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On conharmonic curvature tensor of 6-dimensional planar Hermitian submanifolds of Cayley algebra

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Abstract. In this paper, we consider the conharmonic curvature tensor of 6-dimensional planar Hermitian submanifolds of the octave algebra. The Hermitian (and in general case, almost Hermitian) structure on a such submanifold is induced by the so-called Gray–Brown 3-fold vector cross products in Cayley algebra. The main result of the work is the calculation of the so-called spectrum of the conharmonic curvature tensor for an arbitrary 6-dimensional planar Hermitian submanifold of the octave algebra. By the concept of the spectrum of a tensor, we mean the minimal set of its components on the space of the associated G-structure that completely determines this tensor.

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1 INTRODUCTION

Conformal transformations of Riemannian structures are an important and meaningful object of differential-geometric research. Of significant interest is a special type of such transformations — conharmonic transformations, that is, conformal transformations that preserve the property of harmonicity of smooth functions. This type of transformations was introduced into consideration in the 50s of the last century by the Japanese mathematician Yoshihito Ishii [18]. It is known that such transformations have a tensor invariant — the so-called conharmonic curvature tensor. Remark that the extension of a Riemannian structure to an almost Hermitian structure allows us to single out several more conharmonic invariants.

Note that a significant contribution to the theory of conharmonic transformations, and in particular to the geometric theory of the conharmonic curvature tensor, was made by the outstanding specialist V.F. Kirichenko, as well as some of his students [21–23].

In this paper, we consider the conharmonic curvature tensor of 6-dimensional planar Hermitian submanifolds of the octave algebra. The Hermitian (and in general, almost Hermitian) structure on such submanifolds is induced by the so-called Gray–Brown 3-fold vector cross products in Cayley algebra [14, 15]. This note is a continuation of the authors' research in the field of geometry of planar Hermitian

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submanifolds of Cayley algebra, which was started more than 25 years ago (see [6–9], etc.).

2 PRELIMINARIES

Let us consider an almost Hermitian manifold, i.e. a $2n$-dimensional manifold M^{2n} with a Riemannian metric $g = \langle \cdot, \cdot \rangle$ and an almost complex structure J. Moreover, the following condition must hold

$$\langle JX,\ JY\rangle = \langle X,\ Y\rangle\,, \quad X, Y \in \aleph(M^{2n}),$$

where $\aleph(M^{2n})$ is the module of smooth vector fields on M^{2n} [17]. All considered manifolds, tensor fields and similar objects are assumed to be smooth of the class C^∞. We recall that the fundamental form (or Kählerian form) of an almost Hermitian manifold is determined by the relation

$$F(X,\ Y) = \langle X,\ JY\rangle\,, \quad X, Y \in \aleph(M^{2n}).$$

The specification of an almost Hermitian structure on a manifold is equivalent to the setting of a G-structure, where G is the unitary group $U(n)$ [4,20]. Its elements are the local frames adapted to the structure (local A-frames). They look as follows:

$$(p,\ \varepsilon_1, \ldots, \varepsilon_n,\ \varepsilon_{\hat{1}}, \ldots, \varepsilon_{\hat{n}});$$

where

$$\varepsilon_a = \frac{1}{2}\,(e_a - i\,J\,e_a)\,;\ \ \varepsilon_{\hat{a}} = \frac{1}{2}\,(e_a + i\,J\,e_a)\,.$$

Here the index a ranges from 1 to n, and we state $\hat{a} = a + n$.

Therefore, the matrices of the operator of the complexified almost complex structure J^C , the complexified Riemannian metric g^C and the complexified fundamental form F^C written in an A-frame look as follows, respectively [2,3]:

$$\left(J^{c}{}_{j}^{k}\right) = \left(\begin{array}{c|c} iI_n & 0 \\ \hline 0 & -iI_n \end{array}\right); (g^{c}{}_{kj}) = \left(\begin{array}{c|c} 0 & I_n \\ \hline I_n & 0 \end{array}\right); (F^{c}{}_{kj}) = \left(\begin{array}{c|c} 0 & iI_n \\ \hline -iI_n & 0 \end{array}\right),$$

where I_n is the identity matrix; $k, j = 1, \ldots, 2n$.

An almost Hermitian manifold is called Hermitian, if its almost complex structure is integrable. The following identity characterizes the Hermitian structure [17,20]:

$$\nabla_X(F)(Y, Z) - \nabla_{JX}(F)(JY, Z) = 0,$$

where $X, Y, Z \in \aleph(M^{2n})$ .The first group of the Cartan structural equations of a Hermitian manifold written in an A-frame looks as follows [2,20]:

$$d\omega^a = \omega^a_b \wedge \omega^b + B^{ab}{}_c\omega^c \wedge \omega_b,$$

$$d\omega_a = -\omega^b_a \wedge \omega_b + B_{ab}{}^c\omega_c \wedge \omega^b,$$

where $\left\{B^{ab}{}_{c}\right\}$ and $\{B_{ab}{}^{c}\}$ are the components of the Kirichenko tensors of M^{2n} [1,4]; $a, b, c = 1, ..., n$.

Let us also recall the explicit form of Gray–Brown 3-fold vector cross products in the octave algebra [14]:

$$P_1(X,\ Y,\ Z\,) = -X(\bar{Y}Z) + \langle X,\ Y\rangle\, Z + \langle Y,\ Z\rangle\, X - \langle Z, X\rangle\, Y;$$

$$P_2(X,\ Y,\ Z\,) = -(X\bar{Y})Z + \langle X,\ Y\rangle\, Z + \langle Y,\ Z\rangle\, X - \langle Z, X\rangle\, Y.$$

Here $\mathbf{O} \equiv \mathbf{R}^8$ is Cayley algebra; $X, Y, Z \in \mathbf{O}$; $\langle\cdot,\ \cdot\rangle$ is the scalar product in $\mathbf{O}$, $X \to \bar{X}$ is the conjugation operator in $\mathbf{O}$.

In the article [19], V.F. Kirichenko obtained structural equations of an arbitrary almost Hermitian structure induced by a 3-fold vector cross product in Cayley algebra on its 6-dimensional submanifold of general type. For the case of a Hermitian structure, these equations were refined [4, 10, 12] in the following form:

$$d\omega^a = \omega^a_b \wedge \omega^b + \frac{1}{\sqrt{2}}\varepsilon^{abh} D_{hc}\omega^c \wedge \omega_b;$$

$$d\omega_a = -\omega^b_a \wedge \omega_b + \frac{1}{\sqrt{2}}\varepsilon_{abh} D^{hc}\omega_c \wedge \omega^b;$$

$$d\omega^a_b = \omega^a_c \wedge \omega^c_b - \left(\frac{1}{2}\delta^{ah}_{bg} D_{hd} D^{gc} + \sum_{\phi} T^{\phi}_{\hat{a}\hat{c}} T^{\phi}_{bd}\right)\omega_c \wedge \omega^d,$$

where $\{\omega^k\}$ are the components of the displacement form and $\{\omega^k_j\}$ are the components of the Riemannian connection of the metric. Here and further $\varphi = 7,\ 8$; $a, b, c, d, g, h = 1, 2, 3$; $\hat{a} = a + 3$; $k, j = 1, 2, 3, 4, 5, 6$. As in [2] and [5], $\omega_a = \omega^{\hat{a}}$. Note that $\varepsilon_{abc} = \varepsilon^{123}_{abc}$, $\varepsilon^{abc} = \varepsilon^{abc}_{123}$ are the components of Kronecker tensor; $\delta^{ah}_{bg} = \delta^a_b\delta^h_g - \delta^a_g\delta^h_b$; $D^{hc} = D_{\hat{h}\hat{c}}$; $D_{cj} = \mp T^8_{cj} + iT^7_{cj}$; $D_{\hat{c}j} = \mp T^8_{\hat{c}j} - iT^7_{\hat{c}j}$, where $\left\{T^{\varphi}_{kj}\right\}$ are the components of the configuration tensor, or of the second fundamental form of the immersion of M^6 into $\mathbf{O}$. A 6-dimensional submanifold is called planar (or flattening) if it is contained in a hyperplane of Cayley algebra. Note that the concept of planar submanifold of the octave algebra was introduced into consideration by V.F. Kirichenko and M. Banaru [11]. It turned out that all 6-dimensional Kählerian submanifolds of the octave algebra (which were completely classified by V.F. Kirichenko [19]) are planar. It should be noted that there are known examples of 6-dimensional planar submanifolds of Cayley algebra with a Hermitian structure different from the Kählerian structure [2, 7, 9].

We also recall that the conharmonic curvature tensor on a Riemannian manifold of dimension m is defined by the equality [18]:

$$Ch(X, Y, Z, W) = R(X, Y, Z, W) -$$

$$-\frac{1}{m-2}\left[\langle X, W\rangle\, Ric\,(Y, Z) - \langle X, Z\rangle\, Ric\,(Y, W) + \langle Y, Z\rangle\, Ric\,(X, W) -\right.$$

$$- \langle Y, W \rangle \, Ric\,(X, Z)] \,.$$

Here R and Ric are the Riemannian curvature tensor and the Ricci tensor, respectively.

The conharmonic curvature tensor has all the classical properties of the Riemannian curvature tensor and the Weyl conformal curvature tensor [21, 23], namely:

$$Ch(X, Y, Z, W) = -Ch\,(X, Y, W, Z);$$

$$Ch(X, Y, Z, W) = -Ch\,(Y, X, W, Z);$$

$$Ch(X, Y, Z, W) + Ch(Y, Z, X, W) + Ch(Z, X, Y, W) = 0;$$

$$Ch(X, Y, Z, W) = Ch(Z, W, X, Y).$$

3 THE MAIN RESULTS

Let us calculate the components of the conharmonic curvature tensor on the space of the associated G-structure for a 6-dimensional planar Hermitian submanifolds of the octave algebra. In terms of covariant components, the formula that defines the conharmonic curvature tensor can be written as follows:

$$Ch_{ijkl} \;=\; R_{ijkl} \;-\; \frac{1}{4}\,(Ric_{jl}\, g_{ik} \;+\; Ric_{ik}\, g_{jl} \;-\; Ric_{jk}\, g_{il} \;-\; Ric_{il}\, g_{jk}).$$

As in the cases of the Riemannian curvature tensor and the Weyl tensor of conformal curvature [20], based on the classical properties of this tensor mentioned above, it is sufficient to find only the components Ch_{abcd}; $Ch_{\hat{a}bcd}$; $Ch_{\hat{a}\hat{b}cd}$; $Ch_{\hat{a}b\hat{c}d}$ that completely determine this tensor.

The components of the Riemannian curvature tensor of a 6-dimensional planar Hermitian submanifold of the octave algebra are known [13]:

$$R_{abcd} = 0, \; R_{\hat{a}bcd} = 0, \; R_{\hat{a}\hat{b}cd} = 0,$$

$$R_{\hat{a}b\hat{c}d} = -2\,|\mu|^2\, T^7_{\hat{a}\hat{c}} T^7_{bd},$$

where μ is a complex constant. The components of the Ricci tensor are also known:

$$Ric_{ab} = 0; \; Ric_{\hat{a}b} = -2\,|\mu|^2\, T^7_{\hat{a}\hat{c}} T^7_{cb}.$$

Taking into account the components of the complexified Riemannian metric [13]:

$$g_{ab} = 0, \quad g_{\hat{a}b} = \delta^a_b, \quad g_{a\hat{b}} = \delta^b_a, \quad g_{\hat{a}\hat{b}} = 0,$$

we get:

$$Ch_{abcd} = R_{abcd} - \frac{1}{4}(Ric_{bd} g_{ac} + Ric_{ac} g_{bd} - Ric_{bc} g_{ad} - Ric_{ad} g_{bc}) = 0;$$

$$Ch_{\hat{a}bcd} = R_{\hat{a}bcd} - \frac{1}{4}(Ric_{bd} g_{\hat{a}c} + Ric_{\hat{a}c} g_{bd} - Ric_{bc} g_{\hat{a}d} - Ric_{\hat{a}d} g_{bc}) = 0;$$

$$Ch_{\hat{a}\hat{b}cd} = R_{\hat{a}\hat{b}cd} - \frac{1}{4}(Ric_{\hat{b}d}g_{\hat{a}c} + Ric_{\hat{a}c}g_{\hat{b}d} - Ric_{\hat{b}c}g_{\hat{a}d} - Ric_{\hat{a}d}g_{\hat{b}c}) =$$
$$= -\frac{1}{2}|\mu|^2 \left(T^7_{\hat{a}\hat{h}}T^7_{hc}\delta^b_d + T^7_{\hat{b}\hat{h}}T^7_{hd}\delta^a_c - T^7_{\hat{a}\hat{h}}T^7_{hd}\delta^b_c - T^7_{\hat{b}\hat{h}}T^7_{hc}\delta^a_d\right);$$
$$Ch_{\hat{a}b\hat{c}d} = R_{\hat{a}b\hat{c}d} - \frac{1}{4}(Ric_{bd}g_{\hat{a}\hat{c}} + Ric_{\hat{a}\hat{c}}g_{bd} - Ric_{b\hat{c}}g_{\hat{a}d} - Ric_{\hat{a}d}g_{b\hat{c}}) =$$
$$= -2|\mu|^2 T^7_{\hat{a}\hat{c}}T^7_{bd} + \frac{1}{2}|\mu|^2 \left(T^7_{\hat{a}\hat{h}}T^7_{hd}\delta^c_b + T^7_{\hat{c}\hat{h}}T^7_{hb}\delta^a_d\right).$$

So the following result is proved.

Theorem 1. *The conharmonic curvature tensor of a 6-dimensional planar Hermitian submanifold of Cayley algebra is defined by the equalities:*

$$Ch_{abcd} = 0;\ Ch_{\hat{a}bcd} = 0;$$

$$Ch_{\hat{a}\hat{b}cd} = -\frac{1}{2}|\mu|^2 \left(T^7_{\hat{a}\hat{h}}T^7_{hc}\delta^b_d + T^7_{\hat{b}\hat{h}}T^7_{hd}\delta^a_c - T^7_{\hat{a}\hat{h}}T^7_{hd}\delta^b_c - T^7_{\hat{b}\hat{h}}T^7_{hc}\delta^a_d\right);$$
$$Ch_{\hat{a}b\hat{c}d} = -2\left|\mu^2\right| T^7_{\hat{a}\hat{c}}T^7_{bd} + \frac{1}{2}\left|\mu^2\right| \left(T^7_{\hat{a}\hat{h}}T^7_{hd}\delta^c_b + T^7_{\hat{c}\hat{h}}T^7_{hb}\delta^a_d\right).$$

We observe that when $\mu = \sqrt{-1}$ (i.e. $\left|\mu^2\right| = 1$) these formulae correspond to the conharmonic curvature tensor of a 6-dimensional Kählerian submanifold of the octave algebra which, as it was mentioned above, are an important particular case of planar Hermitian submanifolds.

A 6-dimensional planar Hermitian submanifold of Cayley algebra is conharmonically flat if and only if all the components of the configuration tensor vanish. That is why this submanifold is Riicci-flat and its scalar curvature also vanishes. It means that this submanifold is locally holomorphically isometric to a trivial 6-dimensional Kählerian flat manifold, namely, to the Euclidean space C^3. So, we can formulate our second result.

Theorem 2. *A 6-dimensional planar Hermitian submanifold of Cayley algebra is a conharmonically flat manifold if and only if it is locally holomorphically isometric to the space C^3 equipped with a standard Kählerian structure.*

4 A REMARK

It is clear that the calculated components of the conharmonic curvature tensor allow us to study the so-called conharmonic analogues of the Gray's identities from [16]. Such analogues were introduced into consideration by V.F. Kirichenko, A. Rustanov and A. Shihab in [21]. We note that the main part of the results obtained in the article [21], as well as in the works [22] and [23], relate to nearly Kählerian manifolds, mainly 4-dimensional. Another possible application of the obtained results is the further development of the theory of 6-dimensional Hermitian submanifolds of the octave algebra.

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