BULETINUL ACADEMIEI DE ŞTIINŢE
A REPUBLICII MOLDOVA. MATEMATICA
Numbers 1(107), 2025, Pages 43–57
ISSN 1024–7696, E-ISSN 2587–4322

About the Laplace transform on $L^1(\mathbb{R}^+)$

Nargiza Boboyarova and Jonibek Abdullayev

Abstract. In the paper some preliminaries are stated, including the basic properties of $\mathscr{L}L^1(\mathbb{R}^+)$. As the appropriate conditions for $g \in \mathscr{L}L^1(\mathbb{R}^+)$, we attempted to take those ones formulated in the paper.

Mathematics subject classification: 44A10, 46E15, 97I80.
Keywords and phrases: Laplace transformation, space $L^1(\mathbb{R}^+)$, Banach spaces..

1 Introduction

The Laplace transform is important in various areas of mathematics: functional analysis, Fourier analysis and solving equations of mathematical physics (see [1–5]). The study of holomorphic Laplace transform has for a long time interested complex analysis. This paper is devoted the to study of the Laplace transform in the space $L^1(\mathbb{R}^+)$ and some problems concerning the algebras of holomorphic functions. The Laplace transform is named in honor of mathematician and astronomer Pierre-Simon Laplace, who used the transform in his work on probability theory. The Laplace transform has many important applications in mathematics, physics, optics, electrical engineering, control engineering, signal processing, and probability theory. In mathematics, it is used for solving differential and integral equations. In physics, it is used for analysis of linear time-invariant systems such as electrical circuits, harmonic oscillators, optical devices, and mechanical systems.

While certain of the properties of the Laplace transformation are so well known that they have become engineering tools, there are others that have received very little attention, and yet are very interesting.

One of these comes about as follows: Suppose that $f \in L^1(\mathbb{R}^+)$ is a complex-valued function and s is a complex parameter. We define the Laplace transform of f as

$$g(s) = Lf(s) = \int_0^\infty e^{-sx} f(x)\,dx,$$

the integral converges at least for $s \in \Sigma = \{z \in \mathbb{C} : \operatorname{Re} z \geq 0\}$, i.e. $D(g) \supset \Sigma$. We shall suppose $D(\mathscr{L}f) = \Sigma$ for $f \in L^1(\mathbb{R}^+)$. If $\mathscr{L}L^1(\mathbb{R}^+) = C_0(\Sigma) \cap Hol(\Sigma^0)$, where

$$C_0(\Sigma) = \left\{ f \in C(\Sigma) : \lim_{|z|\to\infty} f(z) = 0,\ z \in \Sigma \right\},$$

DOI: https://doi.org/10.56415/basm.y2025.i1.p43

we could use the Open Mapping Theorem (see e.g.[6]) and obtain that L^{-1} is continuous. So $\mathscr{L}$ would be a homeomorphism of $\mathscr{L}L^1(\mathbb{R}^+)$ onto $C_0(\Sigma) \cap Hol\left(\Sigma^0\right)$. However, holds

$$\mathscr{L}L^1\left(\mathbb{R}^+\right) \underset{\neq}{\subset} C_0(\Sigma) \cap Hol\left(\Sigma^0\right)$$

(see [6], p.215, Ex.2). This is the starting point of our work. It means that for any

$$g \in C_0(\Sigma) \cap Hol\left(\Sigma^0\right) \setminus vL^1\left(\mathbb{R}^+\right)$$

and for any $\{g_n\}_{n=1}^{\infty} \subset C_0(\Sigma) \cap Hol\left(\Sigma^0\right)$ such that $g_n \rightrightarrows g$ in Σ and $\left\{\mathscr{L}^{-1}g_n\right\}_{n=1}^{\infty} \subset L^1(\mathbb{R}^+)$, the sequence $\left\{L^{-1}g_n\right\}_{n=1}^{\infty}$ has no partial limit in $L^1(\mathbb{R}^+)$. The simplest way how to verify it is to prove $\lim\limits_{n\to\infty}\left\|L^{-1}g_n\right\| = \infty$.

We tried to find a necessary and sufficient condition for $g \in C_0(\Sigma) \cap Hol\left(\Sigma^0\right)$ to be an element of $\mathscr{L}L^1(\mathbb{R}^+)$, i.e. for the existence of $f \in L^1(\mathbb{R}^+)$ such that $g = \mathscr{L}f$. Further, we wanted to define a norm on $\mathscr{L}L^1(\mathbb{R}^+)$ such that $\mathscr{L}$ is a homeomorphism of $L^1(\mathbb{R}^+)$ onto $\mathscr{L}L^1(\mathbb{R}^+)$ which can be calculated without an explicit determination of $\mathscr{L}^{-1}g$. In the paper we showed some ways how to do it.

In the paper some preliminaries are stated, including the basic properties of $\mathscr{L}L^1(\mathbb{R}^+)$. As the appropriate conditions for $g \in \mathscr{L}L^1(\mathbb{R}^+)$, we attempted to take those ones formulated in the paper. We call special attention to the works of D. Widder [7–9], who has great results in this way. But his theorems are, in general, very involved and provide no practical criteria.

2 Laplace transform of the space $L^1(\mathbb{R}^+)$

Let $\mathbb{C}$ be the set of all complex numbers, $\Sigma = \{s \in \mathbb{C} : \operatorname{Re} s \geq 0\}$ be the right half-plane of $\mathbb{C}$ and Σ^0 be its interior. Let $C(\Sigma)$ be the set of all continuous functions $f : \Sigma \to \mathbb{C}$,

$$C_b(\Sigma) = \{f \in C(\Sigma) : f \text{ is bounded in } \Sigma\},$$

$$C(\Sigma \cup \{\infty\}) = C_c(\Sigma) = \{f \in C(\Sigma) : \lim_{|z|\to\infty} f(z) = c,\ c \in \mathbb{C},\ z \in \Sigma\}$$

and

$$C_0(\Sigma) = \{f \in C(\Sigma) : \lim_{|z|\to\infty} f(z) = 0,\ z \in \Sigma\}.$$

Denote by $Hol(\Sigma^0)$ the set of all analytic functions on Σ^0.

Suppose that $f \in L^1(\mathbb{R}^+)$ is a complex-valued function and s is a complex parameter. We define the Laplace transform of f as

$$g(s) = \mathscr{L}f(s) = \int_0^{\infty} e^{-sx} f(x)dx,$$

the integral converges at least for $s \in \Sigma$, i.e. $D(g) \subset \Sigma$. We shall suppose $D(\mathscr{L}f) = \Sigma$ for $f \in L^1(\mathbb{R}^+)$.

Now, we develop some useful properties of the Laplace transform.

Proposition 1. *The Laplace transform is an injective linear mapping* $L^1(\mathbb{R}^+)$ *to* $C_0(\Sigma)\cap Hol(\Sigma^0)$.

Proof. Since the integral

$$g(s)=\int\limits_0^\infty e^{-sx}f(x)dx$$

uniformly converges on Σ, it follows that $g\in C(\Sigma)\cap Hol(\Sigma^0)$. Since

$$|g(s)|\le\int\limits_0^\infty |f(x)|dx=\|f\|_1$$

we have $g\in C_b(\Sigma)$. Now we will prove that $g\in C_0(\Sigma)\subset C_b(\Sigma)$.
Let $s=\sigma+it\in\Sigma^0$, then

$$|g(s)|\le\int\limits_0^\infty |f(x)|e^{-\sigma x}dx$$

and the relation

$$\lim_{\sigma\to+\infty}\int\limits_0^\infty |f(x)|e^{-\sigma x}dx=0$$

follows from Lebesgue's Dominated Convergence Theorem. Thus for any $\varepsilon>0$ there exists $\sigma_0>0$ such that $|g(\sigma+it)|<\varepsilon$ for $\sigma>\sigma_0$. Let $\sigma\in\langle 0;\sigma_0\rangle$. We can find $A>0$ such that

$$\int\limits_A^\infty |f(x)|dx<\frac{\varepsilon}{4}.$$

Since the set $AC(\langle 0;A\rangle)$ of all absolutely continuous functions on $\langle 0;A\rangle$ is dense in $L^1(0;A)$ (cf.[6], 5.14, p.120), there exists $\psi\in AC(\langle 0;A\rangle)$ such that

$$\int\limits_0^A |f(x)-\psi(x)|dx<\frac{\varepsilon}{4}.$$

Since

$$|g(\sigma+it)|\le\int\limits_A^\infty |f(x)|dx+\int\limits_0^A |f(x)-\psi(x)|dx+$$

$$+\left|\int\limits_0^A \psi(x)e^{-\sigma x}\cos tx\,dx\right|+\left|\int\limits_0^A \psi(x)e^{-\sigma x}\sin tx\,dx\right|,$$

it's enough to estimate the last two integrals. Using integration by parts we get

$$\int_0^A \psi(x)e^{-\sigma x}\cos tx dx = \left[\psi(x)e^{-\sigma x}\frac{\sin tx}{t}\right]_0^A - \frac{1}{t}\int_0^A (\psi'(x)-\sigma\psi(x))e^{-\sigma x}\sin tx dx$$

and so

$$\left|\int_0^A \psi(x)e^{-\sigma x}\cos tx dx\right| \le \frac{1}{|t|}(|\psi(A)|+|\psi(0)|)+\frac{1}{|t|}\int_0^A (|\psi'(x)|+\sigma_0|\psi(x)|)dx = \frac{K}{|t|} < \frac{\varepsilon}{4}$$

for $|t| > t_0 = \frac{4K}{\varepsilon}$. For the integral

$$\int_0^A \psi(x)e^{-\sigma x}\sin tx dx$$

holds the same. Thus for $\sigma \in \langle 0;\sigma_0\rangle$ and $|t| > t_0$ we have also $|g(\sigma+it)| < \varepsilon$. The linearity of $\mathscr{L}$ is obvious. The transform $\mathscr{L}$ is injective according to ([6], 9.12, p.208.)

□

Let us define in $C_b(\Sigma)$ the norm

$$\|g\| = \sup_{s\in\Sigma}|g(s)|.$$

For $g \in C_c(\Sigma)$ we have

$$\|g\| = \max_{s\in\Sigma\cup\{\infty\}}|g(s)|.$$

Theorem 1. *The space* $C_0(\Sigma)\cap Hol(\Sigma^0)$ *is a Banach space.*

Proof. We know that $C(K)$ is a Banach space with the norm

$$\|g\| = \max_{s\in K}|g(s)|$$

for the compact set K . It follows $C_c(\Sigma)$ is a Banach space. In particular, $C_0(\Sigma)$ is a closed subspace of $C_c(\Sigma)$. Since every closed subspace of a Banach space is a Banach space, it follows $C_0(\Sigma)$ is a Banach space.

Note also that for $\{g_n\} \subset C_0(\Sigma)\cap Hol(\Sigma^0)$ such that $g_n \rightrightarrows g$ in Σ, we have $g \in C_0(\Sigma)\cap Hol(\Sigma^0)$. Hence $C_0(\Sigma)\cap Hol(\Sigma^0)$ is a Banach space as a closed subspace of $C_0(\Sigma)$.

□

Proposition 1 contains some properties of $\mathscr{L}$. We have $\mathscr{L}L^1(\mathbb{R}^+) \subset C_0(\Sigma)\cap Hol(\Sigma^0)$ according to Proposition 1. Moreover, the next theorem holds.

Theorem 2. *The mapping* $\mathscr{L} : L^1(\mathbb{R}^+) \to C_0(\Sigma) \cap Hol(\Sigma^0)$ *is continuous.*

Proof. Let $f_n, f \in L^1(\mathbb{R}^+)$, $g_n = \mathscr{L} f_n$, $g = \mathscr{L} f$, $g_n, g \in C_0(\Sigma) \cap Hol(\Sigma^0)$.

$f_n \to f$ in $L^1(\mathbb{R}^+)$, $\|f_n - f\|_1 = \int\limits_0^\infty |f_n(t) - f(t)| dt \to 0$. Then

$$|g_n(s) - g(s)| = \left| \int\limits_0^\infty e^{-st}(f_n(t) - f(t)) dt \right| \leq \int\limits_0^\infty |e^{-st}||f_n(t) - f(t)| dt \leq$$

$$\leq \int\limits_0^\infty |f_n(t) - f(t)| dt = \|f_n - f\|_1$$

for all $s \in \Sigma$. It follows

$$\|g_n - g\| = \max_{s \in \Sigma} |g_n(s) - g(s)| \leq \|f_n - f\|_1.$$

Hence $\|f_n - f\|_1 \to 0$ as $n \to \infty$, which implies

$$\|\mathscr{L} f_n - \mathscr{L} f\| = \|g_n - g\| \to 0 \quad \text{as} \quad n \to \infty.$$

□

The question of equality of the sets $\mathscr{L} L^1(\mathbb{R}^+)$ and $C_0(\Sigma) \cap Hol(\Sigma^0)$ appears.

It seems the answer is negative. To prove this fact we have to find some function $F \in C_0(\Sigma) \cap Hol(\Sigma^0)$ such that $F \notin \mathscr{L} L^1(\mathbb{R}^+)$, i.e. $F \in C_0(\Sigma) \cap Hol(\Sigma^0) - \mathscr{L} L^1(\mathbb{R}^+)$. We can use the open mapping theorem (see [3]): the continuous injective mapping $\mathscr{L} : L^1(\mathbb{R}^+) \to C_0(\Sigma) \cap Hol(\Sigma^0)$ is on, it follows $\mathscr{L}^{-1}$ is continuous.

If $\mathscr{L} L^1(\mathbb{R}^+) = C_0(\Sigma) \cap Hol(\Sigma^0)$, we could use the Open Mapping Theorem (see [6], p.115, 5.9) and obtain that $\mathscr{L}^{-1}$ is continuous. So $\mathscr{L}$ would be a homeomorphism of $\mathscr{L} L^1(\mathbb{R}^+)$ onto $C_0(\Sigma) \cap Hol(\Sigma^0)$. However,

$$\mathscr{L} L^1(\mathbb{R}^+) \subsetneq C_0(\Sigma) \cap Hol(\Sigma^0)$$

holds (see Example p.215, Ex.2). It means that for any

$$g \in C_0(\Sigma) \cap Hol(\Sigma^0) \setminus \mathscr{L} L^1(\mathbb{R}^+)$$

and for any

$$\{g_n\}_{n=1}^\infty \subset C_0(\Sigma) \cap Hol(\Sigma^0)$$

such that $g_n \rightrightarrows g$ in Σ and $\{\mathscr{L}^{-1} g_n\}_{n=1}^\infty \subset L^1(\mathbb{R}^+)$, the sequence $\{\mathscr{L}^{-1} g_n\}_{n=1}^\infty$ has no partial limit in $L^1(\mathbb{R}^+)$. The simplest way how to verify it is to prove $\lim\limits_{n \to \infty} \|\mathscr{L}^{-1} g_n\| = \infty$.

3 Growth conditions on $g^{(n)}$ for $g \in \mathscr{L}L^1(\mathbb{R}^+)$

For $\varepsilon > 0$ put $\Sigma_\varepsilon = \{s \in \mathbb{C} : \operatorname{Re} s \geq \varepsilon\}$. Let $n \in \mathbb{N}$, $f \in L^1(\mathbb{R}^+)$. Since the function $x^n f(x) e^{-\varepsilon x}$ is an element of $L^1(\mathbb{R}^+)$ for all $\varepsilon > 0$, it follows the integral

$$\int_0^\infty x^n f(x) e^{-sx} dx$$

converges uniformly in Σ_ε for all $\varepsilon > 0$. Therefore we have

$$g^{(n)}(s) = (-1)^n \int_0^\infty x^n f(x) e^{-sx} dx$$

for $s \in \Sigma^0$. Using Proposition 1 for the function

$$g^{(n)}(s+\varepsilon) = (-1)^n \mathscr{L}[x^n f(x) e^{-\varepsilon x}](s), \quad s \in \Sigma$$

we get the following

Proposition 2. *For all* $n \in \mathbb{N}$ *and for any* $\varepsilon > 0$ *we have* $g^{(n)} \in C_0(\Sigma_\varepsilon)$.

Definition 1. Let $g \in Hol(\Sigma^0)$, $n \in \mathbb{N}$. We define on Σ the function h_n by the formulas

$$h_n(s) = h_n(\sigma + it) = \begin{cases} 0, & \sigma = 0 \\ \dfrac{\sigma^n}{n!} g^{(n)}(s), & \sigma > 0. \end{cases}$$

Proposition 3. *Let* $f \in L^1(\mathbb{R}^+)$, $g = \mathscr{L}f$, h_n *are as in Definition 1. Then*
a) $h_n \in C_0(\Sigma)$ *for all* $n \in \mathbb{N}$ *. In particular,*

$$\lim_{\sigma \to 0+} \sigma^n g^{(n)}(\sigma + it) = \lim_{\sigma \to \infty} \sigma^n g^{(n)}(\sigma + it) = 0$$

uniformly in $t \in \mathbb{R}$.
b) $\|h_n\| = O\left(\dfrac{1}{\sqrt{n}}\right)$, $n \to \infty$.

Proof. For $n \in \mathbb{N}$ and $s = \sigma + it \in \Sigma^0$ the relation

$$\sigma^n |g^{(n)}(s)| = \left| \int_0^\infty (\sigma x)^n f(x) e^{-sx} dx \right| \leq \int_0^\infty (\sigma x)^n |f(x)| e^{-\sigma x} dx$$

holds. Using Lebesgue's Dominated Convergence Theorem we obtain that the limit of right hand side of this inequality is equal to 0 as $\sigma \to 0+$ or $\sigma \to +\infty$. Thus for any $\varepsilon > 0$ there exists $\delta \in (0;1)$ such that $\sigma^n |g^{(n)}(s)| < \varepsilon$ if $\sigma = \operatorname{Re} s \in (0;\delta) \cup (\frac{1}{\delta}; +\infty)$. According to Proposition 2 we have

$$\sigma^n |g^{(n)}(s)| \leq \left(\frac{1}{\delta}\right)^n |g^{(n)}(s)| < \varepsilon$$

if $\sigma = \operatorname{Re} s \in \langle \delta; \frac{1}{\delta} \rangle$ and $|t| = |\operatorname{Im} s|$ is large enough.

Since $\max\limits_{x \geq 0} x^n e^{-x} = n^n e^{-n}$ we can estimate

$$|h_n(s)| \leq \frac{n^n e^{-n}}{n!} \|f\|_1$$

for all $n \in \mathbb{N}$, $s \in \Sigma$. By Stirling's formula $\left(n^n e^{-n} \sim \dfrac{n!}{\sqrt{2\pi n}}, \ n \to \infty\right)$ we obtain

$$\|h_n\| = \max_{s \in \Sigma^0} \left| \frac{\sigma^n}{n!} g^{(n)}(s) \right| = O\left(\frac{1}{\sqrt{n}}\right), \quad n \to \infty.$$

□

To prove the existence of $g \in C_0(\Sigma) \cap Hol(\Sigma^0) \setminus \mathscr{L}L^1(\mathbb{R}^+)$ it is sufficient to find a convergent sequence $\{g_n\}_{n=1}^{\infty}$ in $C_0(\Sigma) \cap Hol(\Sigma^0)$ such that $\{\mathscr{L}^{-1} g_n\} \subset L^1(\mathbb{R}^+)$ is not fundamental in $L^1(\mathbb{R}^+)$.

4 Integrability conditions on $g^{(n)}$ for $g \in \mathscr{L}L^1(\mathbb{R}^+)$

Proposition 4. *Let* $f \in L^1(\mathbb{R}^+)$, $g = \mathscr{L}f$. *Then* $g'(\sigma) \in L^1(\mathbb{R}^+)$ *(i.e. the restriction of* g' *to* $(0; +\infty)$ *is an integrable function or* g *has the finite variation along the real half-axis in* Σ *).*

We can prove the more general case of this proposition.

Proposition 5. *Let* $f \in L^1(\mathbb{R}^+)$, $g = \mathscr{L}f$. *Then*

$$\frac{\sigma^{n-1}}{(n-1)!} g^{(n)}(\sigma) \in L^1(\mathbb{R}^+)$$

for all $n \in \mathbb{N}$, *and*

$$\int\limits_0^\infty \frac{\sigma^{n-1}}{(n-1)!} |g^{(n)}(\sigma)| d\sigma = O(1), \quad n \to \infty.$$

Proof. Firstly, we prove this theorem in the case $f \geq 0$. Then

$$|g^{(n)}(\sigma)| = \int\limits_0^\infty x^n f(x) e^{-\sigma x} dx = (-1)^n g^{(n)}(\sigma)$$

for $\sigma > 0$ and using Propositions 1 and 3 we get

$$\int\limits_0^\infty \frac{\sigma^{n-1}}{(n-1)!} |g^{(n)}(\sigma)| d\sigma = (-1)^n \int\limits_0^\infty \frac{\sigma^{n-1}}{(n-1)!} g^{(n)}(\sigma) d\sigma =$$

$$= (-1)^n \left(\left[\frac{\sigma^{n-1}}{(n-1)!} g^{(n-1)}(\sigma) \right]_0^\infty - \int_0^\infty \frac{\sigma^{n-2}}{(n-2)!} g^{(n-1)}(\sigma) d\sigma \right) =$$

$$= (-1)^{n-1} \int_0^\infty \frac{\sigma^{n-2}}{(n-2)!} g^{(n-1)}(\sigma) d\sigma = \dots = - \int_0^\infty g'(\sigma) d\sigma = g(0).$$

For a real valued function $f \in L^1(\mathbb{R}^+)$ denote

$$f(x) = f^+(x) - f^-(x), \quad x > 0, \quad f^+, f^- \geq 0$$

be the positive and negative parts of f, respectively. Then

$$f^+, f^- \in L^1(\mathbb{R}^+), \quad f^+ + f^- = |f|$$

and

$$\int_0^\infty \frac{\sigma^{n-1}}{(n-1)!} |g^{(n)}(\sigma)| d\sigma = \int_0^\infty \frac{\sigma^{n-1}}{(n-1)!} |(\mathscr{L} f^+)^{(n)}(\sigma) - (\mathscr{L} f^-)^{(n)}(\sigma)| d\sigma \leq$$

$$\leq \int_0^\infty \frac{\sigma^{n-1}}{(n-1)!} (|(\mathscr{L} f^+)^{(n)}(\sigma)| + |(\mathscr{L} f^-)^{(n)}(\sigma)|) d\sigma = \int_0^\infty \frac{\sigma^{n-1}}{(n-1)!} |(\mathscr{L} f^+)^{(n)}(\sigma)| d\sigma +$$

$$+ \int_0^\infty \frac{\sigma^{n-1}}{(n-1)!} |(\mathscr{L} f^-)^{(n)}(\sigma)| d\sigma = \mathscr{L} f^+(0) + \mathscr{L} f^-(0) = \mathscr{L}|f|(0) = \int_0^\infty |f(x)| dx = \|f\|_1$$

by the previous part of the proof.

Finally, for a complex valued function $f \in L^1(\mathbb{R}^+)$ it is enough to use the fact that the function $f \in L^1(\mathbb{R}^+)$ if and only if $\{\operatorname{Re} f, \ \operatorname{Im} f\} \subset L^1(\mathbb{R}^+)$.

□

We shall try to find the property characterizing $\mathscr{L} L^1(\mathbb{R}^+)$ in the sense that every function $g \in C_0(\Sigma) \cap Hol(\Sigma^0)$ has this property if and only if $g \in \mathscr{L} L^1(\mathbb{R}^+)$. As the appropriate conditions we shall try to take those ones proved in Proposition 3 and Proposition 5. Moreover, we shall try to find the norm on $\mathscr{L} L^1(\mathbb{R}^+)$, which is equivalent to $\|\mathscr{L}^{-1} g\|_1$ and can be calculated without an explicit determination of $\mathscr{L}^{-1} g$.

In the fifth section for any $g \in C_0(\Sigma) \cap Hol(\Sigma^0)$ one approximating sequence $\{g_n\}_{n=1}^\infty$ is constructed. It is enough to prove $\{\mathscr{L}^{-1} g_n\}_{n=1}^\infty \subset L^1(\mathbb{R}^+)$ has no partial limit in $L^1(\mathbb{R}^+)$ for this only sequence $\{g_n\}_{n=1}^\infty$ if $g \in C_0(\Sigma) \cap Hol(\Sigma^0) \setminus \mathscr{L} L^1(\mathbb{R}^+)$.

5 Growth conditions on $g^{(n)}$ for $g \in C_b(\Sigma) \cap Hol(\Sigma^0)$

In this section the growth conditions on $g^{(n)}$ are shown for $g \in C_b(\Sigma) \cap Hol(\Sigma^0)$ and $g \in C_0(\Sigma) \cap Hol(\Sigma^0)$, where

$$C_b(\Sigma) = \{f \in C(\Sigma) \ : \ f \text{ is bounded in } \Sigma\}.$$

5.1 General case

Let $g \in C_c(\Sigma) \cap Hol(\Sigma^0)$. Put

$$f(z) = g\left(\frac{1-z}{1+z}\right) \text{ if } |z| \le 1,\ z \ne -1,\ f(-1) = g(\infty).$$

Then $f \in C(\overline{U}) \cap Hol(U)$, where $C_0(\overline{U}) = \{f \in C(\overline{U}); f(-1) = 0\}$, and $U = \{z \in \mathbb{C}; |z| < 1\}$ is the unit ball in $\mathbb{C}$. It means that the values of f in U can be uniquely determined by the values of $f|_{\partial U}$ and there exists unique $h \in C_0(\overline{U})$ harmonic in U such that $\widetilde{h}|_{\partial U} = |f||_{\partial U}$.

Denote $g_0(t) = g(it)$, $t \in \mathbb{R}$. Transforming the formula for f and U to the corresponding one for g and Σ we obtain

$$g(x+iy) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{x g_0(t) dt}{x^2 + (y-t)^2}$$

for $x > 0$. This formula holds for $g \in C_b(\Sigma) \cap Hol(\Sigma^0)$, too (cf.[10], the first theorems on p.134 and p.138; the convergence is only locally uniform here).

and $h \in C_0(\Sigma)$ if $h(x+iy) = |g(x+iy)| = |g_0(y)|$ for $x = 0$ and

$$\widetilde{m}\left(\frac{1-z}{1+z}\right) = h(x+iy) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{x|g_0(t)|dt}{x^2 + (y-t)^2};$$

the function h is harmonic in Σ^0 but we don't use this property in our proof.

Proposition 6. *Let for $g \in C_b(\Sigma) \cap Hol(\Sigma^0)$ and $n \in \mathbb{N}$ the function h_n be as in Definition 1. Then*
a) $h_n \in C_b(\Sigma)$,
b) $\|h_n\| = \sup_{s \in \Sigma^0} |h_n(s)| = O(4^n),\ n \to \infty.$

Proof. It holds

$$\frac{\partial^n}{\partial y^n}\left(\frac{1}{x^2 + (y-t)^2}\right) = \frac{P_n(x, y-t)}{(x^2 + (y-t)^2)^{n+1}}$$

for all $n \in \mathbb{N}$ and $x > 0$, where P_n are polynomials of two variables x, u, defined inductively by

$$P_0(x,u) = 1,\ P_{n+1}(x,u) = (-1)^{n+1}[2(n+1)uP_n(x,u) - (x^2+u^2)P_n'(x,u)],$$

(here "$'$" means the derivative by u). It is clear that P_n is a homogeneous polynomial of degree n. Let's denote

$$P_n(x,u)=\sum_{k=0}^{n} c_{n,k}x^{n-k}u^k.$$

We have the estimate $\sum\limits_{k=0}^{n}|c_{n,k}|\le 4^n n!$ for the height of P_n. Really, this estimate holds for $n=0$. Supposing it is satisfied for n, we obtain by the inductive definition of P_{n+1}

$$\sum_{k=0}^{n+1}|c_{n+1,k}|\le 2(n+1)\sum_{k=0}^{n}|c_{n,k}|+2\sum_{k=0}^{n}|kc_{n,k}|\le 4(n+1)\sum_{k=0}^{n}|c_{n,k}|\le 4^{n+1}(n+1)!.$$

It means

$$x|P_n(x,u)|\le\sum_{k=0}^{n}|c_{n,k}|x^{n+1-k}|u|^k\le 4^n n!(x^2+u^2)^{\frac{n+1}{2}}$$

and

$$\left|\frac{\partial^n}{\partial y^n}\left(\frac{x}{x^2+(y-t)^2}\right)\right|\le\frac{4^n n!}{(x^2+(y-t)^2)^{\frac{n+1}{2}}}\le\frac{4^n n!}{(\triangle^2+(y-t)^2)^{\frac{n+1}{2}}}$$

for $x\ge\triangle\ge 0$. The last function is integrable over $\mathbb{R}$ for any $n\in\mathbb{N}$ and $y\in\mathbb{R}$. So

$$g^{(n)}(x+iy)=\frac{1}{\pi}\int\limits_{-\infty}^{\infty}\frac{xP_n(x,y-t)}{(x^2+(y-t)^2)^{n+1}}g_0(t)dt,$$

$$|h_n(x+iy)|=\frac{x^n}{n!}|g^{(n)}(x+iy)|=\frac{x}{\pi n!}\left|\int\limits_{-\infty}^{\infty}\frac{x^nP_n(x,y-t)}{(x^2+(y-t)^2)^{n+1}}g_0(t)dt\right|\le$$

$$\le\frac{x}{\pi n!}\int\limits_{-\infty}^{\infty}\frac{\sum\limits_{k=0}^{n}|c_{n,k}|x^{2n-k}|y-t|^k}{(x^2+(y-t)^2)^{n+1}}|g_0(t)|dt\le\frac{x}{\pi n!}\int\limits_{-\infty}^{\infty}\frac{(x^2+(y-t)^2)^n\sum\limits_{k=0}^{n}|c_{n,k}|}{(x^2+(y-t)^2)^{n+1}}|g_0(t)|dt\le$$

$$\le\frac{4^n}{\pi}\int\limits_{-\infty}^{\infty}\frac{x}{x^2+(y-t)^2}|g_0(t)|dt\le\frac{4^nK}{\pi}\int\limits_{-\infty}^{\infty}\frac{x}{x^2+(y-t)^2}dt=4^nK$$

for $x+iy\in\Sigma^0$, if $K=\sup\limits_{s\in\Sigma^0}|g(s)|$. It follows

$$h_n\in C_b(\Sigma^0)\text{ and }\|h_n\|=\sup_{s\in\Sigma^0}|h_n(s)|=O(4^n),\ n\to\infty.$$

It remains to prove h_n is continuous at any point of $\partial\Sigma$ with respect to Σ.

Let $y_0 \in \mathbb{R}$. Choose $\varepsilon > 0$. There exists $\delta > 0$ such that

$$|g_0(t) - g_0(y_0)| < \frac{\varepsilon}{3 \cdot 4^n}$$

if $|t - y_0| < \delta$. We can find $\triangle > 0$ such that

$$\int_{-\infty}^{-\triangle} \frac{du}{1+u^2} = \int_{\triangle}^{\infty} \frac{du}{1+u^2} < \frac{\varepsilon\pi}{4^n 6K}.$$

As h_n is the same for both functions g and $g - g_0(y_0)$, we have

$$|h_n(x+iy)| \le \frac{4^n}{\pi} \int_{-\infty}^{\infty} \frac{x}{x^2+(y-t)^2} |g_0(t) - g_0(y_0)| dt \le$$

$$\le \frac{4^n}{\pi} \int_{-\infty}^{\infty} |g_0(xu+y) - g_0(y_0)| \frac{du}{1+u^2} =$$

$$= \frac{4^n}{\pi} \left(\int_{-\infty}^{-\triangle} + \int_{-\triangle}^{\triangle} + \int_{\triangle}^{\infty} \right) |g_0(xu+y) - g_0(y_0)| \frac{du}{1+u^2} <$$

$$< \frac{\varepsilon}{3} + \frac{4^n}{\pi} \int_{-\triangle}^{\triangle} |g_0(xu+y) - g_0(y_0)| \frac{du}{1+u^2} + \frac{\varepsilon}{3} < \frac{2\varepsilon}{3} + \frac{\varepsilon}{3\pi} \int_{-\infty}^{\infty} \frac{du}{1+u^2} = \varepsilon$$

if $|xu + y - y_0| < \delta$ for all $u \in \langle -\triangle, \triangle \rangle$. This condition is satisfied for $|x| < \frac{\delta}{2\triangle}$, $|y - y_0| < \frac{\delta}{2}$. These inequalities describe a neighborhood $U(iy_0)$. It holds $|h_n(s)| < \varepsilon$ for $s \in U(iy_0) \cap \Sigma$.

□

5.2 Special case: $g \in C_0(\Sigma) \cap Hol(\Sigma^0)$

The property a) of Proposition 3 doesn't characterize $\mathscr{L}L^1(\mathbb{R}^+)$ as all functions $g \in C_0(\Sigma) \cap Hol(\Sigma^0)$ satisfy this condition.

Remark 1. It holds $h_n \in C_0(\Sigma), n \in \mathbb{N}$, for any function $g \in C_0(\Sigma) \cap Hol(\Sigma^0)$.

For $g \in C(\Sigma) \cap Hol(\Sigma^0)$ denote $g_0(t) = g(it)$, $t \in R$, $\widetilde{C}(\mathbb{R}) = \{f; f = g_0$ for $g \in C(\Sigma) \cap Hol(\Sigma^0)\}$. Clearly, $\widetilde{C}(\mathbb{R}) \subset C(\mathbb{R})$.

Let $g_0 \in \widetilde{C}_b(\mathbb{R})$. Then the function $g \in C(\Sigma) \cap Hol(\Sigma^0)$ satisfies the relation

$$g(s) = g(x+iy) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{x g_0(t) dt}{x^2 + (y-t)^2}.$$

Moreover, we have

$$\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{xg_0(t)dt}{x^2+(y-t)^2}\to g_0(y)$$

as $x\to 0+$ for all $y\in\mathbb{R}$, if $g_0\in C_0(\Sigma)\cap Hol(\Sigma^0)$.

Proposition 7. *Let for* $g\in C_0(\Sigma)\cap Hol(\Sigma^0)$ *and* $n\in\mathbb{N}$ *the function* h_n *be defined in Definition 1. Then*
a) $h_n\in C_0(\Sigma)$,
b) $\|h_n\|=\max\limits_{\sigma\in\Sigma^0}|h_n(s)|=O(4^n),\ n\to\infty.$

Proof. According to Proposition 6 we know only $h_n\in C_b(\Sigma)$. But

$$|h_n(x+iy)|\le\frac{4^n}{\pi}\int_{-\infty}^{\infty}\frac{x}{x^2+(y-t)^2}|g_0(t)|dt=4^nh(x+iy),$$

where $h\in C_0(\Sigma)$ (and is harmonic in Σ^0), such that $h(iy)=|g_0(y)|$ for all $y\in\mathbb{R}$. It follows $h_n\in C_0(\Sigma)$, too.

□

It holds

$$\frac{\partial^n}{\partial y^n}\left(\frac{1}{x^2+(y-t)^2}\right)=\frac{P_n(x,y-t)}{(x^2+(y-t)^2)^{n+1}}$$

two variables $x,\ u$ are defined inductively by

$$P_0(x,u)=1,\ P_{n+1}(x,u)=(-1)^{n+1}[2(n+1)uP_n(x,u)-(x^2+u^2)P_n'(x,u)]$$

(here is means the derivative by u). It is clear that P_n is a homogenous polynomial of degree n. Let's denote

$$P_n(x,u)=\sum_{k=0}^{n}c_{n,k}x^{n-k}u^k.$$

We have the estimate $\sum\limits_{k=0}^{n}|c_{n,k}|\le 4^nn!$ for the height of P_n. Really, for n we obtain by the inductive definition of P_{n+1}

$$\sum_{k=0}^{n+1}|c_{n+1,k}|\le 2(n+1)\sum_{k=0}^{n}|c_{n,k}|+2\sum_{k=0}^{n}|kc_{n,k}|\le 4(n+1)\sum_{k=0}^{n}|c_{n,k}|\le 4^{n+1}(n+1)!.$$

It means

$$x|P_n(x,u)|\le\sum_{k=0}^{n}|c_{n,k}|x^{n+1-k}|u|^k\le 4^nn!(x^2+u^2)^{\frac{n+1}{2}}$$

and
$$\left|\frac{\partial^n}{\partial y^n}\left(\frac{x}{x^2+(y-t)^2}\right)\right| \leq \frac{4^n n!}{(x^2+(y-t)^2)^{\frac{n+1}{2}}} \leq \frac{4^n n!}{(\triangle^2+(y-t)^2)^{\frac{n+1}{2}}}$$
for $x \geq \triangle \geq 0$. The last function is integrable over $\mathbb{R}$ for any $n \in \mathbb{N}$ and $y \in \mathbb{R}$. So
$$g^{(n)}(x+iy) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{xP_n(x,y-t)}{(x^2+(y-t)^2)^{n+1}} g_0(t)dt,$$
$$|h_{(n)}(x+iy)| = \frac{x^n}{n!}|g^{(n)}(x+iy)| = \frac{x}{n!}\left|\int_{-\infty}^{\infty} \frac{x^n P_n(x,y-t)}{(x^2+(y-t)^2)^{n+1}}|g_0(t)|dt\right| \leq$$
$$\leq \frac{x}{n!}\int_{-\infty}^{\infty} \frac{\sum_{k=0}^{n}|c_{n,k}|x^{2n-k}|y-t|^k}{(x^2+(y-t)^2)^{n+1}}|g_0(t)|dt \leq \frac{x}{n!}\int_{-\infty}^{\infty} \frac{(x^2+(y-t)^2)^n \sum_{k=0}^{n}|c_{n,k}|}{(x^2+(y-t)^2)^{n+1}}|g_0(t)|dt \leq$$
$$\leq 4^n \int_{-\infty}^{\infty} \frac{x}{x^2+(y-t)^2}|g_0(t)|dt = 4^n h(x+iy),\ x > 0.$$
It follows $\lim_{s\to 0} h_n(s) = 0,\ s \in \Sigma^0$, and $\|h_n\| = \sup_{s\in\Sigma^0}|h_n(s)| = O(4^n),\ n \to \infty$. It remains to prove h_n is continuous in any point $it_0 \in \partial\Sigma,\ t_0 \in \mathbb{R}$, with respect to Σ. As $g_{t_0} \in C_0(\Sigma) \cap Hol(\Sigma^0)$ for all $t_0 \in \mathbb{R}$, it is enough to prove
$$\lim_{\substack{s\to 0\\ s\in\Sigma^0}} (\operatorname{Re} s)^n g^{(n)}(s) = 0$$
for all $n \in \mathbb{N}$. Define
$$h(s) = g(\frac{1}{s}) - g(0),\ s \in \Sigma \cup \{\infty\}.$$
Then $h \in C_0(\Sigma) \cap Hol(\Sigma^0)$, and so
$$\lim_{z\to\infty} (\operatorname{Re} z)^n h^{(n)}(z) = 0,\ z \in \Sigma,$$
for all $n \in \mathbb{N}$. Specially for $n = 1$ we have
$$0 = \lim_{\substack{z\to\infty\\ z\in\Sigma^0}} \operatorname{Re} z[g(\frac{1}{z})]' = -\lim_{\substack{z\to\infty\\ z\in\Sigma^0}} \frac{(\operatorname{Re} z)}{z^2} g'(\frac{1}{z}) = -\lim_{\substack{s\to 0\\ s\in\Sigma^0}} s^2(\operatorname{Re}\frac{1}{s})g'(s) =$$
$$= -\lim_{\substack{s\to 0\\ s\in\Sigma^0}} \frac{s^2}{|s|^2}\operatorname{Re} s\ g'(s),\ z = \frac{1}{s}.$$
It follows
$$\lim_{\substack{s\to 0\\ s\in\Sigma^0}} \operatorname{Re} s\ g'(s) = 0.$$

Suppose $\lim\limits_{\substack{s\to 0\\ s\in\Sigma^0}} (\operatorname{Re} s)^j [g(s)]^{(j)} = 0,\ s \in \Sigma^0$, for $j = 1, 2, ..., n-1$. Then

$$0 = \lim_{\substack{z\to\infty\\ z\in\Sigma^0}} (\operatorname{Re} z)^n [g(\frac{1}{z})]^{(n)} = \lim_{\substack{z\to\infty\\ z\in\Sigma^0}} (\operatorname{Re} z)^n \left[(-\frac{1}{z^2})^n g^{(n)}(\frac{1}{z}) + \sum_{j=1}^{n-1} A_{jn} z^{-n-j} g^{(j)}(\frac{1}{z})\right] =$$

$$= \lim_{\substack{s\to 0\\ s\in\Sigma^0}} (\operatorname{Re}\frac{1}{s})^n \left[(-1)^n s^{2n} g^{(n)}(s) + \sum_{j=1}^{n-1} A_{jn} s^{n+j} g^{(j)}(s)\right] =$$

$$= (-1)^n \lim_{\substack{s\to 0\\ s\in\Sigma^0}} \frac{s^{2n}}{|s|^{2n}} (\operatorname{Re} s)^n g^{(n)}(s) +$$

$$+ \sum_{j=1}^{n-1} A_{jn} \lim_{\substack{s\to 0\\ s\in\Sigma^0}} \frac{(\operatorname{Re} s)^{n-j} s^{n+j}}{|s|^{2n}} (\operatorname{Re} s)^j g^{(j)}(s) =$$

$$= (-1)^n \lim_{\substack{s\to 0\\ s\in\Sigma^0}} \frac{s^{2n}}{|s|^{2n}} (\operatorname{Re} s)^n g^{(n)}(s),\ z = \frac{1}{s},$$

for some $A_{jn} \in \mathbb{R}$. It follows

$$\lim_{\substack{s\to 0\\ s\in\Sigma^0}} (\operatorname{Re} s)^n g^{(n)}(s) = 0.$$

The proof is finished.

□

Proposition 6 means that the property $h_n \in C_0(\Sigma),\ n \in \mathbb{N}$, of the function $g \in C_0(\Sigma) \cap Hol(\Sigma^0)$

References

[1] K. P. Isaev, R. S. Yulmukhametov, *Laplace transforms of functionals on Bergman spaces*, Izv. RAN. Ser. Mat., 68:1 (2004), 5–42; Izv. Math., **68**:1 (2004), 3–41

[2] A. V. Pavlov, *About the equality of the transform of Laplace to the transform of Fourier*, Probl. Anal. Issues Anal., **5**(23):1 (2016), 21–30.

[3] M. N. Kuznetsova, *Laplace transformation and nonlinear hyperbolic equations*, Ufimsk. Mat. Zh., **1**:3 (2009), 87–96.

[4] V. Z. Meshkov, I. P. Polovinkin, *A Criterion for the Compactness of the Support of a Generalized Function (Distribution) in Terms of the Fourier–Laplace Transform*, Mat. Zametki, **96**:1 (2014), 83–87; Math. Notes, **96**:1 (2014), 95–98.

[5] Nargiza Boboyarova, Jonibek Abdullayev *Algebras of primitive functions*, Asia Pac. J. Math., 2024 **11**:2.

[6] Rudin W., *Real and Complex Analysis*. McGraw-Hill Inc, New York, 1974.

[7] D. Widder, *The Laplace Transform*, Princeton University Press, 1966.

[8] D. Widder, *A Symbolic Form and Inversion Formula for a Laplace Transform*, The American Mathematical Monthly, Vol. **55**, No 8,pp.489-491, 1948.

[9] D. Widder, *A Classification of Generating Functions*, Transactions of the American Mathematical Society, Vol. **39**, No.2, pp. 244-298, 1936.

[10] Koosis P., *Introduction to H^p Spaces.* Cambridge University Press. 1980.

Boboyarova Nargiza Ashurovna
Urgench technological university RANCH, house 26
Khanka street, Urgench city, 220100, Uzbekistan
E-mail: *boboyarova@gmail.com*

Received January 19, 2024

Abdullayev Jonibek Shokirovich
Urgench State University, 14, Kh.Alimdjan str, Urgench
city, 220100, Uzbekistan
E-mail: *jonibekabdullayev1201@gmail.com*