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Completeness of the factor group of a complete topological group and completeness of the factor ring of a complete topological ring by a compact normal subgroup and a compact ideal, respectively.

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Abstract. If M is a subgroup of an Abelian group $G(+)$ and τ is a group topology on the group G such that (G, τ) is a complete topological group and M is a compact or it is locally compact subgroup of the topological group (G, τ), then the factor group $(\overline{G}, \overline{\tau}) = (G, \tau)/M$ is a complete topological group.

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1 Introduction

The question of preserving the completeness of topological groups and the completeness of topological rings under various constructions is one of the areas of research in topological algebra (see, for example, [2]). This article is devoted to the study of the question of preserving the completeness of topological groups and topological rings when taking factor groups and factor rings and is a continuation of the research that was presented in the articles [1] and [3].

The main results of the article are Theorems 3.5 and 3.6.

2 Notations and preliminaries

This section of the article presents some well-known definitions and results that are necessary for presentation of the main results.

Definition 2.1. As usual, a collection $\mathcal{F}$ of subsets of a set X is called a *filter* of the set X if the following conditions are satisfied:
1. $\mathcal{F} \neq \emptyset$ and $\emptyset \notin \mathcal{F}$;
2. If $A \in \mathcal{F}$ and if B is a subset of the set X such that $A \subseteq B$, then $B \in \mathcal{F}$;
3. If $A \in \mathcal{F}$ and $B \in \mathcal{F}$, then $A \bigcap B \in \mathcal{F}$.

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Definition 2.2. If $\mathcal{F}_1$ and $\mathcal{F}_2$ are filters of a set X and $\mathcal{F}_1 \subseteq \mathcal{F}_2$, then we say that $\mathcal{F}_1 \leq \mathcal{F}_2$.

Definition 2.3. A filter $\mathcal{F}$ of a set X is called an *ultrafilter* if it is maximal in the set of all filters of the set X.

Theorem 2.1. (see [4], p. 55, Corollary) *If $\mathcal{F}$ is an ultrafilter of a set X and if $\{A_1, A_2, \dots, A_n\}$ is a finite collection of subsets of the set X such that $\bigcup\limits_{i=1}^{n} A_i \in \mathcal{F}$, then $A_k \in \mathcal{F}$ for some natural number $1 \leq k \leq n$.*

Theorem 2.2. (see [4], p. 54, Theorem 2) *For any filter $\mathcal{F}_1$ of a set X there exists an ultrafilter $\widetilde{\mathcal{F}}$ of a set such that $\mathcal{F} \leq \widetilde{\mathcal{F}}$.*

Definition 2.4. A collection S of subsets of a set X is called *a basis of a filter* $\mathcal{F}$ of the set X if $S \subseteq \mathcal{F}$ and for any set $A \in \mathcal{F}$ there exists a set $B \in S$ such that $B \subseteq A$.

Theorem 2.3. (see [4], p. 50, Proposition 1). *A collection $S \neq \emptyset$ of subsets of a set X is a basis of some filter $\mathcal{F}$ of the set X if $\emptyset \notin S$ and for any sets $A \in S$ and $B \in S$ there exists a set $C \in S$ such that $C \subseteq A \bigcap B$.*

Definition 2.5. As usual, a partially ordered set $(\Gamma, \leq)$ is called *a directed set* if for any two elements a and b from Γ there exists an element $c \in \Gamma$ such that $a \leq c$ and $b \leq c$.

Definition 2.6. If $(\Gamma, \leq)$ is a directed set, then, as usual, a sequence of elements $\{g_\gamma | \gamma \in \Gamma\}$ of a topological Abelian group (G, τ) is called *a Cauchy sequence by the directed set* $(\Gamma, \leq)$ in the topological group (G, τ) if for any neighborhood V of zero of the topological group (G, τ) there exists an element $\gamma_0 \in \Gamma$ such that $g_\gamma - g_{\gamma_0} \in V$ for any element $\gamma \in \Gamma$ such that $\gamma_0 \leq \gamma$.

Definition 2.7. If $(\Gamma, \leq)$ is a directed set, then an element g of a topological Abelian group (G, τ) is called a *limit of a Cauchy sequence* $\{g_\gamma | \gamma \in \Gamma\}$ by this directed set if for any neighborhood V of zero in the topological group (G, τ) there exists an element $\gamma_0 \in \Gamma$ such that $g - g_\gamma \in V$ for any element $\gamma \in \Gamma$ such that $\gamma_0 \leq \gamma$.

In this case, we will write that $g = lim\{g_\gamma | \gamma \in \Gamma\}$.

Definition 2.8. A topological Abelian group (G, τ) is called *a complete group* if for any directed set $(\Gamma, \leq)$, any Cauchy sequence $\{g_\gamma | \gamma \in \Gamma\}$ has a limit by this directed set in the topological group (G, τ).

Definition 2.9. If (G, τ) is a topological Abelian group and $\mathcal{F}$ is some filter of the set G, then filter $\mathcal{F}$ is called *a Cauchy filter* in the topological group (G, τ) if for any neighborhood V of zero of the topological group (G, τ) there exists a set $A \in \mathcal{F}$ such that $A - A \subseteq V$.

Definition 2.10. Let (X, τ) be a topological space and let $\mathcal{F}$ be filter of the set X, then an element $a \in X$ is called *a limit of the filter* $\mathcal{F}$ in the topological space

(X, τ) if for any neighborhood V of the point a in the topological space (X, τ) there exists a set $F \in \mathcal{F}$ such that $F \subseteq V$.

Theorem 2.4. (see, for example, [2], Theorem 3.2.17). *A topological Abelian group (G, τ) is a complete group if and only if any Cauchy filter $\mathcal{F}$ of the topological group (G, τ) has a limit in topological group (G, τ).*

Definition 2.11. A topological group (G, τ) is called *separable* if the topological space (G, τ) is Hausdorff.

Theorem 2.5. (see [5], Theorem 16) *A locally compact, separable Abelian group has a basis of the filter of neighborhoods of zero that consists of subgroups.*

Definition 2.12. A subset S of a topological ring (R, τ) is called *bounded* if for any neighborhood V of zero of the topological ring (R, τ) there exists a neighborhood U of zero of the topological ring such that $U \cdot S \subseteq V$ and $S \cdot U \subseteq V$.

Definition 2.13. A topological ring (R, τ) is called *locally bounded* if it has a neighborhood of zero that is a bounded set.

3 Main Results.

This section presents results that address this issue, and some of them (Theorems 2.1, 2.2, 2.3, 2.4) were published earlier in other papers and are given in this article for the sake of completeness of the state of the issue.

Theorem 3.1. (see [2], Theorem 4.1.48). *For any separable topological Abelian group (G, τ), there exist a complete separable topological Abelian group $(\widehat{G}, \widehat{\tau})$ and a closed subgroup M in the topological group $(\widehat{G}, \widehat{\tau})$ such that topological groups (G, τ) and $(\widehat{G}, \widehat{\tau})/M$ are isomorphic.*

Remark 3.1. Unfortunately, there is an error in the proof of the analogue of Theorem 2.1 for topological rings in the monograph [2]. This error is partially corrected by the following

Theorem 3.2. (see [3]) *For any separable locally bounded topological ring (R, τ) there exist a complete separable topological ring $(\widehat{R}, \widehat{\tau})$ and a closed ideal M in the topological ring $(\widehat{R}, \widehat{\tau})$ such that the topological rings (R, τ) and $(\widehat{R}, \widehat{\tau})/M$ are isomorphic.*

Theorem 3.3. (see [2], Proposition 3.2.22). *If M is a subgroup of an Abelian group G and τ is a separable group topology on the group G such that the factor group $(G, \tau)/M$ and the subgroup $(M, \tau|_M)$ of the topological group (G, τ) are complete topological groups, then the topological group (G, τ) is also complete.*

Theorem 3.4. (see [2], Proposition 3.2.20) *If a separable topological Abelian group (G, τ) is a complete topological group and satisfies the first axiom of countability, i.e. it has a countable basis of the filter of neighborhoods of zero, then the factor group $(G, \tau)/M$ is complete for any closed subgroup M of the topological group (G, τ).*

Theorem 3.5. *If M is a subgroup of an Abelian group $G(+)$ and τ is a group topology on the group G such that (G,τ) is a complete topological group and M is a compact subgroup of the topological group (G,τ), then the factor group $(\overline{G},\bar{\tau}) = (G,\tau)/M$ is a complete topological group.*

Proof. Let $f : G \to \overline{G}$ be a canonical homomorphism and let $(\Gamma,\leq)$ be an arbitrary directed set.

If $\{\bar{g}_\gamma | \gamma \in \Gamma\}$ is any Cauchy sequence by the directed set $(\Gamma,\leq)$ in the topological group $(\overline{G},\bar{\tau})$, then for each $\gamma \in \Gamma$ we choose some element $g_\gamma \in f^{-1}(\bar{g}_\gamma)$ and for each $\lambda \in \Gamma$ we consider the set $S_\lambda = \{g_\gamma | \gamma \geq \lambda\}$. Then (see Theorem 2.3) the collection $\{S_\lambda | \lambda \in \Gamma\}$ is a basis of some filter $\mathcal{F}$ of the set G.

We fix some ultrafilter $\widehat{\mathcal{F}}$ of the set G such that $\widehat{F} \geq \mathcal{F}$ (for the existence of the ultrafilter $\widehat{\mathcal{F}}$ see Theorem 2.2).

We will carry out the further proof of the theorem in several Steps.

Step 1. We check that for any neighborhood V of zero of a topological group (G,τ) there exist elements $\gamma_V \in \Gamma$ and $m_V \in M$ such that $g_{\gamma_V} + V + m_V \in \widehat{\mathcal{F}}$ and $\bar{g}_\gamma \in \bar{g}_{\gamma_V} + f(V)$ for any $\gamma \in \Gamma$ and $\gamma \geq \gamma_V$.

Let V be an arbitrary neighborhood of zero in a topological group (G,τ), and let W be a neighborhood of zero in the topological group (G,τ) such that $W + W \subseteq V$. Without loss of generality, we can assume that W is an open set in a topological group (G,τ). Because M is a compact subgroup of the topological group (G,τ) and $M \subseteq \bigcup\limits_{m\in M} (m + W)$ then there exists a finite set $\{m_1, m_2, \dots, m_n\} \subseteq M$ such that $M \subseteq W + \{m_1, m_2, \dots, m_n\}$.

Since $\{\bar{g}_\gamma | \gamma \in \Gamma\}$ is a Cauchy sequence by the directed set $(\Gamma,\leq)$ in the topological group $(\bar{G},\bar{\tau})$, then there exists an element $\gamma_V \in \Gamma$ such that $\bar{g}_\gamma - \bar{g}_{\gamma_V} \in f(W)$ for any $\gamma \in \Gamma$ such that $\gamma \geq \gamma_V$. Then $f(g_\gamma - g_{\gamma_V}) = \bar{g}_\gamma - \bar{g}_{\gamma_V} \in f(V)$ for any $\gamma \in \Gamma$ such that $\gamma \geq \gamma_V$, and hence

$$g_\gamma - g_{\gamma_V} \in W + M \subseteq W + W + \{m_1, m_2, \dots, m_n\} \subseteq V + \{m_1, m_2, \dots, m_n\} \subseteq \bigcup_{i=1}^{n} (V + m_i)$$

for any $\gamma \in \Gamma$ such that $\gamma \geq \gamma_V$, i.e. $S_{\gamma_V} = \{g_\gamma | \gamma \geq \gamma_V\} \subseteq \bigcup\limits_{i=1}^{n} (V + m_i)$

Since (see above, definition of filter F) $S_{\gamma_V} \in \mathcal{F}$ and since $\mathcal{F}$ is a filter of the set G, then $\bigcup\limits_{i=1}^{n} (V + m_i) \in \mathcal{F} \subseteq \widehat{\mathcal{F}}$, and since $\widehat{\mathcal{F}}$ is an ultrafilter, then (see Theorem 2.1) there exists an element $m_V \in \{m_1, m_2, \dots, m_n\}$ such that $m_V + V \in \widehat{\mathcal{F}}$.

Step 2. Check that $\widehat{\mathcal{F}}$ is a Cauchy filter in the topological group (G,τ).

Indeed, if U is an arbitrary neighborhood of zero in the topological group (G,τ), and W is a neighborhood of zero in the topological group (G,τ) such that $W - W \subseteq U$, then $(g_{\gamma_W} + W + m_W) - (g_{\gamma_W} + W + m_W) = W - W \subseteq U$. From the arbitrariness of the neighborhood U of zero in the topological group (G,τ) and the fact that $g_{\gamma_W} + W \in \widehat{\mathcal{F}}$ it follows, that $\widehat{\mathcal{F}}$ is a Cauchy filter in the topological group (G,τ).

Since the topological group (G, τ) is a complete group, then $\bigcap_{A \in \widehat{\mathcal{F}}} [A]_{(G,\tau)} \neq \emptyset$, and let $g_0 \in \bigcap_{A \in \widehat{\mathcal{F}}} [A]_{(G,\tau)}$.

Step 3. Check that $\bar{g}_0 = f(g_0)$ is the limit of the Cauchy sequence $\{\bar{g}_\gamma \in \Gamma\}$ in the topological group $(\bar{G}, \bar{\tau})$ by the directed set $(\Gamma, \leq)$.

Indeed, if $\bar{V}$ is an arbitrary neighborhood of zero in the topological group $(\bar{G}, \bar{\tau})$, then $V = f^{-1}(\bar{V})$ is a neighborhood of zero in the topological group (G, τ), and hence there exists a neighborhood U of zero in the topological group (G, τ) such that $U - U - U \subseteq V$. Then (see step 1 above) $g_{\gamma_0} + U + m_U \in \widehat{\mathcal{F}}$.

Since $g_0 \in \bigcap_{A \in \widehat{\mathcal{F}}} [A]_{(G,\tau)}$, then $g_0 \in [g_U + U + m_U]_{(G,\tau)} \subseteq g_U + U + m_U + U$, and hence $g_{\gamma_U} + m_U \in g_0 - U - U$. Then (see above, definition of the element $\gamma_U \in \Gamma$)

$$\bar{g}_\gamma \in \bar{g}_{\gamma_U} + f(U) = f(g_{\gamma_U}) + f(U) = f(g_{\gamma_U} + m_U) \subseteq f(g_0 - U - U) + f(U) \subseteq$$

$$f(g_0 - U - U + U) \subseteq f(g_0 + V) = \bar{g}_0 + \bar{V}$$

for any $\gamma \geq \gamma_U$.

Since the neighborhood V of zero in the topological group (G, τ) is arbitrary, it follows that the element $\bar{g}_0$ is the limit of the Cauchy sequence $\{\bar{g}_\gamma \in \Gamma\}$ in the topological group $(\bar{G}, \bar{\tau})$ by the directed set $(\Gamma, \leq)$, and since the sequence $\{\bar{g}_\gamma \in \Gamma\}$ is arbitrary, then the topological group is complete.

This completely proves the theorem.

Theorem 3.6. *Let A and B be subgroups of an Abelian group G and let τ be a group topology on the group G such that the topological group (G, τ) is a complete topological group and A is a closed subgroup of the topological group (G, τ). If B is an open subgroup in the topological group $(A, \tau|_A)$ and $(B, \tau|_B)$ is a compact group, then the factor group $(G, \tau)/A$ is a complete group.*

Proof. Since $(B, \tau|_B)$ is a compact subgroup of the topological group (G, τ), then by Theorem 3.5 the topological group $(G, \tau)/B)$ is a complete group. Since B is an open subgroup of the topological group $(A, \tau|_A)$, then the factor group $(A, \tau|_A)/B$ is a discrete group, and hence, A/B is a discrete subgroup of the topological group $(G, \tau)/B$. Then, according to Theorem 3.2, the topological group $((G, \tau)/B)/(A/B)$ is a complete group, and hence, the topological group $(G, \tau)/A = ((G, \tau)/B)/(A/B)$ is a complete group.

This completes the proof of the theorem.

Corollary 3.1. *If a complete topological Abelian group (G, τ) contains a subgroup A such that the topological group $(A, \tau|_A)$ is a locally compact totally disconnected group, then the factor group $(G, \tau)/A$ is a complete topological group.*

Remark 3.2. Since the completeness of a topological ring is determined by the completeness of its additive group, then for topological rings it are true results to the results presented above for topological groups..

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