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Reverse Hardy inequalities via μ-proportional generalized fractional integral operators

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Abstract. In this paper, we present a further improvement of the reverse Hardy type inequality via ${}_{a^+}\mathfrak{I}_{\mu}^{\Phi}$ and ${}_{b^-}\mathfrak{I}_{\mu}^{\Phi}$, the proportional generalized fractional integral operators with respect to another strictly increasing continuous function μ. We obtain a new result by using two parameters of integrability p and q, some special cases are mentioned according to the choice of the function Φ.

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1 Introduction

The classical Hardy inequality [7, 8], has attracted the attention of many mathematicians and has been the subject of various extensions and refinements.

If φ is a measurable function with non-negative values, $p > 1$, then

$$\int_0^\infty \left(\frac{1}{\tau}\int_0^\tau \varphi(t)dt\right)^p d\tau \leq \left(\frac{p}{p-1}\right)^p \int_0^\infty \varphi^p(\tau)d\tau.$$

If the right-hand side is finite, equality holds if and only if $\varphi(\tau) = 0$ almost everywhere.

Later, Littlewood and Hardy [9] established the reverse of above inequality in this way.

Let $\varphi : [0,\infty) \longrightarrow [0,\infty)$. If $0 < p < 1$, then

$$\int_0^\infty \frac{1}{\tau^p}\left(\int_\tau^\infty \varphi(t)dt\right)^p d\tau \geq \left(\frac{p}{p-1}\right)^p \int_0^\infty \varphi^p(\tau)d\tau.$$

This reverse inequality has drawn attention in recent years, either for time scales or on $\mathbb{R}$. Fractional calculus is one of the almost powerful branches of mathematics, it has become significant because of its important use in various fields such as physics, engineering, computing, etc. Much distinct integral fractional operators have been established in this area by dealing with integral inequalities, such as the Riemann–Liouville, Hadamard, Katugampola and particularly the proportional fractional integral which was introduced in the context of generalized fractional operators in relation to another function (for more details see [15, 16]).

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The authors in [1,2] mentioned the following definition of the left– and right-sided μ-Riemann-Liouville fractional integrals of a function φ, respectively.

For an integrable function φ on the interval $[\theta_1, \theta_2]$ and for an increasing function μ, where $\mu \in \mathcal{C}^1[\theta_1, \theta_2]$ such that $\mu'(t) \neq 0$, for all $t \in [\theta_1, \theta_2]$.

Definition 1. The left– and right-sided Riemann–Liouville fractional integrals of a function φ with respect to the function μ on $[\theta_1, \theta_2]$ are defined, respectively, as follow

$$I^{\beta,\mu}_{\theta_1^+}\varphi(\tau) = \frac{1}{\Gamma(\beta)} \int_{\theta_1}^{\tau} \mu'(s) \left(\mu(\tau) - \mu(s)\right)^{\beta-1} \varphi(s)ds, \quad \tau > \theta_1,$$

$$I^{\beta,\mu}_{\theta_2^-}\varphi(\tau) = \frac{1}{\Gamma(\beta)} \int_{\tau}^{\theta_2} \mu'(s) \left(\mu(s) - \mu(\tau)\right)^{\beta-1} \varphi(s)ds, \quad \tau < \theta_2.$$

In [3], the author presented a generalization of the reverse Hardy's inequality. Let φ, g be positive functions defined on $[\theta_1, \theta_2]$ and
$F(\tau) = \int_{\theta_1}^{\tau} \varphi(t)dt$. If g is non-decreasing then
(i) for $p \geq 1$,

$$p \int_{\theta_1}^{\theta_2} \frac{F(\tau)}{g(\tau)} d\tau \leq (\theta_2 - \theta_1)^p \int_{\theta_1}^{\theta_2} \frac{\varphi^p(\tau)}{g(\tau)} d\tau - \int_{\theta_1}^{\theta_2} \frac{(\tau - \theta_1)^p}{g(\tau)} \varphi^p(\tau)d\tau,$$

(ii) for $0 < p < 1$,

$$p \int_{\theta_1}^{\theta_2} \frac{F^p(\tau)}{g(\tau)} d\tau \geq \frac{(\theta_2 - \theta_1)^p}{g(\theta_2)} \int_{\theta_1}^{\theta_2} \varphi^p(\tau)d\tau - \frac{1}{g(\theta_2)} \int_{\theta_1}^{\theta_2} (\tau - \theta_1)^p \varphi^p(\tau)d\tau.$$

Moreover, a new version of the reverse Hardy's inequality with two parameters has been presented on time scales in [5]. Motivated by the above literature, in this article we formulate and prove several inverse Hardy-type results by using an analogue of the fractional integration operator given in [6]. Furthermore, we establish new versions of the Hardy-type inverse inequality in fractional calculus by employing the μ-proportional generalized fractional integral operators involving two parameters p, q.

2 μ-proportional generalized fractional integral operators

In this section, we present a definition of the μ-proportional generalized fractional integral of a function with respect to the function μ. Let $0 < \theta_1 < \tau < \theta_2 < +\infty$.

Definition 2. Let $\alpha \geq 0$, $\mu \in \mathcal{C}^1[\theta_1, \theta_2]$ be strictly increasing and positive function, such that $\mu'(s) \neq 0$, for all $s \in [\theta_1, \theta_2]$. The left– and right–sided μ-generalized fractional integral's of a function φ with respect to the function μ on $[\theta_1, \theta_2]$ are defined, respectively, as follows:

$${}_{\theta_1^+}\mathfrak{I}^{\Phi}_{\mu}\varphi(\tau) = \int_{\theta_1}^{\tau} \mu'(s)\Phi^{\alpha}(\mu(\tau) - \mu(s))\varphi(s)ds, \quad \tau > \theta_1,$$

$$_{\theta_2^-}\mathfrak{I}_\mu^\Phi \varphi(\tau) = \int_\tau^{\theta_2} \mu'(s)\Phi^\alpha(\mu(s)-\mu(\tau))\varphi(s)ds, \quad \tau < \theta_2,$$

where $\Phi : (0,\infty) \to (0,\infty)$ is an increasing function satisfying the following condition.

$$\exists\, c, \eta > 0, \ \int_0^z \Phi^\alpha(\tau + k)d\tau = c\,\Phi^{\alpha+\eta}(z+k), \ \text{ for } k \geq 0. \tag{1}$$

When $\varphi(s) = 1$, we denote

$$_{\theta_1^+}\mathfrak{I}_\mu^\Phi \mathbf{1}(\tau) = \int_{\theta_1}^{\tau} \mu'(s)\Phi^\alpha(\mu(\tau)-\mu(s))ds, \quad \tau > \theta_1,$$

and

$$_{\theta_2^-}\mathfrak{I}_\mu^\Phi \mathbf{1}(\tau) = \int_\tau^{\theta_2} \mu'(s)\Phi^\alpha(\mu(s)-\mu(\tau))ds \quad \tau < \theta_2.$$

The most important characteristic of μ-proportional generalized fractional integrals $_{\theta_1^+}\mathfrak{I}_\mu^\Phi$ and $_{\theta_2^-}\mathfrak{I}_\mu^\Phi$ is that they give, depending on the choice of the function φ, certain types of μ–fractional integrals cited below.

1. Setting $\Phi(\tau) = \tau$ and $\alpha = 0$, then μ-proportional generalized fractional integrals $_{\theta_1^+}\mathfrak{I}_\mu^\Phi$ and $_{\theta_2^-}\mathfrak{I}_\mu^\Phi$ are identical with the μ–Riemann integral operator:

$$I_{\theta_1^+}\varphi(\tau) = \int_{\theta_1}^{\tau} \varphi(s)d\mu(s), \quad \tau > \theta_1,$$

$$I_{\theta_2^-}\varphi(\tau) = \int_{\tau}^{\theta_2} \varphi(s)d\mu(s), \quad \tau < \theta_2.$$

2. Setting $\Phi^\alpha(\tau) = \dfrac{\tau^{\frac{\alpha}{k}-1}}{k\Gamma_k(\alpha)}$, then μ-proportional generalized fractional integrals $_{\theta_1^+}\mathfrak{I}_\mu^\Phi$ and $_{\theta_2^-}\mathfrak{I}_\mu^\Phi$ are identical with the μ-fractional integral of the k-Riemann-Liouville operator of order $\alpha > 0$.

$$I_{\theta_1^+,k}^{\alpha,\mu}\varphi(\tau) = \frac{1}{k\Gamma_k(\alpha)}\int_{\theta_1}^{\tau} (\mu(\tau)-\mu(s))^{\frac{\alpha}{k}-1}\varphi(s)d\mu(s), \quad \tau > \theta_1,$$

$$I_{\theta_2^-,k}^{\alpha,\mu}\varphi(\tau) = \frac{1}{k\Gamma_k(\alpha)}\int_{\tau}^{\theta_2} (\mu(s)-\mu(\tau))^{\frac{\alpha}{k}-1}\varphi(s)d\mu(s), \quad \tau < \theta_2.$$

3. Setting $\Phi^\alpha(\tau) = \dfrac{1}{k\Gamma_k(\alpha)}\dfrac{\left(\ln\frac{\mu(\tau)}{\mu(\tau)-\tau}\right)^{1-\frac{\alpha}{k}}}{\mu(\tau)-\tau}$, then μ-generalized fractional integrals $_{\theta_1^+}\mathfrak{I}_\mu^\Phi$ and $_{\theta_2^-}\mathfrak{I}_\mu^\Phi$ are identical with the μ-fractional integral of the k-Hadamard operator of order $\alpha > 0$.

$$I_{\theta_1^+,k}^{\alpha}\varphi(\tau) = \frac{1}{k\Gamma_k(\alpha)}\int_{\theta_1}^{\tau} \left(\ln\frac{\mu(\tau)}{\mu(s)}\right)^{\frac{\alpha}{k}-1}\varphi(s)\frac{d\mu(s)}{s}, \quad \tau > \theta_1,$$

$$I_{\theta_2^-,k}^{\alpha}\varphi(\tau) = \frac{1}{k\Gamma_k(\alpha)}\int_{\tau}^{\theta_2} \left(\ln\frac{\mu(s)}{\mu(\tau)}\right)^{\frac{\alpha}{k}-1}\varphi(s)\frac{d\mu(s)}{s}, \quad \tau < \theta_2.$$

4. Setting
$$\Phi^{\alpha}(\tau)=\frac{(\rho+1)^{1-\alpha}}{\Gamma(\alpha)}\frac{(\mu(\tau)-\tau)^{\rho}}{(\mu(\tau)^{\rho+1}-(\mu(\tau)-\tau)^{\rho+1})^{1-\alpha}}$$
and $\alpha>0$, then μ–generalized fractional integrals ${}_{\theta_1^+}\mathfrak{I}_{\mu}^{\Phi}$ and ${}_{\theta_2^-}\mathfrak{I}_{\mu}^{\Phi}$ are identical with the μ-fractional integrals of the Katugampola operator:
$$I^{\alpha}_{\theta_1^+,\rho}\varphi(\tau)=\frac{(\rho+1)^{1-\alpha}}{\Gamma(\alpha)}\int_{\theta_1}^{\tau}\frac{\mu^{\rho}(s)\varphi(s)}{[\mu^{\rho+1}(\tau)-\mu(s)^{\rho+1}]^{1-\alpha}}d\mu(s),\tau>\theta_1,$$
$$I^{\alpha}_{\theta_2^-,\rho}\varphi(\tau)=\frac{(\rho+1)^{1-\alpha}}{\Gamma(\alpha)}\int_{\tau}^{\theta_2}\frac{\mu^{\rho}(s)\varphi(s)}{[\mu^{\rho+1}(s)-\mu^{\rho+1}(\tau)]^{1-\alpha}}d\mu(s),\tau<\theta_2.$$
5. Setting $\Phi^{\alpha}(\tau)=(\mu(\tau)-\tau)^{\alpha-1}$ and $\alpha\in(0,1)$, then μ-generalized fractional integrals ${}_{\theta_1^+}\mathfrak{I}_{\mu}^{\Phi}$ is identical with the μ–fractional conformal integrals operator:
$$I^{\alpha}_{\theta_1^+}\varphi(\tau)=\int_{\theta_1}^{\tau}\mu^{\alpha-1}(s)\varphi(s)d\mu(s),\ \tau>\theta_1.$$

Remark 1. Choose $\mu(s)=s$, we get respectively the Riemann integral operator, $k-$Riemann-Liouville integral operator, k-Hadamard integral operator, Katugampola integral operator and fractional conformal integral operator.

We need the following Lemma and Proposition to prove our results.

Lemma 1. *([4]) Let $1<p\le q<\infty$ and φ, w be non-negative measurable functions on $[\theta_1,\,\theta_2]$. We suppose that, $0<\int_{\theta_1}^{\theta_2}\varphi^r(s)w(s)ds<\infty$, for $r>1$, then*
$$\int_{\theta_1}^{\theta_2}\varphi^{p}(s)w(s)ds\le\left(\int_{\theta_1}^{\theta_2}w(s)ds\right)^{\frac{q-p}{q}}\left(\int_{\theta_1}^{\theta_2}\varphi^{q}(s)w(s)ds\right)^{\frac{p}{q}}. \tag{2}$$

Proof. If $p=q$, then we get equality and for $p\neq q$, we use Hölder's integral inequality with $\frac{q}{p}>1$. We have
$$\begin{aligned}\int_{\theta_1}^{\theta_2}\varphi^{p}(s)w(s)ds &=\int_{\theta_1}^{\theta_2}\left(w^{\frac{q-p}{q}}(s)\right)\left(\varphi^{p}(s)w^{\frac{p}{q}}(s)\right)ds\\ &\le\left(\int_{\theta_1}^{\theta_2}w(s)ds\right)^{\frac{q-p}{q}}\left(\int_{\theta_1}^{\theta_2}\varphi^{q}(s)w(s)ds\right)^{\frac{p}{q}}.\end{aligned}$$
□

Proposition 1. *([5]) Let $0<B<A$ be two positive real values, then*
$$\textit{for}\quad p\ge1:\qquad A-B\le(A^p-B^p)^{\frac{1}{p}}, \tag{3}$$
$$\textit{for}\quad 0<p<1:\qquad A-B\ge(A^p-B^p)^{\frac{1}{p}}.$$

3 Inverse Hardy type inequalities for μ-proportional generalized fractional integral operators

Let $0 \leq \theta_1 < \theta_2 < +\infty$.

Theorem 1. *Let $1 \leq p \leq q < \infty$, $\alpha \geq 0$ and $\varphi > 0, g > 0$ on $[\theta_1, \theta_2]$, such that g is non-decreasing. If $\mu'(\tau) \geq 1$, then $\exists\, c, \eta_0, \eta_1 > 0$, the inequality*

$$\int_{\theta_1}^{\theta_2} \frac{\left[{}_{\theta_1^+}\mathfrak{I}_\mu^\Phi \varphi(\tau)\right]^p}{g(\tau)} d\tau \leq c \left({}_{\theta_1^+}\mathfrak{I}_\mu^\Phi \frac{1}{g(\theta_2)}\right)^{\frac{q-p}{q}} \tag{4}$$

$$\times \left\{ \Phi^{\frac{q}{p}((\alpha+\eta_0)(p-1)+\eta_1)}(\mu(\theta_2) - \mu(\theta_1))\, {}_{\theta_1^+}\mathfrak{I}_\mu^\Phi \left[\frac{\varphi^q(\theta_2)}{g(\theta_2)}\right] \right.$$

$$\left. - {}_{\theta_1^+}\mathfrak{I}_\mu^\Phi \left[\frac{\varphi^q(\theta_2)}{g(\theta_2)} \Phi^{\frac{q}{p}((\alpha+\eta_0)(p-1)+\eta_1)}(\mu(\theta_2) - \mu(\theta_1))\right] \right\}^{\frac{p}{q}},$$

hold.

Proof. For $\tau > \theta_1$, we put $\mu(\tau) - \mu(s) = t$, then

$$\begin{aligned} {}_{\theta_1^+}\mathfrak{I}_\mu^\Phi \mathbf{1}(\tau) &= \int_{\theta_1}^{\tau} \mu'(s)\Phi^\alpha(\mu(\tau) - \mu(s))ds = \int_0^{\mu(\tau)-\mu(\theta_1)} \Phi^\alpha(t)dt \\ &= c_0\, \Phi^{\alpha+\eta_0}(\mu(\tau) - \mu(\theta_1)), \quad c_0, \eta_0 > 0. \end{aligned}$$

Setting $\tau = \theta_2$, we have

$${}_{\theta_1^+}\mathfrak{I}_\mu^\Phi \mathbf{1}(\theta_2) = c_0\, \Phi^{\alpha+\eta_0}(\mu(\theta_2) - \mu(\theta_1)).$$

For $1 \leq p \leq q < \infty$, by using Hölder inequality for $\dfrac{1}{p} + \dfrac{1}{p'} = 1$, we have

$$\begin{aligned} \int_{\theta_1}^{\theta_2} \frac{\left[{}_{\theta_1^+}\mathfrak{I}_\mu^\Phi \varphi(\tau)\right]^p}{g(\tau)} d\tau &= \int_{\theta_1}^{\theta_2} g^{-1}(\tau) \left(\int_{\theta_1}^{\tau} \mu'(s)\Phi^\alpha(\mu(\tau) - \mu(s))\varphi(s)ds\right)^p d\tau \\ &= \int_{\theta_1}^{\theta_2} g^{-1}(\tau) \left(\int_{\theta_1}^{\tau} \left[\mu'(s)\Phi^\alpha(\mu(\tau) - \mu(s))\right]^{\frac{1}{p'}} \right. \\ &\quad \left. \times \left[\mu'(s)\Phi^\alpha(\mu(\tau) - \mu(s))\right]^{\frac{1}{p}} \varphi(s)ds\right)^p d\tau \end{aligned}$$

$$\leq \int_{\theta_1}^{\theta_2} g^{-1}(\tau)\left(\int_{\theta_1}^{\tau}\mu'(s)\Phi^{\alpha}(\mu(\tau)-\mu(s))ds\right)^{p-1}$$

$$\times\left(\int_{\theta_1}^{\tau}\mu'(s)\Phi^{\alpha}(\mu(\tau)-\mu(s))\varphi^{p}(s)ds\right)d\tau$$

$$=\int_{\theta_1}^{\theta_2}\int_{\theta_1}^{\tau} g^{-1}(\tau)\left[c_0\,\Phi^{\alpha+\eta_0}(\mu(\tau)-\mu(\theta_1))\right]^{p-1}$$

$$\times\mu'(s)\Phi^{\alpha}(\mu(\tau)-\mu(s))\varphi^{p}(s)dsdx$$

$$=c_0^{p-1}\int_{\theta_1}^{\theta_2} g^{-1}(\tau)\Phi^{(\alpha+\eta_0)(p-1)}(\mu(\tau)-\mu(\theta_1))$$

$$\times\textstyle\int_{\theta_1}^{\tau}\mu'(s)\Phi^{\alpha}(\mu(\tau)-\mu(s))\varphi^{p}(s)dsdx.$$

Since g is non-decreasing on $[s,\theta_2]$ and $\mu(\tau)-\mu(s)\leq\mu(\theta_2)-\mu(s)$, hence by the change of integration order, we obtain

$$\int_{\theta_1}^{\theta_2}\frac{\left[{}_{\theta_1^+}\mathfrak{I}_{\mu}^{\Phi}\varphi(\tau)\right]^{p}}{g(\tau)}d\tau \leq c_0^{p-1}\int_{\theta_1}^{\theta_2}\frac{\mu'(s)\,\varphi^{p}(s)}{g(s)}\Phi^{\alpha}(\mu(\theta_2)-\mu(s)) \tag{5}$$

$$\times\left[\int_{s}^{\theta_2}\Phi^{(\alpha+\eta_0)(p-1)}(\mu(\tau)-\mu(\theta_1))d\tau\right]ds.$$

From the hypothesis (1) and $\mu'(\tau)\geq 1$, we deduce that

$$\int_{s}^{\theta_2}\Phi^{(\alpha+\eta_0)(p-1)}(\mu(\tau)-\mu(\theta_1))d\tau$$

$$\leq\int_{s}^{\theta_2}\mu'(\tau)\,\Phi^{(\alpha+\eta_0)(p-1)}(\mu(\tau)-\mu(\theta_1))d\tau$$

$$=c_1\left[\Phi^{(\alpha+\eta_0)(p-1)+\eta_1}(\mu(\tau)-\mu(\theta_1))\right]_{s}^{\theta_2},$$

Denote $\Psi=\Phi^{\frac{(\alpha+\eta_0)(q-1)+\eta_1}{P}}$, apply the inequality (3) for $\frac{q}{p}\geq 1$ and

$$A=\left[\Phi^{\frac{(\alpha+\eta_0)(p-1)+\eta_1}{P}}(\mu(\theta_2)-\mu(\theta_1))\right]^{p}=\Psi^{p}(\mu(\theta_2)-\mu(\theta_1)),$$

$$B=\left[\Phi^{\frac{(\alpha+\eta_0)(p-1)+\eta_1}{P}}(\mu(s)-\mu(\theta_1))\right]^{p}=\Psi^{p}(\mu(s)-\mu(\theta_1)),$$

we get

$$\int_{s}^{\theta_2}\Phi^{(\alpha+\eta_0)(p-1)}(\mu(\tau)-\mu(\theta_1))d\tau \tag{6}$$

$$\leq c_1 \left(\Psi^q(\mu(\theta_2)-\mu(\theta_1))-\Psi^q(\mu(s)-\mu(\theta_1))\right)^{\frac{p}{q}}.$$

Put (6) in (5) and apply (2), therefore

$$\int_{\theta_1}^{\theta_2} \frac{\left[{}_{\theta_1^+}\mathfrak{I}_\mu^\Phi \varphi(\tau)\right]^p}{g(\tau)} d\tau \leq c_0^{p-1}\, c_1 \int_{\theta_1}^{\theta_2} \frac{\mu'(s)\,\varphi^p(s)}{g(s)} \Phi^\alpha(\mu(\theta_2)-\mu(s))$$

$$\times \left[\left(\Psi^q(\mu(\theta_2)-\mu(\theta_1))-\Psi^q(\mu(s)-\mu(\theta_1))\right)^{\frac{p}{q}}\right] ds$$

$$= c_0^{p-1}\, c_1 \int_{\theta_1}^{\theta_2} \mu'(s)\Phi^\alpha(\mu(\theta_2)-\mu(s))g^{-1}(s)$$

$$\times \left[\varphi(s)\left(\Psi^q(\mu(\theta_2)-\mu(\theta_1))-\Psi^q(\mu(s)-\mu(\theta_1))\right)^{\frac{1}{q}}\right]^p ds$$

$$\leq c_0^{p-1}\, c_1 \left(\int_{\theta_1}^{\theta_2} \mu'(s)\Phi^\alpha(\mu(\theta_2)-\mu(s))g^{-1}(s) ds\right)^{\frac{q-p}{q}}$$
$$\times \left(\int_{\theta_1}^{\theta_2} \mu'(s)\Phi^\alpha(\mu(\theta_2)-\mu(s))\frac{\varphi^q(s)}{g(s)}\right.$$

$$\left.\times \left[\Psi^q(\mu(\theta_2)-\mu(\theta_1))-\Psi^q(\mu(s)-\mu(\theta_1))\right] ds\right)^{\frac{p}{q}}$$

$$= c_0^{p-1}\, c_1 \left({}_{\theta_1^+}\mathfrak{I}_\mu^\Phi g^{-1}(\theta_2)\right)^{\frac{q-p}{q}}$$

$$\times \left[\Psi^q(\mu(\theta_2)-\mu(\theta_1)) \int_{\theta_1}^{\theta_2} \mu'(s)\Phi^\alpha(\mu(\theta_2)-\mu(s))\frac{\varphi^q(s)}{g(s)} ds\right.$$

$$\left. - \int_{\theta_1}^{\theta_2} \mu'(s)\Phi^\alpha(\mu(\theta_2)-\mu(s))\frac{\varphi^q(s)}{g(s)}\Psi^q(\mu(s)-\mu(\theta_1))\, ds\right]^{\frac{p}{q}}$$

$$= c_0^{p-1}\, c_1 \left({}_{\theta_1^+}\mathfrak{I}_\mu^\Phi g^{-1}(\theta_2)\right)^{\frac{q-p}{q}} \left[\Psi^q(\mu(\theta_2)-\mu(\theta_1))\, {}_{\theta_1^+}\mathfrak{I}_\mu^\Phi \left(\tfrac{\varphi^q(\theta_2)}{g(\theta_2)}\right)\right.$$

$$\left. - {}_{\theta_1^+}\mathfrak{I}_\mu^\Phi \left(\tfrac{\varphi^q(\theta_2)}{g(\theta_2)}\Psi^q(\mu(\theta_2)-\mu(\theta_1))\right)\right]^{\frac{p}{q}}.$$

Putting $c = c_0^{p-1}\, c_1$, we get the required inequality (4).

□

We present some results which are special cases of Theorem 1 in the corollaries mentioned below.

1. Setting $\Phi(\tau)=\mu(\tau)=\tau$ and $\alpha=0$, then we get $c_0=\eta_0=\eta_1=1$, $c_1=\frac{1}{p}$ and

$$\mathcal{R}_{\theta_1^+}\varphi(\tau) = \int_{\theta_1}^{\tau} \varphi(t)dt, \quad \tau > \theta_1.$$

Corollary 1. *(Hardy type inequality via Riemann integral operator.)*
Let $1 \leq p \leq q < \infty$ and $\varphi > 0, g > 0$ on $[\theta_1, \theta_2]$, such that g is non-decreasing, then

$$p \int_{\theta_1}^{\theta_2} \frac{\left(\mathcal{R}_{\theta_1^+}(\varphi(\tau))\right)^p}{g(\tau)} d\tau \leq \left(\mathcal{R}_{\theta_1^+}\left(\frac{1}{g(\theta_2)}\right)\right)^{\frac{q-p}{q}} \tag{7}$$

$$\times \left\{(\theta_2 - \theta_1)^q \mathcal{R}_{\theta_1^+}\left(\frac{\varphi^q(\theta_2)}{g(\theta_2)}\right) - \mathcal{R}_{\theta_1^+}\left(\frac{\varphi^q(\theta_2)}{g(\theta_2)}(\theta_2 - \theta_1)^q\right)\right\}^{\frac{p}{q}}.$$

The inequality (7) via Riemann operator with two parameters $0 < p \leq q$ coincides with [5, Corollary 4.1], taking $q = p$, we get [3, Theorem 2.2].

2. Setting $\Phi^\alpha(\tau) = \dfrac{\tau^{\frac{\alpha}{k}-1}}{k\Gamma_k(\alpha)}$, $\alpha > 0$ and $\mu(\tau) = \tau$, then we get $c_0 = \eta_0 = c_1 = \eta_1 = k$ and

$$\mathcal{L}^k_{\theta_1^+}\varphi(\tau) = \frac{1}{k\Gamma_k(\alpha)} \int_{\theta_1}^{\tau} (\tau - t)^{\frac{\alpha}{k}-1} \varphi(t) dt, \quad \tau > \theta_1.$$

Corollary 2. *(Hardy type inequality via a k-Riemann-Liouville operator.) Under the assumptions of Corollary 1, we have*

$$\int_{\theta_1}^{\theta_2} \frac{\left(\mathcal{L}^k_{\theta_1^+}(\varphi(\tau))\right)^p}{g(\tau)} d\tau \leq k^p \left(\mathcal{L}^k_{\theta_1^+}\left(\frac{1}{g(\theta_2)}\right)\right)^{\frac{q-p}{q}} \tag{8}$$

$$\times \left(\frac{1}{k\Gamma_k(\frac{q}{p}(\alpha(p-1) + kp))}\right)^{\frac{p}{q}}$$

$$\times \left\{(\theta_2 - \theta_1)^{(\frac{q\alpha(p-1)}{pk} + q - 1)} \mathcal{R}_{\theta_1^+}\left(\frac{\varphi^q(\theta_2)}{g(\theta_2)}\right) - \mathcal{L}^k_{\theta_1^+}\left(\frac{\varphi^q(\theta_2)}{g(\theta_2)}(\theta_2 - \theta_1)^{(\frac{q\alpha(p-1)}{pk} + q - 1)}\right)\right\}^{\frac{p}{q}},$$

where the k-gamma function verified for all $\alpha > 0$, $k > 0$,

$$\Gamma_k(\alpha) = \int_0^\infty t^{\alpha-1} e^{-\frac{t^k}{k}} dt, \qquad \Gamma_k(\alpha + k) = \alpha\Gamma_k(\alpha).$$

The inequality (8) is a new result via k-Riemann-Liouville operator on $[\theta_1, \tau]$ with two parameters $0 < p \leq q$, if we put $k = 1$, we get a new result of Riemann-Liouville.

4 Conclusions

In this paper, by using μ-proportional generalized fractional integral operators, a generalization of the Inverse Hardy Inequality is obtained. From Theorem 1, we obtain several corollaries that establish new extensions for different integral operators. Our work technique can be used to obtain inverse inequalities of another nature, for example, in [12] new inequalities of the Hardy type are obtained, so obtaining their inverse inequalities remains an open problem.

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