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Some integrals for cosine families of bounded linear operators on some non-Archimedean Banach spaces

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Abstract. In this paper, we define and study the Volkenborn integral and the Shnirelman integral for cosine families of bounded linear operators on some non-Archimedean Banach spaces over $\mathbb{Q}_p$ and $\mathbb{C}_p$ respectively. We give some functional calculus for cosine families of infinitesimal generator A such that A is a nilpotent operator on some non-Archimedean Banach spaces over $\mathbb{C}_p$. Many results are proved and examples are given to support our work.

Mathematics subject classification: 47D03, 47D09, 47S10.
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1 Introduction and preliminaries

Throughout this paper, $\mathbb{K}$ is a complete non-Archimedean valued field with a non-trivial valuation $|\cdot|$, X is a non-Archimedean Banach space over $\mathbb{K}$, $B(X)$ denotes the set of all bounded linear operators on X, I is the unit operator of X and $\mathbb{Q}_p$ is the field of p-adic numbers ($p \geq 2$ being a prime) equipped with a p-adic valuation $|\cdot|_p$ and $\mathbb{Z}_p$ denotes the ring of p-adic integers. We denote the completion of algebraic closure of $\mathbb{Q}_p$ under the p-adic valuation $|\cdot|_p$ by $\mathbb{C}_p$.

The theory of classical C_0-cosine families of bounded linear operators was initiated by M. Sova [10]. From [10, Theorem 2.17], if A is the infinitesimal generator of a C_0-cosine family $(C(s))_{s\in\mathbb{R}^+}$, then A is closed. By Lemma 2.14 of [10], we have

$$\int_0^t (t-s)C(s)xds \in D(A)$$

and

$$A(\int_0^t (t-s)C(s)Axds) = C(t)x - x$$

for all $x \in X$ and $t \in \mathbb{R}^+$. This is thanks to the Haar measure on the topological group $(\mathbb{R}, +)$.

In non-Archimedean operator theory, A. El Amrani, A. Blali, J. Ettayb and M. Babahmed [5] introduced the concept of cosine families of bounded linear operators on non-Archimedean Banach spaces. Recently, J. Ettayb [7] studied the Volkenborn

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integral and the Shnirelman integral for groups of bounded linear operators on finite-dimensional non-Archimedean Banach spaces over $\mathbb{Q}_p$ and $\mathbb{C}_p$ respectively. He gave some functional calculus for groups of infinitesimal generator A such that A is a nilpotent operator on finite-dimensional non-Archimedean Banach spaces. Let $r > 0$ and $\Omega_r = \{t \in \mathbb{K} : |t| < r\}$ be the open ball of $\mathbb{K}$ centred at 0 with radius r [5], we have the following definition.

Definition 1. [5] A function $C : \Omega_r \longrightarrow B(X)$ is called a C_0 or strongly continuous operator cosine function on X if

(i) $C(0) = I$,

(ii) For every $t, s \in \Omega_r$, $C(t+s) + C(t-s) = 2C(t)C(s)$,

(iii) For each $x \in X$, $t \longrightarrow C(t)x$ is continuous on Ω_r.

A cosine family of bounded linear operators $(C(t))_{t\in\Omega_r}$ is uniformly continuous if $\lim\limits_{t\to 0} \|C(t) - I\| = 0$.
The linear operator A defined by

$$D(A) = \{x \in X : \lim_{t\to 0} 2\frac{C(t)x - x}{t^2} \text{ exists}\}$$

and

$$\text{for each } x \in D(A),\ Ax = \lim_{t\to 0} 2\frac{C(t)x - x}{t^2}$$

is called the infinitesimal generator of the cosine family $(C(t))_{t\in\Omega_r}$.

In this paper, we extend the Volkenborn integral and the Shnirelman integral for studying the C_0-cosine families of bounded linear operators on some non-Archimedean Banach spaces and we show some results about it.

In the next definition, the notation $gcd(n,p)$ stands for the greatest common divisor of the integers n and p.

Definition 2. [4] Let $f(z)$ be a $\mathbb{C}_p$-valued function defined for all $z \in \mathbb{C}_p$ such that $|z - a|_p = r$ where $a \in \mathbb{C}_p$ and $r > 0$ with $r \in |\mathbb{C}_p|_p$. Let $\Gamma \in \mathbb{C}_p$ such that $|\Gamma|_p = r$. Then the Shnirelman integral of f is defined as the following limit, if it exists,

$$\int_{a,\Gamma} f(z)dz = \lim_{n\to\infty}{}' \frac{1}{n} \sum_{\zeta^n=1} f(a + \zeta\Gamma),$$

where $\lim'$ indicates that the limit is taken over n such that $gcd(n,p) = 1$.

Theorem 1. *[1] Let $f(z) = \sum\limits_{n\in\mathbb{N}} a_n f_n(z)$ where the series on the right converges uniformly to $f(z)$ for all points $z \in \mathbb{C}_p$ such that $|z-a|_p = |\gamma|_p$. Suppose that for all $n \in \mathbb{N}$, $\int_{a,\gamma} f_n(z)dz$ exists. Then $\int_{a,\gamma} f(z)dz$ exists and $\int_{a,\gamma} f(z)dz = \sum\limits_{n\in\mathbb{N}} a_n \int_{a,\gamma} f_n(z)dz$.*

Lemma 1. *[1] Let p be any integer such that $0 <\mid p \mid< n$. Then*

$$\sum_{i=1}^{n} \xi_i^{(n)p} = 0.$$

Now, let $f(z) = \sum_{k=0}^{\infty} a_k z^k$ be a power series converging for all $z \in \mathbb{C}_p$ such that $|z|_p < R \left(R > 0\right)$, we have the following:

Theorem 2. *[1] If $|a|_p < R$ and $|\gamma|_p < R$, then*

$$\int_{a,\gamma} f(z)dz = f(a).$$

Corollary 1. *[1] With the same hypothesis as in Theorem 2, then*

$$\int_{a,\gamma} (z-a)f(z)dz = 0.$$

Theorem 3. *[1] Let $f(z) = \sum_{k=0}^{\infty} a_k z^k$ be a power series converging for all $z \in \mathbb{C}_p$ such that $|z|_p < R$ $(R > 0)$. Suppose that $x, r \in \mathbb{C}_p$ such that $|x|_p, |r|_p < R$. Then,*

$$\int_{0,r} \frac{zf(z)}{z-x} dz = \begin{cases} f(x) & \text{if } |x|_p < |r|_p, \\ 0 & \text{if } |x|_p > |r|_p. \end{cases}$$

Theorem 4. *[1] With the same hypothesis as in Theorem 3, we have:*

$$\int_{0,r} \frac{zf(z)}{(z-x)^{n+1}} dz = \frac{f^n(x)}{n!} \quad \text{for} \quad |x|_p < |r|_p.$$

Theorem 5. *[9] Additive, translation invariant and bounded $\mathbb{Q}_p$-valued measure on clopens of $\mathbb{Z}_p$ is the zero measure.*

We denote $C(\mathbb{Z}_p, \mathbb{Q}_p)$ the space of all functions defined and continuous from $\mathbb{Z}_p$ into $\mathbb{Q}_p$.

Theorem 6. *[9] Let $f \in C(\mathbb{Z}_p, \mathbb{Q}_p)$. The function defined on $\mathbb{N}$ by*

$$F(0) = 0,\ F(n) = f(0) + f(1) + \cdots + f(n-1),$$

is uniformly continuous. The extended function is denoted by $Sf(x)$ (called indefinite sum of f). If f is strictly differentiable, so is Sf.

We denote $C_s^1(\mathbb{Z}_p, \mathbb{Q}_p)$ the space of all functions defined and strictly differentiable in $\mathbb{Z}_p$ taking values in $\mathbb{Q}_p$. For more details, we refer to [9].

Definition 3. [9] The Volkenborn integral of $f \in C^1_s(\mathbb{Z}_p, \mathbb{Q}_p)$ is defined by

$$\int_{\mathbb{Z}_p} f(t)dt = \lim_{n\to\infty} p^{-n} \sum_{j=0}^{p^n-1} f(j) = \lim_{n\to\infty} \frac{Sf(p^n) - Sf(0)}{p^n} = (Sf)'(0).$$

Lemma 2. *[9] For all $t \in \Omega^*_{p^{\frac{-1}{p-1}}}$,*

$$\int_{\mathbb{Z}_p} e^{tu} du = \frac{t}{e^t - 1}.$$

From Lemma 2 and setting for all $t \in \Omega_{p^{\frac{-1}{p-1}}}$, $cosh(t) = \frac{e^t+e^{-t}}{2}$, we have the following lemma.

Lemma 3. *For all $t \in \Omega^*_{p^{\frac{-1}{p-1}}}$,*

$$\int_{\mathbb{Z}_p} cosh(tu)du = \frac{1}{2}\left(\frac{t}{e^t - 1} - \frac{t}{e^{-t} - 1}\right).$$

Recall that $c_0(\mathbb{K})$ is the set of all sequences $(x_i)_{i\in\mathbb{N}}$ in $\mathbb{K}$ such that $\lim_{i\to\infty} x_i = 0$. Moreover, the space $c_0(\mathbb{K})$ equipped with the norm $\|(x_i)_{i\in\mathbb{N}}\| = \sup_{i\in\mathbb{N}} |x_i|$ is a non-Archimedean Banach space. For more details on non-Archimedean Banach spaces and free Banach spaces, we refer to [4].

2 Integral for C_0-cosine families on some non-Archimedean Banach spaces over $\mathbb{C}_p$

We have the following definition.

Definition 4. Let X be a non-Archimedean Banach space over $\mathbb{C}_p$ and let $r > 0$ be a real number. A one-parameter family $(C(t))_{t\in\Omega_r}$ of bounded linear operators on X is said to be an analytic cosine family on X if

(i) $C(0) = I$.

(ii) For all $t, s \in \Omega_r$, $C(t + s) + C(t - s) = 2C(t)C(s)$.

(iii) For all $x \in X$, $t \to C(t)x$ is analytic on Ω_r.

In the next definition, the notation $gcd(n, p)$ stands for the greatest common divisor of the integers n and p. From Definition 4.10 of [4], we obtain:

Definition 5. Let $(C(t))_{t\in\Omega_r}$ be an analytic cosine family of bounded linear operators on $c_0(\mathbb{C}_p)$. The cosine family $(C(t))_{t\in\Omega_r}$ is said to be integrable in the sense of Schnirelman if for all $a \in \Omega_r$ and $\gamma \in \Omega_r\backslash\{0\}$, the sequence $(S_n)_n \subset B(c_0(\mathbb{C}_p))$ defined by

$$S_n = \frac{1}{n} \sum_{\zeta^n=1} C(a + \zeta\gamma),$$

converges strongly as $n \to \infty$ $\Big($the limit is taken over n such that $gcd(n,p)=1\Big)$ to a bounded linear operator. More precisely

$$\int_{a,\gamma} C(t)dt = \lim_{n\to\infty}{}' \frac{1}{n} \sum_{\zeta^n=1} C(a+\zeta\gamma),$$

where $\lim'$ indicates that the limit is taken over n such that $gcd(n,p)=1$.

Lemma 4. *Let $(C(t))_{t\in\Omega_r}$ be an analytic cosine family on $c_0(\mathbb{C}_p)$ such that $\int_{a,\gamma} C(t)dt$ exists and $\sup_{t\in\Omega_r}\|C(t)\|\leq M$ where $a\in\Omega_r$ and $\gamma\in\Omega_r\backslash\{0\}$. Then*

(i) For all $x\in c_0(\mathbb{C}_p)$, $\left\|\int_{a,\gamma} C(t)xdt\right\| \leq M\|x\|$.

(ii) For all $a\in\Omega_r$ and $x\in c_0(\mathbb{C}_p)$, $\int_{a,\gamma} C(t)xdt = C(a)xdt$.

Proof.

(i) It suffices to apply Definition 5.

(ii) By Definition 5, $(C(t))_{t\in\Omega_r}$ is analytic and Theorem 2. □

Definition 6. Let $A\in B(c_0(\mathbb{C}_p))$. A is said to be nilpotent of index d if there is an integer number $d\leq n$ such that $A^d = 0_{c_0(\mathbb{C}_p)}$ and $A^{d-1}\neq 0_{c_0(\mathbb{C}_p)}$ $\Big($where $0_{c_0(\mathbb{C}_p)}$ denotes the null operator from $c_0(\mathbb{C}_p)$ into $c_0(\mathbb{C}_p)\Big)$.

Example 1. Let $A\in B(c_0(\mathbb{C}_p))$ be defined by

$$Ae_1 = e_2, \quad Ae_2 = e_3 \text{ and } Ae_i = 0 \quad \text{for all } i\geq 3.$$

Then $A\neq 0$ and $A^3=0$. Consequently A is nilpotent of index 3.

Proposition 1. *Let A be a nilpotent operator of index n on $c_0(\mathbb{C}_p)$ such that $\|A\| < p^{\frac{-1}{p-1}}$. Then $C(t)=\sum_{k=0}^{n-1}\frac{t^{2k}A^k}{(2k)!}$ is an analytic cosine family on $c_0(\mathbb{C}_p)$.*

Proof. Since A is a nilpotent operator of index n on $c_0(\mathbb{C}_p)$. Set $C(t)=\sum_{k=0}^{n-1}\frac{t^{2k}A^k}{(2k)!}$, then $(C(t))_t$ is analytic on $c_0(\mathbb{C}_p)$, since for each $k\in\{1,\cdots,n\}$, $\frac{t^{2k}}{(2k)!}$ is analytic. □

Theorem 7. *Let A be a nilpotent operator of index n on $c_0(\mathbb{C}_p)$. Set $C(t)=\sum_{k=0}^{n-1}\frac{t^{2k}A^k}{(2k)!}$. Then for all $x\in c_0(\mathbb{C}_p)$, $\int_{a,\gamma} C(t)xdt = C(a)x$.*

Proof. Let $C(t) = \sum_{k=0}^{n-1} \frac{t^{2k}A^k}{(2k)!}$. Using Proposition 1 and Theorem 2, we have for all $x \in c_0(\mathbb{C}_p)$,

$$\begin{aligned}\int_{a,\gamma} C(t)xdt &= \sum_{k=0}^{n-1} \frac{A^k}{(2k)!} \int_{a,\gamma} t^{2k} xdt \\ &= \sum_{k=0}^{n-1} \frac{a^{2k}A^k}{(2k)!} x = C(a)x.\end{aligned}$$

□

Corollary 2. *Under the hypothesis of Theorem 7, then for all $x \in c_0(\mathbb{C}_p)$,*

$$\int_{a,\gamma} (t-a)C(t)xdt = 0.$$

Remark 1. Let $A \in B(c_0(\mathbb{C}_p))$ be a nilpotent operator, then e^{tA} is integrable in the sense of Shnirelman.

Set for all $\lambda \in \rho(A)$, $R(\lambda, A) = (\lambda I - A)^{-1}$ where $\rho(A)$ is the resolvent set of the linear operator A defined on $c_0(\mathbb{C}_p)$, we have the following:

Proposition 2. *Let $A \in B(c_0(\mathbb{C}_p))$. If A is a nilpotent operator of index n, then for all $\lambda \in \mathbb{C}_p^*$, $R(\lambda, A)$ exists. Furthermore, for each $\lambda \in \mathbb{C}_p^*$, we have*

$$R(\lambda, A) = \sum_{k=0}^{n-1} \frac{A^k}{\lambda^{k+1}}.$$

Proof. The proof is similar to the proof of Proposition 2 of [7]. □

We have the following proposition.

Proposition 3. *Let A be a nilpotent operator of index n on $c_0(\mathbb{C}_p)$ and $r = \frac{-1}{p-1}$. Then*

$$\text{for all } t \in \Omega_r,\ e^{tA} = \int_{0,\gamma} \lambda e^{\lambda t} R(\lambda, A) d\lambda \text{ where } \gamma \in \Omega_r \backslash \{0\}.$$

Proof. The proof is similar to the proof of Proposition 3 of [7]. □

By Proposition 3 and setting $C_1(t) = \frac{e^{tA}+e^{-tA}}{2}$, we have the following:

Proposition 4. *Let A be a nilpotent operator on $c_0(\mathbb{C}_p)$ and $r = \frac{-1}{p-1}$. Then*

$$\text{for all } t \in \Omega_r, C_1(t) = \int_{0,\gamma} \lambda cosh(\lambda t) R(\lambda, A) d\lambda \text{ where } \gamma \in \Omega_r \backslash \{0\}.$$

Proof. By Proposition 3, we have $e^{tA} = \int_{0,\gamma} \lambda e^{\lambda t} R(\lambda, A) d\lambda$ and $e^{-tA} = \int_{0,\gamma} \lambda e^{-\lambda t} R(\lambda, A) d\lambda$. Since $cosh(\lambda t) = \frac{e^{t\lambda}+e^{-t\lambda}}{2}$ and $C_1(t) = \frac{e^{tA}+e^{-tA}}{2}$, we have

$$\begin{aligned} C_1(t) &= \frac{e^{tA}+e^{-tA}}{2} \\ &= \frac{1}{2}\left(\int_{0,\gamma} \lambda e^{\lambda t} R(\lambda, A) d\lambda + \int_{0,\gamma} \lambda e^{-\lambda t} R(\lambda, A) d\lambda\right) \\ &= \int_{0,\gamma} \lambda cosh(\lambda t) R(\lambda, A) d\lambda. \end{aligned}$$

□

Proposition 5. *Let A and B be two nilpotent operators on $c_0(\mathbb{C}_p)$ and let $C_1(t)$ and $C_2(t)$ be two C_0-cosine families of infinitesimal generators A and B respectively. If $R(\lambda, A)$ and $R(\lambda, B)$ commute, then $C_1(t)$ and $C_2(t)$ commute.*

Proof. By Proposition 4, we have $C_1(t) = \int_{0,\gamma} \lambda cosh(\lambda t) R(\lambda, A) d\lambda$ and $C_2(t) = \int_{0,\gamma} \lambda cosh(\lambda t) R(\lambda, B) d\lambda$. Assume that $R(\lambda, A)$ and $R(\lambda, B)$ commute, then

$$\begin{aligned} C_1(t)C_2(t) &= \int_{0,\gamma} \lambda cosh(\lambda t) R(\lambda, A) d\lambda \int_{0,\gamma} \lambda cosh(\lambda t) R(\lambda, B) d\lambda \\ &= \int_{0,\gamma}\int_{0,\gamma} \lambda cosh(\lambda t) R(\lambda, A) \lambda cosh(\lambda t) R(\lambda, B) d\lambda d\lambda \\ &= \int_{0,\gamma}\int_{0,\gamma} \lambda cosh(\lambda t) R(\lambda, B) \lambda cosh(\lambda t) R(\lambda, A) d\lambda d\lambda \\ &= C_2(t)C_1(t). \end{aligned}$$

□

We have the following:

Proposition 6. *Let A and $(A_k)_{k\in\mathbb{N}}$ be two nilpotent operators on $c_0(\mathbb{C}_p)$. If $R(\lambda, A_k) \to R(\lambda, A)$ as $k \to \infty$, then e^{tA_k} converges to e^{tA} as $k \to \infty$.*

Proof. The proof is similar to the proof of Proposition 5 of [7]. □

One can see the following proposition.

Proposition 7. *Let A and $(A_k)_{k\in\mathbb{N}}$ be two nilpotent operators on $c_0(\mathbb{C}_p)$. Set for all $k \in \mathbb{N}$, $C_k(t) = \frac{e^{tA_k}+e^{-tA_k}}{2}$ and $C(t) = \frac{e^{tA}+e^{-tA}}{2}$. If $R(\lambda, A_k) \to R(\lambda, A)$ as $k \to \infty$, then $C_k(t)$ converges to $C(t)$ as $k \to \infty$.*

Remark 2. The results mentioned above remain valid for any free Banach space over $\mathbb{C}_p$.

3 Integral of cosine families of linear operators on free Banach spaces over $\mathbb{Q}_p$

In this section, we assume that $\mathbb{K} = \mathbb{Q}_p$ and X is a free Banach space over $\mathbb{Q}_p$, we have the following definition:

Definition 7. Let $f \in C_s^1(\mathbb{Z}_p, X)$. The sequence $(S_m)_m \subset B(X)$ defined by

$$S_m = p^{-m} \sum_{j=0}^{p^m-1} f(j),$$

converges strongly as $m \to \infty$ to a bounded linear operator. More precisely

$$\int_{\mathbb{Z}_p} f(t)dt = \lim_{m\to\infty} p^{-m} \sum_{j=0}^{p^m-1} f(j).$$

Set $B_r(X) = \{A \in B(X) : 0 < \|A\| < r\}$ where $r = p^{\frac{-1}{p-1}}$. We have the following results.

Proposition 8. *Let $A \in B_r(X)$ be an invertible diagonal operator, then $(e^{tA})_{t\in\mathbb{Z}_p}$ is a C^1-function and $(e^A - I)^{-1} \in B(X)$.*

Proof. The proof is similar to the proof of Proposition 6 of [7]. □

Proposition 9. *Let $A \in B_r(X)$ be an invertible diagonal operator such that $\int_{\mathbb{Z}_p} e^{tA}dt$ exists. Then for all $x \in X$, $(e^A - I)\int_{\mathbb{Z}_p} e^{tA}xdt = Ax$.*

Proof. The proof is similar to the proof of Proposition 7 of [7]. □

Set $C_1(t) = \frac{e^{tA}+e^{-tA}}{2}$, we have the following proposition.

Proposition 10. *Let $A \in B_r(X)$ be an invertible diagonal operator such that $\int_{\mathbb{Z}_p} e^{tA}dt$ exists. Then for all $x \in X$, $\int_{\mathbb{Z}_p} C_1(t)xdt = \frac{A}{2}\left((e^A - I)^{-1} - (e^{-A} - I)^{-1}\right)x$.*

Proof. By Proposition 9, for all $x \in X$, $\int_{\mathbb{Z}_p} e^{tA}xdt = A(e^A - I)^{-1}x$ and $\int_{\mathbb{Z}_p} e^{-tA}xdt = -A(e^{-A} - I)^{-1}x$. Then

$$\begin{aligned}\int_{\mathbb{Z}_p} C_1(t)xdt &= \frac{1}{2}\left(\int_{\mathbb{Z}_p} e^{tA}xdt + \int_{\mathbb{Z}_p} e^{-tA}xdt\right)\\ &= \frac{A}{2}\left((e^A - I)^{-1} - (e^{-A} - I)^{-1}\right)x.\end{aligned}$$

□

Example 2. Let $r = \frac{-1}{p-1}$, $(\lambda_i)_{i\in\mathbb{N}} \in \Omega_r^*$ and let $A \in B(c_0(\mathbb{Q}_p))$ be defined by

$$\text{for all } i \in \mathbb{N},\ Ae_i = \lambda_i e_i$$

where $(e_i)_{i\in\mathbb{N}}$ is the canonical basis of $c_0(\mathbb{Q}_p)$. Then for all $t \in \mathbb{Z}_p$, we have

$$\text{for all } i \in \mathbb{N},\ e^{tA}e_i = e^{\lambda_i t}e_i.$$

Hence for all $i \in \mathbb{N}$,

$$\int_{\mathbb{Z}_p} e^{tA}e_i dt = \int_{\mathbb{Z}_p} e^{\lambda_i t}e_i dt.$$

Thus for all $x = \sum_{i\in\mathbb{N}} x_i e_i \in X$, we have

$$\begin{aligned}\int_{\mathbb{Z}_p} e^{tA}x dt &= \sum_{i\in\mathbb{N}} \frac{\lambda_i}{e^{\lambda_i}-1}x_i e_i \\ &= (e^A - I)^{-1}Ax.\end{aligned}$$

Set $C_1(t) = \frac{e^{tA}+e^{-tA}}{2}$ and $cosh(at) = \frac{e^{ta}+e^{-at}}{2}$. From Example 2, we have the following example.

Example 3. Let $r = \frac{-1}{p-1}$, $(\lambda_i)_{i\in\mathbb{N}} \in \Omega_r^*$ and let $A \in B(c_0(\mathbb{Q}_p))$ be defined by

$$\text{for all } i \in \mathbb{N},\ Ae_i = \lambda_i e_i$$

where $(e_i)_{i\in\mathbb{N}}$ is the canonical basis of $c_0(\mathbb{Q}_p)$. Then for all $t \in \mathbb{Z}_p$, we have

$$\text{for all } i \in \mathbb{N},\ C_1(t)e_i = cosh(\lambda_i t)e_i.$$

Hence for all $i \in \mathbb{N}$,

$$\int_{\mathbb{Z}_p} C_1(t)e_i dt = \int_{\mathbb{Z}_p} cosh(\lambda_i t)e_i dt.$$

Then for all $i \in \mathbb{N}$,

$$\int_{\mathbb{Z}_p} C_1(t)e_i dt = \left(\frac{\lambda_i}{2(e^{\lambda_i}-1)} - \frac{\lambda_i}{2(e^{-\lambda_i}-1)}\right)e_i.$$

Thus for all $x = \sum_{i\in\mathbb{N}} x_i e_i \in X$, we have

$$\int_{\mathbb{Z}_p} C_1(t)x dt = \frac{A}{2}\left((e^A - I)^{-1} - (e^{-A} - I)^{-1}\right)x.$$

Definition 8. Let $A \in B(X)$. A is said to be a scalar multiple of identity operator on X if $A = aI$ for some $a \in \mathbb{Q}_p$ and I is the identity operator on X.

Example 4. Let A be an invertible scalar multiple of identity operator on X such that $A = aI$ where $a \in \Omega_r^*$ with $r = \frac{-1}{p-1}$. Hence for all $t \in \mathbb{Z}_p$, $C(t) = cosh(ta)I$, then for all $x \in X$ and $a \in \Omega_r^*$, we have

$$\int_{\mathbb{Z}_p} C(t)x dt = \left(\frac{a}{e^a - 1} - \frac{a}{e^{-a}-1}\right)x = \frac{A}{2}\left((e^A - I)^{-1} - (e^{-A} - I)^{-1}\right)x.$$

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