a} + \sum\limits_{y\in X} p^a_{x,y}\varepsilon_y \right\}, & \forall x \in X_1; \\ \varepsilon_x + \omega_x = \min\limits_{a\in A(x)} \left\{ r_{x,a} + \sum\limits_{y\in X} p^a_{x,y}\varepsilon_y \right\}, & \forall x \in X_2; \end{cases} \tag{8}$$

has a solution under the set of solutions of the system of equations

$$\begin{cases} \omega_x = \max\limits_{a\in A(x)} \left\{ \sum\limits_{y\in X} p^a_{x,y}\omega_y \right\}, & \forall x \in X_1; \\ \omega_x = \min\limits_{a\in A(x)} \left\{ \sum\limits_{y\in X} p^a_{x,y}\omega_y \right\}, & \forall x \in X_2, \end{cases} \tag{9}$$

i.e. the system of equations (13) has such a solution ω^*_x, $x \in X$ *for which there exists a solution* ε^*_x, $x \in X$ *of the system of equations*

$$\begin{cases} \varepsilon_x + \omega^*_x = \max\limits_{a\in A(x)} \left\{ r_{x,a} + \sum\limits_{y\in X} p^a_{x,y}\varepsilon_y \right\}, & \forall x \in X_1; \\ \varepsilon_x + \omega^*_x = \min\limits_{a\in A(x)} \left\{ r_{x,a} + \sum\limits_{y\in X} p^a_{x,y}\varepsilon_y \right\}, & \forall x \in X_2. \end{cases}$$

The optimal pure stationary strategies s^{1^*}, s^{2^*} *of the players can be found by fixing arbitrary maps* $s^{1^*}(x) \in A(x)$ *for* $x \in X_1$ *and* $s^{2^*}(x) \in A(x)$ *for* $x \in X_2$ *such that*

$$s^{1^*}(x) \in \left\{ arg \max_{a\in A(x)} \left\{ \sum_{y\in X} p^a_{x,y}\omega^*_y \right\}\right\} \bigcap \left\{ arg \max_{a\in A(x)} \left\{ r_{x,a} + \sum_{y\in X} p^a_{x,y}\varepsilon^*_y \right\}\right\}, x \in X_1,$$

$$s^{2^*}(x) \in \left\{ arg \min_{a\in A(x)} \left\{ \sum_{y\in X} p^a_{x,y}\omega^*_y \right\}\right\} \bigcap \left\{ arg \min_{a\in A(x)} \left\{ r_{x,a} + \sum_{y\in X} p^a_{x,y}\varepsilon^*_y \right\}\right\}, x \in X_2,$$

and $\omega_x(s^{1^*}, s^{2^*}) = \omega^*_x$, $\forall x \in X$, *i.e.*

$$\omega_x(s^{1^*}, s^{2^*}) = \max_{s^1\in S^1} \min_{s^2\in S^2} \omega_x(s^1, s^2) = \min_{s^2\in S^2} \max_{s^1\in S^1} \omega_x(s^1, s^2), \qquad \forall x \in X.$$

The full proof of this theorem is presented in [6, 10]. Based on this theorem all optimal pure stationary strategies of the players can be found.

2.6 Pure stationary equilibria for a discounted stochastic positional game

The existence of stationary Nash equilibria for discounted stochastic games in general has been proven in [3]. Here we show that for an arbitrary stochastic positional games with discounted payoffs there exists a Nash equilibria in pure stationary strategies. This fact follows from the theorem bellow the full proof of which can be found in [9].

Theorem 3. *Let a discounted stochastic positional game that is determined by the tuple* $(\{X_i\}_{i=\overline{1,m}}, \{A(x)\}_{x\in X}, \{r^i\}_{i=\overline{1,m}}, p, \lambda)$ *be given. Then there exist the values* σ_x^i *for* $x \in X,\ i = 1, 2, \ldots, m$ *that satisfy the following conditions:*

1) $r^i_{x,a} + \lambda \sum\limits_{y\in X} p^a_{x,y}\sigma^i_y - \sigma^i_x \le 0, \quad \forall x \in X_i, \quad \forall a \in A(x),\ i = 1, 2, \ldots, m,$

2) $\max\limits_{a\in A(x)} \{r^i_{x,a} + \lambda \sum\limits_{y\in X} p^a_{x,y}\sigma^i_y - \sigma^i_x\} = 0, \quad \forall x \in X_i,\ i = 1, 2, \ldots, m;$

3) *on each position set* $X_i, i \in \{1, 2, \ldots, m\}$ *there exists a map* $s^{i*}: X_i \to \cup_{x\in X_i} A(x)$ *such that*

$$s^{i*}(x) = a^* \in arg \max_{a\in A(x)} \left\{r^i_{x,a} + \lambda \sum_{y\in X} p^a_{x,y}\sigma^i_y - \sigma^i_x\right\}$$

and

$$r^j_{x,a^*} + \lambda \sum_{y\in X} p^{a^*}_{x,y}\sigma^j_y - \sigma^j_x = 0, \quad \forall x \in X_i, \quad j = 1, 2, \ldots, m,$$

where $s^* = (s^{1*}, s^{2*}, \ldots, s^{m*})$ *represents a stationary Nash equilibrium in pyre strategies for the discounted stochastic positional game determined by the tuple* $(\{X_i\}_{i=\overline{1,m}},\ \{A(x)\}_{x\in X}, \{r^i\}_{i=\overline{1,m}},\ p,\ \lambda)$ *and such an equilibrium is a pure stationary Nash equilibrium for the game with an arbitrary starting position* $x \in X$.

Based on this theorem all optimal pure stationary strategies of the players in a discounted stochastic positional game can be found.

3 Stationary Nash equilibria in mixed stationary strategies for dynamic positional games on graphs

It is easy to observe that the considered dynamic positional games with average and and discounted payoffs on graph $G = (X, E)$ with given position sets $X_1, X_2, \ldots X_m$ and rewards functions $r^i : E \to R$ represent the particulary cases of the stochastic positional games with average and discounted payoffs for a Markov decision process from previous section. Indeed, for the dynamic positional game on graph G with average payoffs (or discounted payoffs) the set of outgoing directed edges $E(x) = \{e = (x,y) \in E | y \in X\}$ in a position $x \in X_i$ of player $i \in \{1, 2, \dots .m\}$ can be regarded as the set of actions $A(x)$ of the stochastic positional game for a Markov process with average payoffs (or discounted payoffs), where an action $a \in A(x)$ corresponds to a directed edge $e = (x,y) \in E(x)$, i.e. $a = (x,y)$ and this action in the game is chosen by the corresponding player with probability $p^a_{x,y} = 1$ and $r^i_{x,a} = r^i_{x,y}$; obviously, if in position $x \in X_i$ the action $a = (x,y) \in A(x) = E(x)$ is chosen with the probability $p^a_{x,y} = 1$, then for the rest of the actions $a = (x,z) \in A(x)$ the corresponding probabilities $p^a_{x,z} = 0$. So, the dynamic positional game on G represents the case of stochastic positional game in which the probability transitions $p^a_{x,y}$ take the values 1 or 0.

The pure and mixed stationary strategies in a positional game on G can be defined in a similar way as for a stochastic positional game, i.e. we identify the set of mixed stationary strategies S^i of player $i \in \{1, 2, \dots, m\}$ in a positional game on G with the set of solutions of the system

$$\begin{cases} \sum\limits_{y \in X(x)} s^i_{x,y} = 1, \quad \forall x \in X_i; \\ \qquad\qquad s^i_{x,y} \geq 0, \quad \forall x \in X_i, \ y \in X(x) \end{cases} \tag{10}$$

where $X(x)$ represents the set of neighbor vertices for the vertex x, i.e. $X(x) = \{y \in X | e = (x,y) \in E\}$.

So, based on this theorem all optimal pure stationary strategies of the players can be found.

3.1 Nash equilibria in mixed stationary strategies for average positional games on graphs

For a given average positional game on graph G we can consider a static game in normal form $\langle \{S^i\}_{i=\overline{1,m}}, \ \{\psi^i_\theta(s)\}_{i=\overline{1,m}} \ \rangle$ that is determined by the tuple $(G, \ \{X_i\}_{i=\overline{1,m}}, \ \{r^i_e\}_{e \in E(x), i=\overline{1,m}}, \{\theta_x\}_{x \in X})$, where

$$\psi^i_\theta(s^1, s^2, \dots, s^m) = \sum_{k=1}^{m} \sum_{x \in X_k} \sum_{y \in X(x)} s^k_{x,y} \cdot r^i_{x,y} \cdot q_x, \qquad i = 1, 2, \dots, m, \tag{11}$$

and q_x for $x \in X$ are determined uniquely from the following system of linear equations

$$\begin{cases} q_y - \sum\limits_{k=1}^{m} \sum\limits_{x \in X_k} s^k_{x,y} q_x = 0, \qquad\qquad \forall y \in X; \\ q_y + w_y - \sum\limits_{k=1}^{m} \sum\limits_{x \in X_k} s^k_{x,y} w_x = \theta_y, \quad \forall y \in X \end{cases} \tag{12}$$

for a fixed profile $s = (s^1, s^2, \dots, s^m) \in S$. The functions $\psi^i_\theta(s^1, s^2, \dots, s^m)$, $i = 1, 2, \dots, m$, represent the payoff functions for the stutic game in normal form

$\langle\{S^i\}_{i=\overline{1,m}},\ \{\psi^i_\theta(s)\}_{i=\overline{1,m}}\ \rangle$ o that corresponds to the average positional game in stationary strategies on graph G. Therefore from Theorem 4 as a corollary we obtain the following result.

Theorem 4. *The game in normal form $\langle\{S^i\}_{i=\overline{1,m}}, \{\psi^i_\theta(s)\}_{i=\overline{1,m}}\rangle$ for the average positional game on graph G has a Nash equilibrium which is a stationary Nash equilibrium in mixed stationary strategies of the average positional game on G with an arbitrary starting state $x \in X$.*

This means that an arbitrary average positional game on a graph possesses a Nash equilibrium in mixed stationary strategies.

3.2 Pure stationary equilibria for a two-player zero-sum average positional game on graphs

A two-player zero-sum average positional on graph $G = (X, E)$, represents the case of the game on G with $m = 2$ and $r^1_e = -r^2_e = r_e$, $\forall e \in E$; this game is determined by the tuple $(X = X_1 \cup X_2, \{r_e\}_{e\in E})$. The existence of equilibria in pure stationary strategies for this case of the game can be derived on the basis of the following theorem.

Theorem 5. *Let a two-player zero-sum average stochastic positional game that is determined by the tuple $(X = X_1 \cup X_2, \{r_{x,a}\}_{x\in X})$ be given. Then the system of equations*

$$\begin{cases} \varepsilon_x + \omega_x = \max\limits_{y\in X(x)} \Big\{ r_{x,y} + \varepsilon_y \Big\}, & \forall x \in X_1; \\ \varepsilon_x + \omega_x = \min\limits_{y\in X(x)} \Big\{ r_{x,y} + \varepsilon_y \Big\}, & \forall x \in X_2; \end{cases}$$

has a solution under the set of solutions of the system of equations

$$\begin{cases} \omega_x = \max\limits_{y\in X(x)} \Big\{ \omega_y \Big\}, & \forall x \in X_1; \\ \omega_x = \min\limits_{y\in X(x)} \Big\{ \omega_y \Big\}, & \forall x \in X_2, \end{cases} \tag{13}$$

i.e. the system of equations (13) has such a solution $\omega^*_x,\ x \in X$ for which there exists a solution $\varepsilon^*_x,\ x \in X$ of the system of equations

$$\begin{cases} \varepsilon_x + \omega^*_x = \max\limits_{y\in X(x)} \Big\{ r_{x,a} + \varepsilon_y \Big\}, & \forall x \in X_1; \\ \varepsilon_x + \omega^*_x = \min\limits_{y\in X(x)} \Big\{ r_{x,a} + \varepsilon_y \Big\}, & \forall x \in X_2. \end{cases}$$

The optimal pure stationary strategies s^{1^*}, s^{2^*} of the players can be found by fixing arbitrary maps $s^{1^*}(x) \in A(x)$ for $x \in X_1$ and $s^{2^*}(x) \in A(x)$ for $x \in X_2$ such that

$$s^{1^*}(x) \in \Big\{ arg \max_{y\in X(x)} \Big\{ \omega^*_y \Big\} \Big\} \bigcap \Big\{ arg \max_{y\in X(x)} \Big\{ r_{x,y} + \varepsilon^*_y \Big\} \Big\}, x \in X_1,$$

$$s^{2^*}(x) \in \left\{arg \min_{y \in X(x)} \left\{\omega^*_y\right\}\right\} \bigcap \left\{arg \min_{y \in X(x)} \left\{r_{x,y} + \varepsilon^*_y\right\}\right\}, x \in X_2,$$

and $\omega_x(s^{1^*}, s^{2^*}) = \omega^*_x, \ \forall x \in X, \ \ i.e.$

$$\omega_x(s^{1^*}, s^{2^*}) = \max_{s^1 \in S^1} \min_{s^2 \in S^2} \omega_x(s^1, s^2) = \min_{s^2 \in S^2} \max_{s^1 \in S^1} \omega_x(s^1, s^2), \qquad \forall x \in X.$$

Proof. The two-player zero-sum average positional game on graph G can be represented as a two-player zero-sum average stochastic positional game with transition probabilities $p^a_{x,y}$ in positions $x \in X$, where $a = (x, y) \in A(x) = E(x)$ and $p^a_{x,y}$ take only the value 1 and 0 and rewards $r_{x,a} = r_{x,y}$ for $(x, y) \in E$. Therefor if we take onto account these conditions in Theorem 2 we obtain Theorem 5. So, the two-player zero-sum average positional game on G has a pure stationary equilibrium and all pure stationary strategies of the players can be found on the basis of this theorem. □

3.3 Pure stationary equilibria for a discounted stochastic positional game on graphs

A discounted positional game on graph G can be represented as a discounted stochastic positional game in similar way as we proceeded in previous subsections for average positional games on G. Therefore from Theorem 3 as a corollary we obtain the following result.

Theorem 6. *Let an m-player discounted stochastic positional game on graph* $G = (X, E)$ *be given, where the elements of this game are determined by the tuple* $(G, \{X_i\}_{i=\overline{1,m}}, \ \{r^i\}_{i=\overline{1,m}}, \ \lambda)$. *Then there exist the values* σ^i_x *for* $x \in X$, $i = 1, 2, \ldots, m$ *that satisfy the following conditions:*

1) $r^i_{x,y} + \lambda\sigma^i_y - \sigma^i_x \le 0, \qquad \forall x \in X_i, \quad \forall y \in E(x), \quad i = 1, 2, \ldots, m,$

2) $\max\limits_{y \in E(x)} \{r^i_{x,y} + \lambda\sigma^i_y - \sigma^i_x\} = 0, \qquad \forall x \in X_i, \qquad i = 1, 2, \ldots, m;$

3) *on each position set* $X_i, i \in \{1, 2, \ldots, m\}$ *there exists a map* $s^{i^*} : X_i \to \cup_{x \in X_i} E(x)$ *such that*

$$s^{i^*}(x) = y^* \in arg \max_{y \in E(x)} \left\{r^i_{x,y} + \lambda\sigma^i_y - \sigma^i_x\right\}$$

and

$$r^j_{x,y^*} + \lambda\sigma^i_{y^*} - \sigma^j_x = 0, \quad \forall x \in X_i, \quad j = 1, 2, \ldots, m,$$

where $s^* = (s^{1^*}, \ s^{2^*}, \ldots, s^{m*})$ *represents a stationary Nash equilibrium in pure strategies for the discounted stochastic positional game on graph G determined by* $(\{X_i\}_{i=\overline{1,m}}, \{r^i\}_{i=\overline{1,m}}, p, \lambda)$ *and such an equilibrium is a pure stationary Nash equilibrium for the game with an arbitrary starting position* $x \in X$.

Based on this theorem all pure stationary strategies of the players can be found.

4 Conclusion

For an arbitrary m-player average dynamic positional game on a graph there exists a Nash equilibrium in mixed stationary strategies and for a two-player zero-sum average positional game on a graph there exists equilibrium in pure stationary strategies. For an m-player discounted positional game there exists a Nash equilibrium in pure stationary strategies.

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