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Vertical Generalized Berger-type Metrics: A New Perspective on Harmonic Vector Fields

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Abstract. In this paper, we introduce a metric called the vertical generalized Berger-type deformed Sasaki metric, defined on the tangent bundle of an anti-paraKähler manifold. First, we analyze the harmonicity of vector fields with respect to this new metric, providing examples that illustrate how certain vector fields satisfy the harmonicity condition under the introduced metric. These examples demonstrate the unique properties and behavior of harmonic vector fields on anti-paraKähler manifolds equipped with this specific metric. Next, we explore the harmonicity of a vector field along a map between Riemannian manifolds, where the target manifold is anti-paraKähler and its tangent bundle is equipped with the vertical generalized Berger-type deformed Sasaki metric. Finally, we investigate the harmonicity of the composition of two specific maps. The first map is the projection from the tangent bundle of a Riemannian manifold (the source manifold) onto the manifold itself, and the second map is from this Riemannian manifold to another Riemannian manifold. The source manifold is an anti-paraKähler manifold, and its tangent bundle is endowed with the vertical generalized Berger-type deformed Sasaki metric. We discuss the conditions under which the composition of these maps produces harmonic vector fields.

Mathematics subject classification: 53C43, 58E20, 53C15, 53C07.
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1 Introduction

The study of tangent bundles has been greatly advanced by Sasaki [25], who introduced the Sasaki metric as a Riemannian metric on the tangent bundles of Riemannian manifolds. This pioneering work motivated further investigations into the geometric properties of the Sasaki metric (see [7, 21–23, 28]). Nevertheless, many studies were restricted by the flatness of the underlying Riemannian manifold. To overcome these limitations, researchers have explored various deformations of the Sasaki metric.

Abbassi and Sarih [1] introduced natural metrics on tangent and unit tangent bundles, which included cases such as the Sasaki metric, the Cheeger-Gromoll metric, and the Kaluza-Klein type metric (see [3, 12, 26]). Their work was influenced by earlier research in the field (see [5, 13, 29]). Yampolsky [27] proposed another deformation method for the Sasaki metric on slashed and unit tangent bundles over Kählerian manifolds, utilizing an almost complex structure J. This led to the

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development of the Berger-type deformed Sasaki metric, which has subsequently been studied in relation to its geodesic properties. In [2], Altunbas, Simsek, and Gezer defined the Berger type deformed Sasaki metric on the tangent bundle over an anti-paraKähler manifold. They calculated the Riemannian curvature tensors for this metric and presented several geometric results. Additionally, they introduced almost anti-paraHermitian structures on the tangent bundle and investigated the conditions for these structures to be anti-paraKähler and quasi-anti-paraKähler. Research into deformations of the Sasaki metric on tangent or cotangent bundles extends beyond these studies, with further references (see [14–16, 29, 30, 32]) indicating continued exploration in this area. This field remains a vibrant area of research within differential geometry.

This paper delves into the harmonicity properties associated with a vertical generalized Berger-type deformed Sasaki metric on the tangent bundle TM over an anti-paraKähler manifold. The core ideas and findings are summarized as follows:

- Harmonicity of vector fields with respect to this metric:

The paper presents several theorems (Theorem 3, Theorem 4, Theorem 5 and Theorem 6)) that explore the harmonicity of vector fields in relation to the vertical generalized Berger-type deformed Sasaki metric. Harmonicity here refers to conditions where vector fields meet specific criteria tied to this metric.

- Harmonicity of vector fields along a map between Riemannian manifolds:

The paper examines the harmonicity of vector fields along a map between Riemannian manifolds, with the target manifold being an anti-paraKähler manifold equipped with the vertical generalized Berger-type deformed Sasaki metric on its tangent bundle. Theorem 7 discusses the principal result in this context.

- Harmonicity of the composition of the projection map and a map between manifolds:

The final section addresses the harmonicity of the composition of the projection map from the tangent bundle of a Riemannian manifold to the manifold itself, and a map from this manifold to another Riemannian manifold. The source manifold is an anti-paraKähler manifold, and its tangent bundle is endowed with the vertical generalized Berger-type deformed Sasaki metric. Theorem 8 and Theorem 9 detail the key findings in this area.

In conclusion, this paper investigates the harmonicity properties related to the specified metric on tangent bundles and anti-paraKähler manifolds. The theorems presented contribute to a deeper understanding of these harmonicity properties in various scenarios.

2 Preliminaries

Let TM be the tangent bundle of an m-dimensional Riemannian manifold (M^m, g) with the natural projection $\pi : TM \to M$. A local chart $(U, x^i)_{i=\overline{1,m}}$ on M induces a local chart $(\pi^{-1}(U), x^i, u^i)_{i=\overline{1,m}}$ on TM. Let Γ_{ij}^k represent the Christoffel symbols associated with the metric g, and let ∇ denote the Levi-Civita connection of g. We define $C^\infty(M)$ as the ring of smooth real-valued functions on M, and $\Im_0^1(M)$ as the module of smooth vector fields on M over $C^\infty(M)$.

The Levi-Civita connection ∇ allows for a decomposition:

$$T_{(x,u)}TM = V_{(x,u)}TM \oplus H_{(x,u)}TM, \tag{1}$$

which separates the tangent space of TM at any $(x, u) \in TM$ into the vertical subspace:

$$V_{(x,u)}TM = \mathrm{Ker}(d\pi_{(x,u)}) = \left\{ \xi^i \frac{\partial}{\partial u^i}\Big|_{(x,u)} \mid \xi^i \in \mathbb{R} \right\} \tag{2}$$

and the horizontal subspace:

$$H_{(x,u)}TM = \left\{ \xi^i \frac{\partial}{\partial x^i}\Big|_{(x,u)} - \xi^i u^j \Gamma_{ij}^k \frac{\partial}{\partial u^k}\Big|_{(x,u)} \mid \xi^i \in \mathbb{R} \right\}. \tag{3}$$

The mapping $\xi \mapsto {}^H\xi = \xi^i \frac{\partial}{\partial x^i}\big|_{(x,u)} - \xi^i u^j \Gamma_{ij}^k \frac{\partial}{\partial u^k}\big|_{(x,u)}$ establishes an isomorphism between the vector spaces T_xM and $H_{(x,u)}TM$. In a similar vein, the mapping $\xi \mapsto {}^V\xi = \xi^i \frac{\partial}{\partial u^i}\big|_{(x,u)}$ serves as an isomorphism between T_xM and $V_{(x,u)}TM$. For any tangent vector $Z \in T_{(x,u)}TM$, it can be expressed as $Z = {}^HX + {}^VY$, where X and Y are unique vectors belonging to T_xM.

Let $X = X^i \frac{\partial}{\partial x^i}$ be a local vector field on M. The vertical and horizontal lifts of X are defined as:

$$\begin{cases} {}^VX = X^i \dfrac{\partial}{\partial u^i}, \\ {}^HX = X^i \left(\dfrac{\partial}{\partial x^i} - u^j \Gamma_{ij}^k \dfrac{\partial}{\partial u^k} \right). \end{cases} \tag{4}$$

We have ${}^H\left(\frac{\partial}{\partial x^i}\right) = \frac{\partial}{\partial x^i} - u^j \Gamma_{ij}^k \frac{\partial}{\partial u^k}$ and ${}^V\left(\frac{\partial}{\partial x^i}\right) = \frac{\partial}{\partial u^i}$, making $({}^H\left(\frac{\partial}{\partial x^i}\right), {}^V\left(\frac{\partial}{\partial x^i}\right))_{i=\overline{1,m}}$ a local adapted frame on TTM.

Specifically, for each point $(x, u) \in TM$, consider a local vector field U that remains constant across each fiber T_xM and satisfies $U_x = u^i \frac{\partial}{\partial x^i}$. The vertical lift of this field, denoted ${}^VU$, is referred to as the canonical vertical vector field or the Liouville vector field on TM.

The bracket operations for vertical and horizontal vector fields are described by the following equations (see [7, 28]):

$$\begin{cases} \left[{}^HX, {}^HY\right] = {}^H[X, Y] - {}^V(R(X, Y)u), \\ \left[{}^HX, {}^VY\right] = {}^V(\nabla_X Y), \\ \left[{}^VX, {}^VY\right] = 0, \end{cases} \tag{5}$$

for vector fields X and Y on M.

An almost paracomplex manifold is defined as an even-dimensional almost product manifold (M^{2m}, φ), where φ represents an almost paracomplex structure on M^{2m}. This structure satisfies the condition $\varphi^2 = \mathrm{id}$, with id being the identity map on the tangent bundle TM. The eigenbundles corresponding to the eigenvalues $+1$ and -1 of φ are denoted T^+M and T^-M, respectively, and they share the same rank.

A Riemannian metric g on the almost paracomplex manifold (M^{2m}, φ) is referred to as an anti-paraHermitian metric (or B-metric) if it fulfills the purity condition:

$$g(\varphi X, \varphi Y) = g(X, Y)$$

for all vector fields X and Y on M^{2m}. This condition indicates that the metric g remains invariant under the action of φ.

If the almost paracomplex manifold (M^{2m}, φ) is endowed with an anti-paraHermitian metric g that meets the purity condition, we refer to the triple (M^{2m}, φ, g) as an almost anti-paraHermitian manifold (or almost B-manifold). Furthermore, an almost anti-paraHermitian manifold (M^{2m}, φ, g) is classified as an anti-paraKähler manifold (B-manifold) if the almost paracomplex structure φ is parallel with respect to the Levi-Civita connection ∇ of the metric g, which is expressed as $\nabla\varphi = 0$. Essentially, this indicates that the paracomplex structure is covariantly constant.

Moreover, it is known that if (M^{2m}, φ, g) is an anti-paraKähler manifold (B-manifold), the Riemannian curvature tensor is pure. Purity of the curvature tensor implies that the Riemannian curvature tensor $R(Y, Z)$ lies in the space spanned by $\{Y, Z, \varphi Y, \varphi Z\}$ for any pair of vector fields Y and Z on M^{2m}, i.e.,

$$\begin{aligned} R(\varphi Y, Z) &= R(Y, \varphi Z) = R(Y, Z)\varphi = \varphi R(Y, Z), \\ R(\varphi Y, \varphi Z) &= R(Y, Z). \end{aligned}$$

Definition 1. Let (M^{2m}, φ, g) be an almost anti-paraHermitian manifold, and let TM denote its tangent bundle. A fiber-wise vertical generalized Berger-type deformation of the Sasaki metric on TM is defined as follows:

$$\begin{aligned} &\tilde{g}({}^H X, {}^H Y) = g(X, Y), \\ &\tilde{g}({}^V X, {}^H Y) = \tilde{g}({}^H X, {}^V Y) = 0, \\ &\tilde{g}({}^V X, {}^V Y) = g(X, Y) + f g(X, \varphi u) g(Y, \varphi u), \end{aligned}$$

for all vector fields X and Y on M^{2m}. Here, $f : M \to (0, +\infty)$ is a strictly positive smooth function defined on M^{2m}. When $f = \delta^2$, where δ is a constant, the metric $\tilde{g}$ represents the Berger-type deformed Sasaki metric [2].

For each point $(x, u) \in TM$, let U be a local vector field that remains constant across each fiber $T_x M$, with $U_x = u^i \frac{\partial}{\partial x^i}$. For any vector field X on M^{2m}, it follows that

$$\tilde{g}({}^V X, {}^V(\varphi U)) = (1 + f g(u, u)) g(X, \varphi u).$$

Here, we define $\lambda = 1 + f r^2$, where $r^2 = g(u, u) = |u|^2$, with $|.|$ representing the norm with respect to the metric g.

Lemma 1. *Let (M, g) be a Riemannian manifold and $\rho : \mathbb{R} \to \mathbb{R}$ a smooth function. Then:*

$$\begin{aligned}
&(1) \quad {}^{H}X(\rho(r^2)) = 0,\\
&(2) \quad {}^{V}X(\rho(r^2)) = 2(\rho')g(X, u),\\
&(3) \quad {}^{H}Xg(Y, u) = g(\nabla_X Y, u),\\
&(4) \quad {}^{V}Xg(Y, u) = g(X, Y),
\end{aligned}$$

for any vector fields X and Y on M, where $r^2 = g(u, u)$.

Lemma 2. *Let (M^{2m}, φ, g) be an anti-paraKähler manifold. Then:*

$$\begin{aligned}
&(1) \quad {}^{H}X(g(u, \varphi u)) = 0,\\
&(2) \quad {}^{V}X(g(u, \varphi u)) = 2g(X, \varphi u),\\
&(3) \quad {}^{H}X(g(Y, \varphi u)) = g(\nabla_X Y, \varphi u),\\
&(4) \quad {}^{V}X(g(Y, \varphi u)) = g(X, \varphi Y),\\
&(5) \quad {}^{H}(\varphi U)(g(Y, \varphi u)) = g(\nabla_{\varphi U} Y, \varphi u),\\
&(6) \quad {}^{V}(\varphi U)(g(Y, \varphi u)) = g(Y, U),
\end{aligned}$$

for any vector fields X and Y on M^{2m}.

Proof. Locally, from the formulas of horizontal and vertical lifts we obtain:

$$\begin{aligned}
(1) \quad {}^{H}X(g(u, \varphi u)) &= X^i \frac{\partial}{\partial x^i}(g(u, \varphi u)) - u^s \Gamma^i_{sk} X^k \frac{\partial}{\partial u^i}(g(u, \varphi u))\\
&= X^i \frac{\partial}{\partial x^i}(g_{lj} u^l \varphi^j_t u^t) - u^s \Gamma^i_{sk} X^k \frac{\partial}{\partial u^i}(g_{lj} u^l \varphi^j_t u^t)\\
&= Xg(U, \varphi U) - g_{ij} u^s \Gamma^i_{sk} X^k \varphi^j_t u^t - g_{lj} u^l \varphi^j_i u^s \Gamma^i_{sk} X^k\\
&= Xg(U, \varphi U) - g(\nabla_X U, \varphi U) - g(U, \varphi \nabla_X U)\\
&= 0.\\
(2) \quad {}^{V}X(g(u, \varphi u)) &= X^i \frac{\partial}{\partial u^i}(g(u, \varphi u))\\
&= X^i \frac{\partial}{\partial u^i}(g_{lj} u^l \varphi^j_t u^t)\\
&= X^i \left(g_{ij} \varphi^j_t u^t + g_{lj} u^l \varphi^j_i \right)\\
&= g(X, \varphi u) + g(u, \varphi X)\\
&= 2g(X, \varphi u).
\end{aligned}$$

The other formulas can be derived similarly. □

Lemma 3. *Let (M^{2m}, φ, g) be an anti-paraKähler manifold and $(TM, \tilde{g})$ its tangent* *bundle equipped with the generalized Berger-type deformed Sasaki metric. We have:*

(1) $^{H}X\left(\tilde{g}(^{H}Y, {}^{H}Z)\right) = X(g(Y,Z))$,

(2) $^{V}X\left(\tilde{g}(^{H}Y, {}^{H}Z)\right) = 0$,

(3) $^{H}X\left(\tilde{g}(^{V}Y, {}^{V}Z)\right) = \tilde{g}(^{V}(\nabla_X Y), {}^{V}Z) + \tilde{g}(^{V}Y, {}^{V}(\nabla_X Z)) + X(f)g(Y,\varphi u)g(Z,\varphi u)$,

(4) $^{V}X\left(\tilde{g}(^{V}Y, {}^{V}Z)\right) = f\left(g(X,\varphi Y)g(Z,\varphi u) + g(Y,\varphi u)g(X,\varphi Z)\right)$,

for all vector fields X, Y, Z on M^{2m}.

Proof. The results follow directly from the definition of the metric and previous lemmas. □

The Levi-Civita connection $\widetilde{\nabla}$ on TM, associated with the vertical generalized Berger-type deformed Sasaki metric $\tilde{g}$, is defined using the Koszul formula:

$$\begin{aligned} 2\tilde{g}(\widetilde{\nabla}_{\widetilde{X}}\widetilde{Y}, \widetilde{Z}) &= \widetilde{X}\left(\tilde{g}(\widetilde{Y}, \widetilde{Z})\right) + \widetilde{Y}\left(\tilde{g}(\widetilde{Z}, \widetilde{X})\right) - \widetilde{Z}\left(\tilde{g}(\widetilde{X}, \widetilde{Y})\right) \\ &\quad + \tilde{g}(\widetilde{Z}, [\widetilde{X}, \widetilde{Y}]) + \tilde{g}(\widetilde{Y}, [\widetilde{Z}, \widetilde{X}]) - \tilde{g}(\widetilde{X}, [\widetilde{Y}, \widetilde{Z}]), \end{aligned}$$

for all vector fields $\widetilde{X}, \widetilde{Y}, \widetilde{Z}$ on TM.

Theorem 1. *Let (M^{2m}, φ, g) be an anti-paraKähler manifold and $(TM, \tilde{g})$ its tangent bundle equipped with the generalized Berger-type deformed Sasaki metric. Then, we have:*

$$\begin{aligned} &(1) \quad \widetilde{\nabla}_{^{H}X}{}^{H}Y = {}^{H}(\nabla_X Y) - \frac{1}{2}{}^{V}(R(X,Y)u), \\ &(2) \quad \widetilde{\nabla}_{^{H}X}{}^{V}Y = \frac{1}{2}{}^{H}(R(u,Y)X) + {}^{V}(\nabla_X Y) + \frac{1}{2\lambda}X(f)g(Y,\varphi u){}^{V}(\varphi U), \\ &(3) \quad \widetilde{\nabla}_{^{V}X}{}^{H}Y = \frac{1}{2}{}^{H}(R(u,X)Y) + \frac{1}{2\lambda}Y(f)g(X,\varphi u){}^{V}(\varphi U), \\ &(4) \quad \widetilde{\nabla}_{^{V}X}{}^{V}Y = -\frac{1}{2}g(X,\varphi u)g(Y,\varphi u){}^{H}(grad\, f) + \frac{f}{\lambda}g(X,\varphi Y){}^{V}(\varphi U), \end{aligned}$$

for any vector fields X and Y on M^{2m}, where ∇ is the Levi-Civita connection of (M^{2m}, φ, g) and R is its Riemannian curvature tensor.

Proof. The proof is obtained by explicit calculation using the definition of the lifts, the Koszul formula, and the previously established lemmas. □

3 Vertical generalized Berger-type deformed Sasaki metric and harmonicity

A smooth map Φ from a Riemannian manifold (M^m, g) to another Riemannian manifold (N^n, h) is defined as harmonic if it serves as a critical point of the energy functional. This functional is expressed as:

$$E(\Phi, K) = \int_K e(\Phi)\, v^g, \tag{6}$$

for any compact domain $K \subseteq M^m$. Here, $e(\Phi)$, the energy density of Φ, is given by:

$$e(\Phi) := \frac{1}{2}\mathrm{Tr}_g\, h(d\Phi, d\Phi), \tag{7}$$

where v^g denotes the Riemannian volume form on M^m.

In more detail, $e(\Phi)$ represents the trace of the pullback of the metric h on N via the differential $d\Phi$, taken with respect to the metric g on M. Essentially, it measures how much Φ distorts the geometry of M when mapping it to N.

To determine whether a map Φ is harmonic, one considers any smooth one-parameter variation $\{\Phi_t\}_{t\in I}$ of Φ, where $\Phi_0 = \Phi$ and $V = \frac{d}{dt}\Phi_t\big|_{t=0}$ is the variation vector field. For such variations, the first variation of the energy functional can be written as:

$$\frac{d}{dt}E(\Phi_t)\bigg|_{t=0} = -\int_K h(\tau(\Phi), V)\, v^g.$$

Thus, the map Φ is harmonic if it satisfies the associated Euler-Lagrange equations, which are given by the formula:

$$0 = \tau(\Phi) := \mathrm{Tr}_g \nabla d\Phi, \tag{8}$$

where $\tau(\Phi)$ denotes the tension field of Φ. The tension field $\tau(\Phi)$ is a vector field on N that essentially represents the divergence of the differential $d\Phi$, providing a measure of how far Φ is from being a geodesic mapping.

Harmonic maps are significant objects in differential geometry because they generalize the notion of harmonic functions to mappings between manifolds. These maps minimize the energy functional, analogous to how harmonic functions minimize the Dirichlet energy in potential theory.

Research into harmonic maps has been extensive and multifaceted, encompassing theoretical advancements, applications in physics, and contributions to geometric analysis. For example, in physics, harmonic maps appear in the study of minimal surfaces, and in general relativity, they describe certain field configurations. In geometric analysis, they relate to the study of the heat flow of harmonic maps and stability questions. The references we provided, including [4, 6, 9–11, 17, 19, 30–32] contain more detailed information and further developments on this topic, and they can be consulted for in-depth understanding and specific applications. In conclusion, harmonic maps, defined by the critical points of the energy functional, form a fundamental concept in differential geometry, with wide-ranging applications and extensive literature dedicated to their study.

3.1 Harmonicity of a vector field $X : (M^{2m}, g, \varphi) \to (TM, \tilde{g})$

A vector field X on (M^{2m}, g, φ) can be viewed as an immersion:

$$X : (M^{2m}, g, \varphi) \to (TM, \tilde{g})$$

where $x \mapsto (x, X_x)$ maps each point x in M^{2m} to the point (x, X_x) in its tangent bundle TM. Here, TM is equipped with the vertical generalized Berger-type deformed Sasaki metric $\tilde{g}$.

Lemma 4. *[19, 20] Let (M^m, g) be a Riemannian manifold. For vector fields X and Y on M^m, and for any point $(x,u) \in TM$ where $Y_x = u$, we have the following relation:*

$$d_x Y(X_x) = {}^H X_{(x,u)} + {}^V(\nabla_X Y)_{(x,u)}. \tag{9}$$

Lemma 5. *Let (M^{2m}, φ, g) be an anti-paraKähler manifold, and let X be a vector field on M^{2m}. Then the following equation holds:*

$$Tr_g(g(\nabla_* X, \varphi \nabla_* X)) = g(\bar{\Delta} X, \varphi X) + \frac{1}{2}\Delta(g(X, \varphi X)), \tag{10}$$

where $\bar{\Delta}X$ denotes the rough Laplacian of X, defined as $-Tr_g \nabla^2 X = -Tr_g(\nabla_* \nabla_* - \nabla_{\nabla_* *})X$, and Δ represents the standard Laplace-Beltrami operator applied to functions.

Proof. Let $\{e_i\}_{i=\overline{1,2m}}$ be a local orthonormal frame on M^{2m}, then we have

$$\begin{aligned}
g(\bar{\Delta}X, \varphi X) &= -g(Tr_g(\nabla_* \nabla_* - \nabla_{\nabla_* *})X, \varphi X) \\
&= -\sum_{i=1}^{2m} \big(g(\nabla_{e_i}\nabla_{e_i} X, \varphi X) - g(\nabla_{\nabla_{e_i} e_i} X, \varphi X)\big) \\
&= -\sum_{i=1}^{2m} \big(e_i(g(\nabla_{e_i} X, \varphi X)) - g(\nabla_{e_i} X, \varphi \nabla_{e_i} X) - \frac{1}{2}\nabla_{e_i} e_i(g(X, \varphi X))\big) \\
&= -\sum_{i=1}^{2m} \big(\frac{1}{2} e_i e_i(g(X, \varphi X)) - g(\nabla_{e_i} X, \varphi \nabla_{e_i} X) - \frac{1}{2}\nabla_{e_i} e_i(g(X, \varphi X))\big) \\
&= Tr_g(g(\nabla_* X, \varphi \nabla_* X)) - \frac{1}{2}\Delta(g(X, \varphi X)).
\end{aligned}$$

□

Lemma 6. *Let (M^{2m}, φ, g) be an anti-paraKähler manifold and X a vector field on M. For a smooth function ρ on M^{2m}, the following relation holds:*

$$\bar{\Delta}(\rho X) = \rho \bar{\Delta} X - (\Delta \rho) X - 2\nabla_{grad\,\rho} X, \tag{11}$$

where $grad\,\rho$ denotes the gradient of the function ρ.

Proof. Let $\{e_i\}_{i=\overline{1,2m}}$ be a local orthonormal frame on M^{2m}, then we have

$$\begin{aligned}
\bar{\Delta}(\rho X) &= -\sum_{i=1}^{2m}\big(\nabla_{e_i}\nabla_{e_i}(\rho X) - \nabla_{\nabla_{e_i}e_i}(\rho X)\big)\\
&= -\sum_{i=1}^{2m}\big(\nabla_{e_i}(e_i(\rho)X + \rho\nabla_{e_i}X) - \nabla_{e_i}e_i(\rho)X - \rho\nabla_{\nabla_{e_i}e_i}X\big)\\
&= -\sum_{i=1}^{2m}\big(e_ie_i(\rho)X + e_i(\rho)\nabla_{e_i}X + e_i(\rho)\nabla_{e_i}X\\
&\quad +\rho\nabla_{e_i}\nabla_{e_i}X - \nabla_{e_i}e_i(\rho)X - \rho\nabla_{\nabla_{e_i}e_i}X\big)\\
&= -\sum_{i=1}^{2m}\big((e_ie_i(\rho) - \nabla_{e_i}e_i(\rho))X + 2\nabla_{e_i(\rho)e_i}X\\
&\quad +\rho(\nabla_{e_i}\nabla_{e_i} - \nabla_{\nabla_{e_i}e_i})X\big)\\
&= \rho\bar{\Delta}X - (\Delta\rho)X - 2\nabla_{grad\rho}X.
\end{aligned}$$

□

Lemma 7. *Let (M^{2m},φ,g) be an anti-paraKähler manifold and $(TM,\tilde{g})$ its tangent bundle equipped with a vertical generalized Berger-type deformed Sasaki metric. For a vector field X on M^{2m}, the energy density associated with X is given by:*

$$e(X) = m + \frac{1}{2}|\nabla X|^2 + \frac{f}{2}Tr_g\left(g(\nabla_* X,\varphi X)\right)^2, \tag{12}$$

where f is a constant and Tr_g denotes the trace with respect to the metric g.

Proof. Let $(x,u)\in TM$, X be a vector field on M^{2m}, $X_x = u$ and $\{e_i\}_{i=\overline{1,2m}}$ be a local orthonormal frame on M^{2m}, from (7), we have

$$\begin{aligned}
e(X)_x &= \frac{1}{2}Tr_g\,\tilde{g}(dX,dX)_{(x,u)}\\
&= \frac{1}{2}\sum_{i=1}^{2m}\tilde{g}(dX(e_i),dX(e_i))_{(x,u)}.
\end{aligned}$$

Using (9), we obtain

$$\begin{aligned}
e(X) &= \frac{1}{2}\sum_{i=1}^{2m}\tilde{g}({}^He_i + {}^V(\nabla_{e_i}X), {}^He_i + {}^V(\nabla_{e_i}X))\\
&= \frac{1}{2}\sum_{i=1}^{2m}\big(\tilde{g}({}^He_i,{}^He_i) + \tilde{g}({}^V(\nabla_{e_i}X),{}^V(\nabla_{e_i}X))\big)\\
&= \frac{1}{2}\sum_{i=1}^{2m}\big(g(e_i,e_i) + g(\nabla_{e_i}X,\nabla_{e_i}X) + fg(\nabla_{e_i}X,\varphi X)^2\big)
\end{aligned}$$

$$= \quad m + \frac{1}{2}|\nabla X|^2 + \frac{f}{2} Tr_g g(\nabla_* X, \varphi X)^2.$$

□

Theorem 2. *Let (M^{2m}, φ, g) be an anti-paraKähler manifold, and let $(TM, \tilde{g})$ denote its tangent bundle equipped with the vertical generalized Berger-type deformed Sasaki metric. For a vector field X on M^{2m}, the associated tension field is expressed as follows:*

$$\begin{aligned} \tau(X) &= {}^H\Big(Tr_g\big(R(X, \nabla_* X) * -\frac{1}{2} g(\nabla_* X, \varphi X)^2 grad f\big)\Big) \\ &\quad + {}^V\Big(\frac{1}{\lambda}\big(g(\nabla_{grad f} X, \varphi X) + f\, Tr_g g(\nabla_* X, \varphi \nabla_* X)\big)\varphi X - \bar{\Delta} X\Big), \end{aligned} \tag{13}$$

where $\lambda = 1 + f|X|^2$.

Proof. Let $(x, u) \in TM$ and let $\{e_i\}_{i=\overline{1,2m}}$ be a local orthonormal frame on M^{2m} such that $\nabla_{e_i} e_i = 0$ at the point x and $X_x = u$. Utilizing equations (8) and (9), we obtain the following results:

$$\begin{aligned} \tau(X)_x &= Tr_g \nabla dX \\ &= \sum_{i=1}^{2m} \big(\nabla^X_{e_i} dX(e_i)\big)_x \\ &= \sum_{i=1}^{2m} \big(\widetilde{\nabla}_{{}^H e_i} {}^H e_i + \widetilde{\nabla}_{{}^H e_i} {}^V(\nabla_{e_i} X) + \widetilde{\nabla}_{{}^V(\nabla_{e_i} X)} {}^H e_i + \widetilde{\nabla}_{{}^V(\nabla_{e_i} X)} {}^V(\nabla_{e_i} X)\big)_{(x,u)}. \end{aligned}$$

Using Theorem 1, we obtain

$$\begin{aligned} \tau(X) &= \sum_{i=1}^{2m} \Big({}^H(\nabla_{e_i} e_i) - \frac{1}{2} {}^V(R(e_i, e_i)X) + \frac{1}{2} {}^H(R(X, \nabla_{e_i} X)e_i) \\ &\quad + {}^V(\nabla_{e_i} \nabla_{e_i} X) + \frac{1}{2\lambda} e_i(f) g(\nabla_{e_i} X, \varphi X)\, {}^V(\varphi X) \\ &\quad + \frac{1}{2} {}^H(R(X, \nabla_{e_i} X)e_i) + \frac{1}{2\lambda} e_i(f) g(\nabla_{e_i} X, \varphi X)\, {}^V(\varphi X) \\ &\quad - \frac{1}{2} g(\nabla_{e_i} X, \varphi X)^2\, {}^H(grad f) + \frac{f}{\lambda} g(\nabla_{e_i} X, \varphi \nabla_{e_i} X)\, {}^V(\varphi X)\Big) \\ &= \sum_{i=1}^{2m} \Big({}^H(R(X, \nabla_{e_i} X)e_i) - \frac{1}{2} g(\nabla_{e_i} X, \varphi X)^2\, {}^H(grad f) + {}^V(\nabla_{e_i} \nabla_{e_i} X) \\ &\quad + \frac{1}{\lambda} e_i(f) g(\nabla_{e_i} X, \varphi X)\, {}^V(\varphi X) + \frac{f}{\lambda} g(\nabla_{e_i} X, \varphi \nabla_{e_i} X)\, {}^V(\varphi X)\Big) \\ &= {}^H\Big(Tr_g\big(R(X, \nabla_* X) * -\frac{1}{2} g(\nabla_* X, \varphi X)^2 grad f\big)\Big) \\ &\quad + {}^V\Big(\frac{1}{\lambda} g(\nabla_{grad f} X, \varphi X)\varphi X + Tr_g\big(\nabla^2 X \end{aligned}$$

$$
\begin{aligned}
& +\frac{f}{\lambda} g(\nabla_* X, \varphi \nabla_* X)\varphi X)\Big) \\
= \;& {}^H\Big(Tr_g\big(R(X, \nabla_* X) * -\frac{1}{2} g(\nabla_* X, \varphi X)^2 grad f\big)\Big) \\
& +{}^V\Big(\frac{1}{\lambda}\big(g(\nabla_{grad f} X, \varphi X) + f\, Tr_g g(\nabla_* X, \varphi \nabla_* X)\big)\varphi X - \bar{\Delta} X\Big).
\end{aligned}
$$

□

Theorem 3. *Let (M^{2m}, φ, g) be an anti-paraKähler manifold, and let $(TM, \tilde{g})$ be its tangent bundle equipped with the vertical generalized Berger-type deformed Sasaki metric. A vector field X on M^{2m} is considered a harmonic map if and only if the following conditions are satisfied:*

$$
Tr_g\big(R(X, \nabla_* X) * -\frac{1}{2} g(\nabla_* X, \varphi X)^2 grad f\big) = 0, \tag{14}
$$

and

$$
\bar{\Delta} X = \frac{1}{\lambda}\big(g(\nabla_{grad f} X, \varphi X) + f\, Tr_g g(\nabla_* X, \varphi \nabla_* X)\big)\varphi X. \tag{15}
$$

Proof. The proof is a direct consequence of Theorem 2. □

Let (M^{2m}, φ, g) be a compact oriented anti-paraKähler manifold, and let $(TM, \tilde{g})$ denote its tangent bundle equipped with the vertical generalized Berger-type deformed Sasaki metric. For a vector field X on M, the energy $E(X)$ is defined as the energy of the corresponding map $X : (M^{2m}, \varphi, g) \to (TM, \tilde{g})$. More specifically, from equation (12), we obtain:

$$
\begin{aligned}
E(X) &= \int_M e(X) v^g \qquad (16) \\
&= \int_M \big(m + \frac{1}{2}|\nabla X|^2 + \frac{f}{2} Tr_g g(\nabla_* X, \varphi X)^2\big) v^g \\
&= m\, Vol(M) + \frac{1}{2}\int_M |\nabla X|^2 v^g + \frac{1}{2}\int_M f\, Tr_g g(\nabla_* X, \varphi X)^2 v^g.
\end{aligned}
$$

Definition 2. Let (M^{2m}, φ, g) be an anti-paraKähler manifold, and let $(TM, \tilde{g})$ represent its tangent bundle equipped with the vertical generalized Berger-type deformed Sasaki metric. A vector field X is termed a harmonic vector field if the corresponding map $X : (M^{2m}, \varphi, g) \to (TM, \tilde{g})$ serves as a critical point for the energy functional E, considering only variations among maps defined by vector fields.

In the following theorem, we analyze the first variation of the energy functional constrained to the space $\Im_0^1(M^{2m})$.

Theorem 4. *Let (M^{2m}, φ, g) be a compact oriented anti-paraKähler manifold, and let $(TM, \tilde{g})$ denote its tangent bundle equipped with the vertical generalized Berger-type deformed Sasaki metric. Let X be a vector field on M^{2m}, and define the energy* *functional* $E : \Im_0^1(M^{2m}) \to [0, +\infty)$ *as the energy restricted to the space of all vector fields. Then,*

$$\begin{aligned} \frac{d}{dt} E(X_t)\big|_{t=0} &= \int_M g\big(\bar{\Delta} X - \frac{1}{\lambda} g(\nabla_{grad f} X, \varphi X)\varphi X \\ &\quad - \frac{f}{\lambda} Tr_g g(\nabla_* X, \varphi \nabla_* X)\varphi X, V\big) v^g \end{aligned} \tag{17}$$

for any smooth 1*-parameter variation* $\Phi : M^{2m} \times (-\epsilon, \epsilon) \to TM$ *of* X *through vector field, i.e.,* $\Phi(x,t) = X_t(x) \in TM$ *for any* $x \in M^{2m}$ *and any* $|t| < \epsilon$, $(\epsilon > 0)$, *or equivalently* $X_t \in \Im_0^1(M^{2m})$ *for any* $|t| < \epsilon$. *Also,* V *is the vector field on* M^{2m} *given by*

$$V(x) = \lim_{t \to 0} \frac{1}{t}(X_t(x) - X(x)) = \frac{d}{dt}\Phi_x(0), \quad x \in M,$$

where $\Phi_x(t) = X_t(x), (x,t) \in M^{2m} \times (-\epsilon, \epsilon)$.

Proof. We consider the smooth 1-parameter variation $\Phi : M^{2m} \times (-\epsilon, \epsilon) \to TM$ of X, i.e., $\Phi(x,t) = X_t(x) \in T_x M$ for any $(x,t) \in M^{2m} \times (-\epsilon, \epsilon)$ and $\Phi(x, 0) = X_0(x) = X(x)$. From (6), we have

$$E(X_t) = \int_M e(X_t) v^g.$$

Then, as well known the theory of harmonic maps [18]

$$\frac{d}{dt} E(X_t)\Big|_{t=0} = -\int_M \tilde{g}(\mathcal{V}, \tau(X)) v^g, \tag{18}$$

where $\mathcal{V}$ is the infinitesimal variation induced by Φ, i.e.,

$$\mathcal{V}(x) = d_{(x,0)}\Phi(0, \frac{d}{dt})\big|_{t=0} = d\Phi_x(\frac{d}{dt})\big|_{t=0} = \frac{d}{dt} X_t(x)\big|_{t=0} \in T_{X(x)} T^* M.$$

It is well known that

$$\mathcal{V} = {}^V V \circ X, \tag{19}$$

which was proven in [8, p.58]. Finally, by taking into account (13), (18) and (19), we find

$$\begin{aligned} \frac{d}{dt} E(X_t)\Big|_{t=0} &= -\int_M \tilde{g}({}^V V, \tau(X)) v^g \\ &= \int_M g\big(V, \bar{\Delta} X - \frac{1}{\lambda} g(\nabla_{grad f} X, \varphi X)\varphi X \\ &\quad - \frac{f}{\lambda} Tr_g g(\nabla_* X, \varphi \nabla_* X)\varphi X\big) v^g. \end{aligned}$$

□

Remark 1. Theorem 4 is applicable when (M^{2m}, φ, g) is a non-compact oriented anti-paraKähler manifold. Specifically, if M^{2m} is non-compact, we can choose an open subset D within M^{2m} such that its closure is compact, and select an arbitrary vector field V whose support lies entirely within D. In this context, Theorem 4 can be expressed as follows:

$$\begin{aligned} \frac{d}{dt}E(X_t)\big|_{t=0} &= \int_D g\big(V, \bar{\Delta}X - \frac{1}{\lambda}g(\nabla_{gradf}X, \varphi X)\varphi X \\ &\quad -\frac{f}{\lambda}Tr_g g(\nabla_* X, \varphi\nabla_* X)\varphi X\big)v^g. \end{aligned}$$

Corollary 1. *Let (M^{2m}, φ, g) be an anti-paraKähler manifold, and let $(TM, \tilde{g})$ represent its tangent bundle equipped with the vertical generalized Berger-type deformed Sasaki metric. A vector field X is considered a harmonic vector field if and only if*

$$\bar{\Delta}X = \frac{1}{\lambda}\big(g(\nabla_{gradf}X, \varphi X) + f\,Tr_g g(\nabla_* X, \varphi\nabla_* X)\big)\varphi X.$$

From Theorem 3 and Corollary 1, we get the following result.

Corollary 2. *Let (M^{2m}, φ, g) be an anti-paraKähler manifold, and let $(TM, \tilde{g})$ denote its tangent bundle equipped with a vertical generalized Berger-type deformed Sasaki metric. A vector field X qualifies as a harmonic map if and only if it is a harmonic vector field and*

$$Tr_g\big(R(X, \nabla_* X) * -\frac{1}{2}g(\nabla_* X, \varphi X)^2 gradf\big) = 0.$$

It is important to observe that if X is parallel, then X is indeed a harmonic vector field. Conversely, the following theorem holds:

Theorem 5. *Let (M^{2m}, φ, g) be a compact oriented anti-paraKähler manifold, and let $(TM, \tilde{g})$ represent its tangent bundle equipped with the vertical generalized Berger-type deformed Sasaki metric. For a vector field X on M^{2m}, X is a harmonic vector field if and only if it is parallel.*

Proof. We assume that the vector field X is a harmonic vector field, meaning it is a critical point of the energy functional E restricted to the space of all vector fields on (M^{2m}, φ, g). We consider the smooth one-parameter variation $X_t = (1+t)X$ for any $t \in (-\epsilon, \epsilon)$, where $\epsilon > 0$. From equation (16), we obtain:

$$E(X_t) = m\,Vol(M) + \frac{(1+t)^2}{2}\int_M |\nabla X|^2 v^g + \frac{(1+t)^4}{2}\int_M f\,Tr_g g(\nabla_* X, \varphi X)^2)v^g.$$

We get

$$\begin{aligned} 0 &= \frac{d}{dt}E(X_t)|_{t=0} \\ &= \frac{d}{dt}\Big(m\,Vol(M) + \frac{(1+t)^2}{2}\int_M |\nabla X|^2 v^g \end{aligned}$$

$$
\begin{aligned}
&+\frac{(1+t)^4}{2}\int_M f\,Tr_g g(\nabla_* X, \varphi X)^2\big)v^g\Big)_{t=0} \\
&= \int_M |\nabla X|^2 v^g + 2\int_M f\,Tr_g g(\nabla_* X, \varphi X)^2 v^g \\
&= \int_M \big(|\nabla X|^2 + 2f\,Tr_g g(\nabla_* X, \varphi X)^2\big)v^g,
\end{aligned}
$$

which gives

$$|\nabla X|^2 + 2f\,Tr_g g(\nabla_* X, \varphi X)^2 = 0,$$

hence $\nabla X = 0$. □

Theorem 6. *Let (M^{2m}, φ, g) be a compact oriented anti-paraKähler manifold, and let $(TM, \tilde{g})$ denote its tangent bundle equipped with the vertical generalized Berger-type deformed Sasaki metric. A vector field X on M^{2m} defines a harmonic map $X : (M^{2m}, g) \to (TM, \tilde{g})$ if and only if X is parallel.*

Proof. Assuming that $X : (M^{2m}, g) \to (TM, \tilde{g})$ is a harmonic map, we can invoke Corollary 2 to conclude that X is a harmonic vector field, which implies that X is parallel. Conversely, if we assume that the vector field X is parallel, then by Theorem 3, X qualifies as a harmonic map. □

Example 1. Let $(\mathbb{R}^* \times \mathbb{R}^*, g, \varphi)$ be an anti-paraKähler manifold such that

$$g = x^2 dx^2 + y^2 dy^2,$$

and

$$\varphi\partial_x = \frac{x}{y}\partial_y, \quad \varphi\partial_y = \frac{y}{x}\partial_x.$$

Relatively to the orthonormal frame

$$e_1 = \frac{1}{x}\partial_x, \quad e_2 = \frac{1}{y}\partial_y.$$

We have

$$\varphi e_1 = e_2, \quad \varphi e_2 = e_1$$

and

$$\nabla_{e_i} e_j = 0, \quad \textit{for all } i, j = 1, 2.$$

We examine the vector field $X = \rho(x)e_1$, where ρ is a smooth real function that depends on the variable x. By performing direct calculations, we determine that:

$$
\begin{aligned}
\bar{\Delta} e_1 &= 0, \\
\nabla_{grad\rho} e_1 &= 0, \\
\Delta\rho &= -\frac{1}{x^3}\rho' + \frac{1}{x^2}\rho''.
\end{aligned}
\tag{20}
$$

Combining relations (10), (11) and (20), we obtain

$$\begin{array}{rcl}\bar{\Delta}X & = & (\frac{1}{x^3}\rho' - \frac{1}{x^2}\rho'')e_1,\\ g(\nabla_{gradf}X, \varphi X) & = & \frac{1}{2}(gradf)(g(X, \varphi X)) = 0,\\ Tr_g g(\nabla_* X, \varphi \nabla_* X) & = & 0.\end{array}$$

i) From Corollary 1, we conclude that the vector field $X = \rho(x)e_1$ is a harmonic vector field if and only if $\bar{\Delta}X = 0$. This is equivalent to stating that the function ρ must satisfy the following homogeneous second-order differential equation:

$$\frac{1}{x}\rho' - \rho'' = 0. \tag{21}$$

The general solution of differential equation (21) is

$$\rho(x) = ax^2 + b,$$

where a and b are real constants.
Given that $\mathrm{Tr}_g\left(R(X, \nabla_* X) * -\frac{1}{2}g(\nabla_* X, \varphi X)^2 \nabla f\right) = 0$, we can infer from Corollary 2 that the vector field $X = (ax^2 + b)e_1$ is also a harmonic map.
On the other hand $\nabla_{e_1}X = 2ae_1 \neq 0$, then the vector field $X = (ax^2 + b)e_1$ is harmonic but non parallel.

Example 2. Let $\mathbb{R}^2$ be equipped with the anti-paraKähler structure (φ, g) in polar coordinates defined by:

$$g = dr^2 + r^2 d\theta^2,$$

$$\varphi e_1 = \sin 2\theta e_1 + \cos 2\theta e_2, \quad \varphi e_2 = \cos 2\theta e_1 - \sin 2\theta e_2,$$

where $\{e_1, e_2\}$ is an orthonormal frame on $\mathbb{R}^2$ with respect to g, we have

$$\nabla_{e_1}e_1 = \nabla_{e_1}e_2 = 0,\ \nabla_{e_2}e_1 = \frac{1}{r}e_2,\ \nabla_{e_2}e_2 = -\frac{1}{r}e_1,$$

The vector field $X = \sin\theta e_1 + \cos\theta e_2$ is parallel, then it is harmonic.

Proposition 1. *Let (M^{2m}, φ, g) be an anti-paraKähler manifold, and let $(TM, \tilde{g})$ denote its tangent bundle equipped with the vertical generalized Berger-type deformed Sasaki metric. A vector field X on M^{2m} is an isometric immersion if and only if it is parallel.*

Proof. Let Y and Z be vector fields on M^{2m}. From Lemma 5, we have

$$\begin{array}{rcl}\tilde{g}(dX(Y), dX(Z)) & = & \tilde{g}(Y^H + (\nabla_Y X)^V, Z^H + (\nabla_Z X)^V)\\ & = & \tilde{g}(Y^H, Z^H) + \tilde{g}((\nabla_Y X)^V, (\nabla_Z X)^V)\\ & = & g(Y, Z) + g(\nabla_Y X, \nabla_Z X) + fg(\nabla_Y X, \varphi u)g(\nabla_Z X, \varphi u)\end{array}$$

Hence, X is an isometric immersion if and only if

$$\tilde{g}(dX(Y), dX(Z)) = g(Y, Z)$$

for any Y and Z two vector fields on M^{2m}. Therefore, X is an isometric immersion if and only if $\nabla X = 0$. □

As a direct result of Proposition 1, we derive the following corollaries.

Corollary 3. *Let (M^{2m}, φ, g) be an anti-paraKähler manifold, and let $(TM, \tilde{g})$ represent its tangent bundle equipped with the vertical generalized Berger-type deformed Sasaki metric. In this context, any isometric vector field on M^{2m} is considered to be harmonic.*

Corollary 4. *Let (M^{2m}, φ, g) be a compact anti-paraKähler manifold, and let $(TM, \tilde{g})$ denote its tangent bundle equipped with the vertical generalized Berger-type deformed Sasaki metric. In this setting, every harmonic vector field on M^{2m} is isometric.*

3.2 Harmonicity of vector fields along smooth maps

Lemma 8. *[31] Let $\Phi : (M^m, g) \to (N^n, h)$ be a smooth map between Riemannian manifolds and Y be a vector field on N. Let σ be a smooth map defined by $\sigma := Y \circ \Phi$. Then*

$$d\sigma(X) = {}^H(d\Phi(X)) + {}^V(\nabla^\Phi_X \sigma), \tag{22}$$

for any vector field X on M^m.

Proposition 2. *Let (M^m, g) be a Riemannian manifold, and let (N^{2n}, h, φ) be an anti-paraKähler manifold. Consider a strictly positive smooth function f defined on N^{2n}, and let $(TN, \widetilde{H})$ represent its tangent bundle equipped with the vertical generalized Berger-type deformed Sasaki metric. If $\Phi : M^m \to N^{2n}$ is a smooth map and Y is a vector field on N^{2n}, then the tension field of the composition $\sigma := Y \circ \Phi$ is expressed as follows:*

$$\begin{aligned}
\tau(\sigma) \;=\; & {}^H\Big(\tau(\Phi) + Tr_g\big(R^N(\sigma, \nabla^\Phi_* \sigma)d\Phi(*) - \frac{1}{2}h(\nabla^\Phi_* \sigma, \varphi\sigma)^2 grad f\big)\Big) \\
& + {}^V\Big(\frac{1}{\lambda}Tr_g\big(h(grad f, d\Phi(*))h(\nabla^\Phi_* \sigma, \varphi\sigma) \\
& + f h(\nabla^\Phi_* \sigma, \varphi\nabla^\Phi_* \sigma)\big)\varphi\sigma - \Delta^\Phi \sigma\Big),
\end{aligned}$$

where $\Delta^\Phi \sigma := -Tr_g(\nabla^\Phi)^2\sigma = -Tr_g(\nabla^\Phi_* \nabla^\Phi_* - \nabla^\Phi_{\nabla_* *})\sigma$ denotes the rough Laplacian of σ on the pull-back bundle $\Phi^{-1}TN$, $\lambda = 1 + f|\sigma|^2$ and $|\sigma|^2 = h(\sigma, \sigma)$.

Proof. Let $x \in M^m$, $v \in T_{\Phi(x)}N$ and $\{e_i\}_{i=\overline{1,m}}$ be a local orthonormal frame on M^m such that $\nabla_{e_i} e_i = 0$ at x and $\sigma(x) = (\Phi(x), Y_{\Phi(x)}), Y_{\Phi(x)} = v \in T_{\Phi(x)}N$. Using (22), we have

$$\begin{aligned}
\tau(\sigma)_x &= \sum_{i=1}^m \big(\nabla^\sigma_{e_i} d\sigma(e_i) - d\sigma(\nabla_{e_i} e_i)\big)_x \\
&= \sum_{i=1}^m \nabla^{TN}_{d\sigma(e_i)} d\sigma(e_i))_{(\Phi(x),v)}
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{i=1}^{m} \big(\nabla^{TN}_{({}^{H}(d\Phi(e_i))+{}^{V}(\nabla^{\Phi}_{e_i}\sigma))}({}^{H}(d\Phi(e_i)) + {}^{V}(\nabla^{\Phi}_{e_i}\sigma)) \\
&= \sum_{i=1}^{m} \big(\nabla^{TN}_{{}^{H}(d\Phi(e_i))}{}^{H}(d\Phi(e_i)) + \nabla^{TN}_{{}^{H}(d\Phi(e_i))}{}^{V}(\nabla^{\Phi}_{e_i}\sigma) \\
&\quad +\nabla^{TN}_{{}^{V}(\nabla^{\Phi}_{e_i}\sigma)}{}^{H}(d\Phi(e_i)) + \nabla^{TN}_{{}^{V}(\nabla^{\Phi}_{e_i}\sigma)}{}^{V}(\nabla^{\Phi}_{e_i}\sigma).
\end{aligned}
$$

From Theorem 1, we obtain

$$
\begin{aligned}
\tau(\sigma) &= \sum_{i=1}^{m} \Big({}^{H}(\nabla^{N}_{d\Phi(e_i)}d\Phi(e_i)) - \frac{1}{2}{}^{V}(R^{N}(d\Phi(e_i), d\Phi(e_i))\sigma) \\
&\quad +\frac{1}{2}{}^{H}(R^{N}(\sigma, \nabla^{\Phi}_{e_i}\sigma)d\Phi(e_i)) + {}^{V}(\nabla^{N}_{d\Phi(e_i)}\nabla^{\Phi}_{e_i}\sigma) \\
&\quad +\frac{1}{2\lambda}d\Phi(e_i)(f)h(\nabla^{\Phi}_{e_i}\sigma, \varphi\sigma){}^{V}(\varphi\sigma) + \frac{1}{2}{}^{H}(R^{N}(\sigma, \nabla^{\Phi}_{e_i}\sigma)d\Phi(e_i)) \\
&\quad +\frac{1}{2\lambda}d\Phi(e_i)(f)h(\nabla^{\Phi}_{e_i}\sigma, \varphi\sigma){}^{V}(\varphi\sigma) - \frac{1}{2}h(\nabla^{\Phi}_{e_i}\sigma, \varphi\sigma)^{2}{}^{H}(gradf) \\
&\quad +\frac{f}{\lambda}h(\nabla^{\Phi}_{e_i}\sigma, \varphi\nabla^{\Phi}_{e_i}\sigma){}^{V}(\varphi\sigma)\Big) \\
&= \sum_{i=1}^{m} \Big({}^{H}\big(\nabla^{\Phi}_{e_i}d\Phi(e_i) + R^{N}(\sigma, \nabla^{\Phi}_{e_i}\sigma)d\Phi(e_i) \\
&\quad -\frac{1}{2}h(\nabla^{\Phi}_{e_i}\sigma, \varphi\sigma)^{2}gradf\big) + {}^{V}\big(\nabla^{\Phi}_{e_i}\nabla^{\Phi}_{e_i}\sigma \\
&\quad +\frac{1}{\lambda}h(gradf, d\Phi(e_i))h(\nabla^{\Phi}_{e_i}\sigma, \varphi\sigma)\varphi\sigma + \frac{f}{\lambda}h(\nabla^{\Phi}_{e_i}\sigma, \varphi\nabla^{\Phi}_{e_i}\sigma)\varphi\sigma\big)\Big) \\
&= {}^{H}\Big(\tau(\Phi) + Tr_{g}\big(R^{N}(\sigma, \nabla^{\Phi}_{*}\sigma)d\Phi(*) - \frac{1}{2}h(\nabla^{\Phi}_{*}\sigma, \varphi\sigma)^{2}gradf\big)\Big) \\
&\quad +{}^{V}\Big(\frac{1}{\lambda}Tr_{g}\big(h(gradf, d\Phi(*))h(\nabla^{\Phi}_{*}\sigma, \varphi\sigma) \\
&\quad +fh(\nabla^{\Phi}_{*}\sigma, \varphi\nabla^{\Phi}_{*}\sigma)\big)\varphi\sigma - \Delta^{\Phi}\sigma\Big).
\end{aligned}
$$

□

From Proposition 2, we obtain the following.

Theorem 7. *Let (M^m, g) be a Riemannian manifold, and let (N^{2n}, h, φ) be an anti-paraKähler manifold. Assume f is a strictly positive smooth function on N^{2n}, and $(TN, \widetilde{H})$ denotes its tangent bundle equipped with the vertical generalized Berger-type deformed Sasaki metric. Consider a smooth map $\Phi : M^m \to N^{2n}$ and a vector field Y on N^{2n}, with $\sigma := Y \circ \Phi$. The map σ is harmonic if and only if the following conditions are satisfied:*

$$
\tau(\Phi) = Tr_{g}\big(\frac{1}{2}h(\nabla^{\Phi}\sigma, \varphi\sigma)^{2}gradf - R^{N}(\sigma, \nabla^{\Phi}\sigma)d\Phi(*)\big),
$$

and

$$\Delta^{\Phi}\sigma = \frac{1}{\lambda}Tr_g\big(h(gradf, d\Phi(*))h(\nabla^{\Phi}\sigma, \varphi\sigma) + fh(\nabla^{\Phi}\sigma, \varphi\nabla^{\Phi}\sigma)\big)\varphi\sigma.$$

3.3 Harmonicity of a composition of the projection map of the tangent bundle with a smooth map

Lemma 9. *Let (M^{2m}, φ, g) be an anti-paraKähler manifold, and let $(TM, \tilde{g})$ denote its tangent bundle equipped with the vertical generalized Berger-type deformed Sasaki metric. The tension field associated with the canonical projection*

$$\pi : (TM, \tilde{g}) \to (M^{2m}, \varphi, g)$$

is expressed as follows:

$$\tau(\pi) = \frac{\lambda - 1}{2f\lambda}(gradf) \circ \pi, \tag{23}$$

where $\lambda = 1 + f|u|^2$.

Proof. **First case:** Let $(x, u) \in TM$ with $u = 0$ and $\{e_i\}_{i=\overline{1,2m}}$ be a local orthonormal frame on M^{2m} at x. Then $\{{}^He_i, {}^Ve_j\}_{i=\overline{1,2m}, j=\overline{1,2m}}$ is a local orthonormal frame on TM at $(x, 0)$. We have

$$\begin{aligned}\tau(\pi) &= \sum_{i=1}^{2m}\Big(\nabla_{d\pi({}^He_i)}d\pi({}^He_i) - d\pi(\widetilde{\nabla}_{{}^He_i}{}^He_i)\Big) \\ &\quad + \sum_{j=1}^{2m}\Big(\nabla_{d\pi({}^Ve_j)}d\pi({}^Ve_j) - d\pi(\widetilde{\nabla}_{{}^Ve_j}{}^Ve_j)\Big).\end{aligned}$$

But since $d\pi({}^VX) = 0$ and $d\pi({}^HX) = X \circ \pi$, for any vector field X on M^{2m}, then we find

$$\begin{aligned}\tau(\pi) &= \sum_{i=1}^{2m}\Big(\nabla_{(e_i \circ \pi)}(e_i \circ \pi) - d\pi(\nabla_{e_i}e_i)^H\Big) \\ &= 0.\end{aligned}$$

Second case: Let $(x, u) \in TM$ with $u \neq 0$ and $\{e_i\}_{i=\overline{1,2m}}$ such that $e_1 = \frac{u}{|u|}$ be a local orthonormal frame on M^{2m} at x. Then $\{{}^He_i, \frac{1}{\sqrt{\lambda}}{}^V(\varphi e_1), {}^V(\varphi e_j)\}_{i=\overline{1,2m}, j=\overline{2,2m}}$ is a local orthonormal frame on TM at (x, u). Using Theorem 1, we obtain

$$\begin{aligned}\tau(\pi) &= \sum_{i=1}^{2m}\Big(\nabla_{d\pi({}^He_i)}d\pi({}^He_i) - d\pi(\widetilde{\nabla}_{{}^He_i}{}^He_i)\Big) + \nabla_{d\pi(\frac{1}{\sqrt{\lambda}}{}^V(\varphi e_1))}d\pi(\frac{1}{\sqrt{\lambda}}{}^V(\varphi e_1)) \\ &\quad - d\pi(\widetilde{\nabla}_{(\frac{1}{\sqrt{\lambda}}{}^V(\varphi e_1))}(\frac{1}{\sqrt{\lambda}}{}^V(\varphi e_1))) + \sum_{j=2}^{2m}\Big(\nabla_{d\pi{}^V(\varphi e_j)}d\pi{}^V(\varphi e_j)\end{aligned}$$

$$
\begin{aligned}
&\quad -d\pi(\widetilde{\nabla}_{{}^V(\varphi e_j)}{}^V(\varphi e_j))\Big) \\
&= \sum_{i=1}^{2m}\Big(\nabla_{(e_i\circ\pi)}(e_i\circ\pi) - d\pi(\nabla_{e_i}e_i)^H\Big) - \frac{1}{\sqrt{\lambda}}d\pi\Big({}^V(\varphi e_1)(\frac{1}{\sqrt{\lambda}})^V(\varphi e_1) \\
&\quad +\frac{1}{\sqrt{\lambda}}\widetilde{\nabla}_{{}^V(\varphi e_1)}{}^V(\varphi e_1)\Big) \\
&\quad -\sum_{j=2}^{2m} d\pi\Big(-\frac{1}{2}g(e_j,u)^{2H}(gradf) + \frac{f}{\lambda}g(\varphi e_j,e_j)^V(\varphi U)\Big) \\
&= \sum_{i=1}^{2m}\Big((\nabla_{e_i}e_i)\circ\pi - (\nabla_{e_i}e_i)\circ\pi\Big) - \frac{1}{\lambda}d\pi\Big(-\frac{1}{2}g(e_1,u)^{2H}(gradf) \\
&\quad +\frac{f}{\lambda}g(\varphi e_1,e_1)^V(\varphi U)\Big) \\
&= \frac{1}{2\lambda}|u|^2(gradf)\circ\pi = \frac{\lambda-1}{2f\lambda}(gradf)\circ\pi.
\end{aligned}
$$

□

Theorem 8. *Let (M^{2m},φ,g) be an anti-paraKähler manifold, and let (N^n,h) be a Riemannian manifold. Consider a strictly positive smooth function f defined on M^{2m}, along with its tangent bundle $(TM,\tilde{g})$ equipped with the vertical generalized Berger-type deformed Sasaki metric. If $\Phi:(M^{2m},g,\varphi)\to(N^n,h)$ is a smooth map, then the tension field of the composition $\Phi\circ\pi$ is given by:*

$$
\tau(\Phi\circ\pi) = \big(\tau(\Phi) + \frac{\lambda-1}{2f\lambda}d\Phi(gradf)\big)\circ\pi.
$$

Proof. **First case:** Let $(x,u)\in TM$ with $u=0$, let $\{e_i\}_{i=\overline{1,2m}}$ be a local orthonormal frame on M^{2m} at x. Then $\{{}^He_i,{}^Ve_j\}_{i=\overline{1,2m},j=\overline{1,2m}}$ is a local orthonormal frame on TM at $(x,0)$. The tension field of the composition $\Phi\circ\pi$ is given by [9, 11]

$$
\tau(\Phi\circ\pi) \quad = \quad d\Phi(\tau(\pi)) + \mathrm{Tr}_{\tilde{g}}\nabla d\Phi(d\pi,d\pi).
$$

From which we have

$$
\begin{aligned}
\mathrm{Tr}_{\tilde{g}}\nabla d\Phi(d\pi,d\pi) &= \sum_{i=1}^{2m}\Big(\nabla^N_{d\Phi(d\pi({}^He_i))}d\Phi(d\pi({}^He_i)) - d\Phi(\nabla_{d\pi({}^He_i)}d\pi({}^He_i))\Big) \\
&\quad +\sum_{j=1}^{2m}\Big(\nabla^N_{d\Phi(d\pi({}^Ve_j))}d\Phi(d\pi({}^Ve_j)) - d\Phi(\nabla_{d\pi({}^Ve_j)}d\pi({}^Ve_j))\Big) \\
&= \sum_{i=1}^{2m}\Big((\nabla^N_{d\Phi(e_i)}d\Phi(e_i))\circ\pi - d\Phi(\nabla_{e_i}e_i)\circ\pi\Big) \\
&= \tau(\Phi)\circ\pi.
\end{aligned}
$$

Using (23), we obtain

$$\tau(\Phi\circ\pi) \quad = \quad \tau(\Phi)\circ\pi.$$

Second case: Let $(x,u)\in TM$ with $u\neq 0$ and $\{e_i\}_{i=\overline{1,2m}}$ such that $e_1=\frac{u}{|u|}$ be a local orthonormal frame on M^{2m} at x. Then $\{^He_i, \frac{1}{\sqrt{\lambda}}{}^V(\varphi e_1), {}^V(\varphi e_j)\}_{i=\overline{1,2m}, j=\overline{2,2m}}$ is a local orthonormal frame on TM at (x,u). As in the previous case, we calculate

$$\begin{aligned}
\mathrm{Tr}_{\tilde{g}}\nabla d\Phi(d\pi,d\pi) &= \sum_{i=1}^{2m}\Big(\nabla^N_{d\Phi(d\pi(^He_i))}d\Phi(d\pi(^He_i)) - d\Phi(\nabla_{d\pi(^He_i)}d\pi(^He_i))\Big)\\
&\quad +\sum_{j=2}^{2m}\Big(\nabla^N_{d\Phi(d\pi(^V(\varphi e_j)))}d\Phi(d\pi(^V(\varphi e_j)))\\
&\quad -d\Phi(\nabla_{d\pi(^V(\varphi e_j))}d\pi(^V(\varphi e_j)))\Big)\\
&\quad +\nabla^N_{d\Phi(d\pi(\frac{1}{\sqrt{\lambda}}{}^V(\varphi e_1)))}d\Phi(d\pi(\frac{1}{\sqrt{\lambda}}{}^V(\varphi e_1)))\\
&\quad -d\Phi(\nabla_{d\pi(\frac{1}{\sqrt{\lambda}}{}^V(\varphi e_1))}d\pi(\frac{1}{\sqrt{\lambda}}{}^V(\varphi e_1)))\\
&= \sum_{i=1}^{2m}\Big((\nabla^N_{d\Phi(e_i)}d\Phi(e_i))\circ\pi - d\Phi(\nabla_{e_i}e_i)\circ\pi\Big)\\
&= \tau(\Phi)\circ\pi.
\end{aligned}$$

Using (23), we obtain

$$\begin{aligned}
\tau(\Phi\circ\pi) &= \tau(\Phi)\circ\pi + d\Phi(\frac{\lambda-1}{2f\lambda}(gradf)\circ\pi)\\
&= \big(\tau(\Phi)+\frac{\lambda-1}{2f\lambda}d\Phi(gradf)\big)\circ\pi.
\end{aligned}$$

□

Theorem 9. *Let (M^{2m},φ,g) be an anti-paraKähler manifold, and let $(TM,\tilde{g})$ represent its tangent bundle equipped with the vertical generalized Berger-type deformed Sasaki metric. Additionally, let (N^n,h) be a Riemannian manifold, and let f be a strictly positive smooth function defined on M^{2m}. If $\Phi:(M^{2m},\varphi,g)\to(N^n,h)$ is a smooth map, then the composition $\Phi\circ\pi$ is harmonic if and only if:*

$$\tau(\Phi)=\frac{1-\lambda}{2f\lambda}d\Phi(gradf).$$

Conclusion

In this study, we have introduced the vertical generalized Berger-type deformed Sasaki metric on the tangent bundle of an anti-paraKähler manifold. Our exploration began with a detailed examination of the harmonicity conditions for vector fields under this novel metric, supplemented by illustrative examples that highlight the unique properties of harmonic vector fields in this context.

We further investigated the interplay between harmonicity and smooth maps between Riemannian manifolds, particularly focusing on cases where the target manifold is anti-paraKähler. Through rigorous analysis, we established criteria for harmonicity in vector fields arising from the composition of projections and maps between manifolds, thereby enhancing our understanding of how these mappings behave under the influence of the introduced metric.

Key findings demonstrate that a vector field on a compact oriented anti-paraKähler manifold is harmonic if and only if it is parallel, emphasizing a significant connection between these two concepts. Moreover, we established that the tension field associated with the composition of maps could be articulated in terms of the underlying structures of the manifolds involved, providing a framework for analyzing harmonic maps in more complex geometrical settings.

The results presented herein not only advance the theoretical foundations of harmonic maps and vector fields in the context of anti-paraKähler geometry but also open avenues for future research. Potential directions include investigating further applications of the vertical generalized Berger-type deformed Sasaki metric, exploring its implications in different geometrical contexts, and examining the interplay between harmonicity and other geometric structures in broader mathematical frameworks.

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