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On $4-T-$quasigroups with exactly 20 distinct parastrophes

Tatiana Rotari, Parascovia Syrbu

Abstract. The $T-$forms and the spectrum of $4-T-$quasigroups with exactly 20 distinct parastrophes are considered in the present work. Characterizations of the spectra of finite binary quasigroups with a prescribed number of distinct parastrophes were given by C.C. Lindner and D. Steedly in [1]. Following the results of C.C. Lindner and D. Steedly, M. MacLeish proved that the maximum number of distinct parastrophes of an $n-$quasigroup (Q, A) is a divizor of $(n+1)!$, and obtained characterizations of the spectrum of finite ternary quasigroups with a prescribed (maximum) number of distinct parastrophes. Binary and ternary linear quasigroups over groups, with a given maximum number of distinct parastrophes have been studied by Belyavskaya, Rotari, Sokhatsky, Pirus, Fryz and others [4-8]. Characterizations of the general $T-$form of a $4-T-$quasigroup with exactly 20 distinct parastrophes and some estimations of their spectrum are given in the present work.

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An $n-$ary groupoid (Q, A) is called an $n-$ary quasigroup (or an $n-$quasigroup) if in the equality $A(x_1, x_2, ..., x_n) = x_{n+1}$ every element of $x_1, x_2, ..., x_n, x_{n+1}$ is uniquely determined by the remaining n ones. If (Q, A) is an $n-$quasigroup and $\sigma \in S_{n+1}$, then the operation ${}^{\sigma}\!A$, defined by the equivalence

$${}^{\sigma}\!A(x_{\sigma 1}, ..., x_{\sigma n}) = x_{\sigma(n+1)} \Leftrightarrow A(x_1, x_2, ..., x_n) = x_{n+1},$$

is called a $\sigma-$parastrophe or, simply, a parastrophe of A.

C. C. Lindner and S. Steedly showed in [1] that the possible number of distinct parastrophes of a binary quasigroup divides 6, i.e. is 1, 2, 3, or 6. Moreover, they proved that there exist finite binary quasigroups with a given maximum number d of distinct parastrophes, where $d \in \{1, 2, 3, 6\}$, of any order $q \geq 4$.

M. McLeish [2,3] considered analogous questions in the ternary case. She showed that the possible number of distinct parastrophes of an $n-$quasigoup devides $(n+1)!$ and obtained a series of estimations of the spectra of finite ternary and $n-$ary quasigroups with a prescribed number of distinct parastrophes. In particular, M. McLeish showed that there exist finite ternary quasigroups:

1) with exactly 3 or 4 distinct parastrophes of any order $q \geq 3$;

2) with exactly 6, 12 or 24 distinct parastrophes of any order $q \geq 4$;

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3) with exactly 1 distinct parastrophe (i.e. a $TS-$quasigroup) of any order $q \geq 1$.

M. McLeish obtained also a partial characterization of the spectrum of finite ternary quasigroups with exactly 2 or 8 distinct parastrophes.

Remark that the possible maximum number of distinct parastrophes of an $n-$quasigroup depend on the order of the existing sugbgroups in S_{n+1}. Namely, if (Q, A) is an $n-$guasigroup, then $H = \{\sigma \in S_{n+1} | {}^{\sigma}\!A = A\}$ is a subgroup of S_{n+1} and the maximum number of distinct parastrophes of (Q, A) is equal to the index of H in S_{n+1}.

Binary quasigoups, which are linear over abelian groups and have a prescribed maximum number of distinct parastrophes, which are also orthogonal, were considered by G. Beleavskaya and T. Popovich (T. Rotari) in [4, 5].

Partial characterizations of ternary quasigroup operations, which are linear over groups and have a given number of distinct parastrophes, are obtained by F. Sokhatsky, Y. Pirus and I. Fryz in [6–8].

It is known that the group S_4 has a total of 30 subgroups which, up to isomorphism, are:

$$\{\varepsilon\},\ \mathbb{Z}_2,\ \mathbb{Z}_3,\ \mathbb{Z}_4,\ K_4,\ S_3,\ D_8,\ A_4,\ S_4. \tag{1}$$

Ternary quasigroups (Q, A), linear over a group $(Q, +)$: $A(x_1, x_2, x_3) = \alpha_1 x_1 + \alpha_2 x_2 + \alpha_3 x_3 + c$, where $\alpha_1, \alpha_2, \alpha_3 \in S_Q$, $\alpha_i(0) = 0$, 0 is the neutral element of $(Q, +)$, $i = \overline{1, 3}$, $c \in Q$, and such that the subgroup $H = \{\sigma \in S_4 | {}^{\sigma}\!A = A\}$ is one of the subgroups given in (1) are considered in [6,7]. Remark that a linear $n-$quasigoup (Q, A) may have different general forms for different isomorphic subgroups $H = \{\sigma \in S_{n+1} | {}^{\sigma}\!A = A\}$.

Characterizations of $T-$forms of the $4-T-$quasigroups with exactly 20 distinct parastrophes and some estimations of their spectrum are given in the present work.

Recall that an $n-$quasigroup is called an $n-T-$quasigroup if there exist an abelian group $(Q, +)$, its automorphisms $\alpha_1, \alpha_2, ..., \alpha_n$ and an element $c \in Q$, such that $A(x_1, x_2, ..., x_n) = \alpha_1 x_1 + \alpha_2 x_2 + ... + \alpha_n x_n + c$. In this case the tuple $((Q, +), \alpha_1, \alpha_2, ..., \alpha_n, c)$ is called a $T-$form of (Q, A) and $(Q, +)$ is called a $T-$group of (Q, A) [7,8].

The possible (maximum) number of distinct parastrophes of a $4-$quasigroup is a divisor of 120 and, if (Q, A) has exactly d distinct parastrophes, then d is the index of the subgroup $H = \{\sigma \in S_5 | {}^{\sigma}\!A = A\}$ in the group S_5. It is known that S_5 has a total of 156 subgroups and that it has no subgroups of order 15, 30 and 40. Hence, there does not exist $4-$quasigroups with exactly 8, 4 or 3 distinct parastrophes.

To obtain a full characterization of the $T-$forms of a $4-T-$quasigroup with exactly 20 distinct parastrophes, we will consider all 30 subgroups of order 6 of the group S_5, namely:

1) 10 subgroups isomorfic to $\mathbb{Z}_6$:

$$H_1 = \langle(123)(45)\rangle,\ H_2 = \langle(124)(35)\rangle,\ H_3 = \langle(125)(34)\rangle,\ H_4 = \langle(134)(25)\rangle,$$

$$H_5 = \langle(135)(24)\rangle,\ H_6 = \langle(145)(34)\rangle, H_7 = \langle(234)(15)\rangle,\ H_8 = \langle(235)(14)\rangle,$$

$$H_9 = \langle(245)(13)\rangle, H_{10} = \langle(345)(12)\rangle;$$

2) 20 subgroups isomorfic to S_3, each generated by two substitutions α and β, of order 3 and, respectively 2, including:

2a) 10 subgroups where β is a transposition

$$H_{11} = \langle(123),(12)\rangle,\ H_{12} = \langle(124),(12)\rangle,\ H_{13} = \langle(134),(13)\rangle,$$

$$H_{14} = \langle(125),(12)\rangle, H_{15} = \langle(135),(13)\rangle,\ H_{16} = \langle(145),(14)\rangle,$$

$$H_{17} = \langle(234),(23)\rangle, H_{18} = \langle(235),(23)\rangle, H_{19} = \langle(245),(24)\rangle,$$

$$H_{20} = \langle(345),(34)\rangle;$$

2b) 10 subgroups where β is a product of two independent transpositions

$$H_{21} = \langle(123),(12)(45)\rangle,\ H_{22} = \langle(124),(12)(35)\rangle,\ H_{23} = \langle(125),(12)(34)\rangle,$$

$$H_{24} = \langle(134),(13)(25)\rangle,\ H_{25} = \langle(135),(13)(24)\rangle,\ H_{26} = \langle(145),(14)(23)\rangle,$$

$$H_{27} = \langle(234),(23)(15)\rangle,\ H_{28} = \langle(235),(23)(14)\rangle,\ H_{29} = \langle(245),(13)(24)\rangle,$$

$$H_{30} = \langle(345),(34)(12)\rangle.$$

Theorem 1. *Let (Q,A) be a $4-T-$quasigroup with the $T-$group $(Q,+)$ and let $H \in \{H_i,\ i = \overline{11,20}\}$, where $H = \{\sigma \in S_5 | {}^{\sigma}\!A = A\}$. Then there exist $\alpha, \beta \in Aut(Q,+)$ and an element $c \in Q$, where $\alpha \neq \beta$, $\alpha \neq I$, $\beta \neq I$, $\alpha c + c \neq 0$, $\beta c + c \neq 0$, $Ix = -x, \forall x \in Q$, 0 is the neutral element of $(Q,+)$, such that (Q,A) has one of the following $T-$forms:*

$T_1 = ((Q,+),\alpha,\alpha,\alpha,\beta,c)$, $T_2 = ((Q,+),\alpha,\alpha,\beta,\alpha,c)$, $T_3 = ((Q,+),\alpha,\beta,\alpha,\alpha,c)$,
$T_4 = ((Q,+),\beta,\alpha,\alpha,\alpha,c)$, $T_5 = ((Q,+),I,I,\alpha,\beta,c)$, $T_6 = ((Q,+),I,\alpha,I,\beta,c)$,
$T_7 = ((Q,+),I,\alpha,\beta,I,c)$, $T_8 = ((Q,+),\alpha,I,I,\beta,c)$, $T_9 = ((Q,+),\alpha,I,\beta,I,c)$,
$T_{10} = ((Q,+),\alpha,\beta,I,I,c)$.

Proof. Let (Q,A) be a $4-T-$quasigroup, $T = ((Q,+),\alpha_1,\alpha_2,\alpha_3,\alpha_4,c)$ be a $T-$form and let $H \in \{H_i,\ i = \overline{11,20}\}$, where $H = \{\sigma \in S_5 | {}^{\sigma}\!A = A\}$.

1) If $H = H_{11} = \langle(123),(12)\rangle = \{\varepsilon,(123),(132),(12),(13),(23)\}$ then we have $(12),(13) \in H$, hence $A = {}^{(12)}A$ and $A = {}^{(13)}A$, i.e.

$$\begin{cases} \alpha_1x_1 + \alpha_2x_2 + \alpha_3x_3 + \alpha_4x_4 + c = \alpha_1x_2 + \alpha_2x_1 + \alpha_3x_3 + \alpha_4x_4 + c \\ \alpha_1x_1 + \alpha_2x_2 + \alpha_3x_3 + \alpha_4x_4 + c = \alpha_1x_3 + \alpha_2x_2 + \alpha_3x_1 + \alpha_4x_4 + c \end{cases} \Leftrightarrow$$

$$\begin{cases} \alpha_1x_1 + \alpha_2x_2 = \alpha_1x_2 + \alpha_2x_1 \\ \alpha_1x_1 + \alpha_3x_3 = \alpha_1x_3 + \alpha_3x_1. \end{cases}$$

Taking $x_1 = 0$, the last two equalities imply $\alpha_1 = \alpha_2 = \alpha_3$. Denoting $\alpha_1 = \alpha_2 = \alpha_3 = \alpha$ and $\alpha_4 = \beta$, the operation A takes the form:

$$A(x_1^4) = \alpha x_1 + \alpha x_2 + \alpha x_3 + \beta x_4 + c,$$

where the 4−tuple (x_1, x_2, x_3, x_4) is denoted by (x_1^4).

As $(14) \notin H_{11}$, we have: $A \neq^{(14)} A \Leftrightarrow \alpha x_1 + \beta x_4 \neq \beta x_1 + \alpha x_4 \Leftrightarrow \alpha(x_1 - x_4) \neq \beta(x_1 - x_4)$ i.e. $\alpha \neq \beta$. Also, $(15) \notin H_{11} \Rightarrow A \neq^{(15)} A \Rightarrow A(A(x_1^4), x_2, x_3, x_4) \neq x_1, \Rightarrow$

$$\alpha(\alpha x_1 + \alpha x_2 + \alpha x_3 + \beta x_4 + c) + \alpha x_2 + \alpha x_3 + \beta x_4 + c \neq x_1.$$

Denoting $\alpha x_2 + \alpha x_3 + \beta x_4 + c$ by y in the last inequality we obtain:

$$\alpha(\alpha x_1 + y) + y \neq x_1,$$

which for $\alpha = I$, implies $-(-x_1 + y) + y \neq x_1 \Leftrightarrow x_1 \neq x_1$, a contradiction, hence $\alpha \neq I$. Analogously, as $(45) \notin H_{11}$ we have:

$$A \neq^{(45)} A \Rightarrow A(x_1, x_2, x_3), A(x_1^4)) \neq x_4 \Leftrightarrow$$

$$\alpha x_1 + \alpha x_2 + \alpha x_3 + \beta(\alpha x_1 + \alpha x_2 + \alpha x_3 + \beta x_4 + c) + c \neq x_4.$$

Denoting $\alpha x_1 + \alpha x_2 + \alpha x_3 + c$ by y in the last inequality we have:

$$y + \beta(y + \beta x_4) \neq x_4,$$

which implies $\beta \neq I$ (as for $\beta = I$ we get $x_4 \neq x_4$, a contradiction).

2) Let $H = H_{12} = \langle(124), (12)\rangle = \{\varepsilon, (124), (142), (12), (24), (14)\}$. Then $(12), (14) \in H$, i.e. $A =^{(12)} A$ and $A =^{(14)} A$, so $\alpha_1 x_1 + \alpha_2 x_2 = \alpha_1 x_2 + \alpha_2 x_1$ and $\alpha_1 x_1 + \alpha_4 x_4 = \alpha_1 x_4 + \alpha_4 x_1$, which imply $\alpha_1 = \alpha_2 = \alpha_4$. Denoting $\alpha_1 = \alpha_2 = \alpha_4 = \alpha$ and $\alpha_3 = \beta$, we get the $T-$form: $T_2 = ((Q, +), \alpha, \alpha, \beta, \alpha, c)$.

As $(13), (15)$ and (35) are not in H_{12}, we have: $A \neq^{(13)} A$, $A \neq^{(15)} A$ and $A \neq^{(35)} A$, hence:

$$\alpha x_1 + \beta x_3 \neq \alpha x_3 + \beta x_1,$$

$$\alpha(\alpha x_1 + \alpha x_2 + \beta x_3 + \alpha x_4 + c) + \alpha x_2 + \beta x_3 + \alpha x_4 + c \neq x_1,$$

$$\alpha x_1 + \alpha x_2 + \beta(\alpha x_1 + \alpha x_2 + \beta x_3 + \alpha x_4 + c) + \alpha x_4 + c \neq x_3,$$

which imply, analogously to item 1), $\alpha \neq \beta, \alpha \neq I$ and $\beta \neq I$, respectively.

3) $H = H_{13} = \langle(134), (13)\rangle = \{\varepsilon, (134), (143), (13), (34), (14)\}$. Using the fact that $(13), (14) \in H_{13}$, we obtain $A =^{(13)} A$ and $A =^{(14)} A$, so $\alpha_1 x_1 + \alpha_3 x_3 = \alpha_1 x_3 + \alpha_3 x_1$ and $\alpha_1 x_1 + \alpha_4 x_4 = \alpha_1 x_4 + \alpha_4 x_1$, which imply $\alpha_1 = \alpha_3 = \alpha_4$. Denoting $\alpha_1 = \alpha_3 = \alpha_4$ by α and α_2 by β, we get the $T-$form: $((Q, +), \alpha, \beta, \alpha, \alpha, c) = T_3$.

Using the fact that $(12), (15)$ and (25) are not in H_{13}, i.e. $A \neq^{(12)} A, A \neq^{(15)} A$ and $A \neq^{(25)} A$, we obtain:

$$\alpha_1 x_1 + \alpha_2 x_2 \neq \alpha_1 x_2 + \alpha_2 x_1,$$

$$\alpha(\alpha x_1 + \beta x_2 + \alpha x_3 + \alpha x_4 + c) + \beta x_2 + \alpha x_3 + \alpha x_4 + c \neq x_1,$$

$$\alpha x_1 + \beta(\alpha x_1 + \beta x_2 + \alpha x_3 + \alpha x_4 + c) + \alpha x_3 + \alpha x_4 + c \neq x_2,$$

which imply $\alpha \neq \beta, \alpha \neq I$ and $\beta \neq I$.

4) $H = H_{14} = \langle(125), (12)\rangle = \{\varepsilon, (125), (12), (152), (25), (15)\}$.

As $(12),(15)\in H_{12}$, we have $A={}^{(12)}A$ and $A={}^{(15)}A$, i.e.

$$\alpha_1x_1+\alpha_2x_2=\alpha_1x_2+\alpha_2x_1, \tag{2}$$

$$\alpha_1(\alpha_1x_1+\alpha_2x_2+\alpha_3x_3+\alpha_4x_4+c)+\alpha_2x_2+\alpha_3x_3+\alpha_4x_4+c=x_1. \tag{3}$$

Taking $x_2=0$ in (2), we obtain

$$\alpha_1=\alpha_2. \tag{4}$$

For $x_1=x_2=x_3=x_4=0$, (3) implies

$$\alpha_1c+c=0. \tag{5}$$

Using (4) and (5), and taking $x_1=x_3=x_4=0$ in (3), we get $\alpha_1(\alpha_2x_2)=I(\alpha_2x_2)$, i.e. $\alpha_1=I$, so

$$\alpha_1=\alpha_2=I. \tag{6}$$

Denoting $\alpha_3=\alpha$ and $\alpha_4=\beta$, we obtain the $T-$form $T_4=((Q,+),I,I,\alpha,\beta,c)$.

Now, using the fact that the transpositions $(23),(35),(14)$ do not belong to H_{14}, we have the inequalities:

$$\alpha_2x_2+\alpha_3x_3\neq\alpha_2x_3+\alpha_3x_2,$$

$$Ix_1+Ix_2+\alpha(Ix_1+Ix_2+\alpha x_3+\beta x_4+c)+\beta x_4+c\neq x_3,$$

$$Ix_1+Ix_2+\alpha x_3+\beta(Ix_1+Ix_2+\alpha x_3+\beta x_4+c)+c\neq x_4,$$

which imply $\alpha\neq\beta,\alpha\neq I$ and $\beta\neq I$, respectively.

5) $H=H_{15}=\langle(135),(13)\rangle=\{\varepsilon,(135),(13),(35),(15),(153)\}$. Taking (13), $(15)\in H_{15}$, we have: ${}^{(13)}A=A$ and ${}^{(15)}A=A$, i.e.

$$\alpha_1x_1+\alpha_3x_3=\alpha_1x_3+\alpha_3x_1, \tag{7}$$

$$\alpha_1(\alpha_1x_1+\alpha_2x_2+\alpha_3x_3+\alpha_4x_4+c)+\alpha_2x_2+\alpha_3x_3+\alpha_4x_4+c=x_1. \tag{8}$$

The equality (7) implies

$$\alpha_1=\alpha_3, \tag{9}$$

and (8) implies $\alpha_1c+c=0$ (for $x_1=x_2=x_3=x_4=0$). Using the equality $\alpha_1c+c=0$ and taking $x_1=x_2=x_3=0$ in (8), we get

$$\alpha_1(\alpha_4x_4)+\alpha_4x_4=0,$$

i.e. $\alpha_1=I$. According to (9),

$$\alpha_1=\alpha_3=I,$$

so, denoting $\alpha_2=\alpha$ and $\alpha_4=\beta$, we get $T_5=((Q,+),I,\alpha,I,\beta,c)$. As $(24),(25)$ and (45) are not in H_{15}, the following inequalities hold:

$$\alpha x_2+\beta x_4\neq\alpha x_4+\beta x_2,$$

$$Ix_1 + \alpha(Ix_1 + \alpha x_2 + Ix_3 + \beta x_4 + c) + Ix_3 + \beta x_4 + c \neq x_2,$$

$$Ix_1 + \alpha x_2 + Ix_3 + \beta(Ix_1 + \alpha x_2 + Ix_3 + \beta x_4 + c) + c \neq x_4,$$

which imply, resctively, $\alpha \neq \beta, \alpha \neq I, \beta \neq I$.

6) $H = H_{16} = \langle(145), (14)\rangle = \{\varepsilon, (145), (154), (14), (14), (45)\}$. From $(14), (15) \in H_{16}$ it follows $A =^{(14)} A$ and $A =^{(15)} A$, i.e.

$$\alpha_1 x_1 + \alpha_4 x_4 = \alpha_1 x_4 + \alpha_4 x_1,$$

$$\alpha_1(\alpha_1 x_1 + \alpha_2 x_2 + \alpha_3 x_3 + \alpha_4 x_4 + c) + \alpha_2 x_2 + \alpha_3 x_3 + \alpha_4 x_4 + c = x_1, \tag{10}$$

which imply, respectively $\alpha_1 = \alpha_4$ and $\alpha_1 c + c = 0$. Using the last equality in (10) and taking $x_1 = x_3 = x_4 = 0$, we have $\alpha_1(\alpha_2 x_2) = I(\alpha_2 x_2), \forall x_2 \in Q$, i.e. $\alpha_1 = I$, hence

$$\alpha_1 = \alpha_4 = I. \tag{11}$$

Denoting $\alpha_2 = \alpha$ and $\alpha_3 = \beta$, we get $T_6 = ((Q, +), I, \alpha, \beta, I, c)$. Now, using the fact that $(23), (25)$ and (35) do not belong to H_{16}, we get the inequalities

$$\alpha x x_2 + \beta x_3 \neq \alpha x_3 + \beta x_2,$$

$$Ix_1 + \alpha(Ix_1 + \alpha x_2 + \beta x_3 + Ix_4 + c) + \beta x_3 + Ix_4 + c \neq x_3,$$

$$Ix_1 + \alpha x_2 + \beta(Ix_1 + \alpha x - 2 + \beta x_3 + Ix_4 + c) + Ix_4 + c \neq x_3,$$

which imply, respectively, $\alpha \neq \beta, \alpha \neq I, \beta \neq I$.

7) $H_{17} = \langle(234), (23)\rangle = \{\varepsilon, (234), (243), (23), (24), (34)\}$. From the fact that $(23), (24), (34) \in H_{17}$ it follows: $A =^{(23)} A, A =^{(24)} A, A =^{(34)} A$, i.e. $\alpha_2 x_2 + \alpha_3 x_3 = \alpha_2 x_3 + \alpha_3 x_2$, $\alpha_2 x_2 + \alpha_4 x_4 = \alpha_2 x_4 + \alpha_4 x_2$, $\alpha_3 x_3 + \alpha_4 x_4 = \alpha_3 x_4 + \alpha_4 x_3$, which imply: $\alpha_2 = \alpha_3 = \alpha_4$. Denoting $\alpha_2 = \alpha_3 + \alpha_4 = \alpha$ and $\alpha_1 = \beta$, we get the $T-$form: $T_7 = ((Q, +), \beta, \alpha, \alpha, \alpha, c)$.

The inequalities $\alpha \neq \beta, \beta \neq I$ and $\alpha \neq I$ follow, respectively, from those given by $(12) \notin H_{17}, (15) \notin H_{17}$, $(25) \notin H_{17}$, as:

$$\beta x_1 + \alpha x_2 \neq \beta x_2 + \alpha x_1,$$

$$\beta(\beta x_1 + \alpha x_2 + \alpha x_3 + \alpha x_4 + c) + \alpha x_2 + \alpha x_3 + \alpha x_4 + c \neq x_1,$$

$$\beta x_1 + \alpha(\beta x_2 + \alpha x_2 + \alpha x_3 + \alpha x_4 + c) + \alpha x_3 + \alpha x_4 + c \neq x_2.$$

8) $H = H_{18} = \langle(235), (23)\rangle = \{\varepsilon, (235), (23), (35), (253), (25)\}$.

From $(23), (35) \in H_{18}$, we have $A =^{(23)} A$ and $A =^{(35)} A$, hence

$$\alpha_2 x_2 + \alpha_3 x_3 = \alpha_2 x_3 + \alpha_3 x_2,$$

$$\alpha_1 x_1 + \alpha_2 x_2 + \alpha_3(\alpha_1 x_1 + \alpha_2 x_2 + \alpha_3 x_3 + \alpha_4 x_4 + c) + \alpha_4 x_4 + c = x_3, \tag{12}$$

which imply, respectively,

$$\alpha_2 = \alpha_3 \tag{13}$$

and $\alpha_3 c + c = 0$. Using the last equality and taking $x_2 = x_3 = x_4 = 0$ in (12), we have $\alpha_3(\alpha_1 x_1) = I(\alpha_1 x_1)$, which implies $\alpha_3 = I$, so (according to (13)), $\alpha_1 = \alpha_3 = I$. Denoting $\alpha_2 = \alpha$ and $\alpha_4 = \beta$, we get $T = T_8 = ((Q,+), \alpha, I, I, \beta, c)$.

Using T_8 and the fact that $(14) \notin H_{18}, (15) \notin H_{18}$ and $(45) \notin H_{18}$, we obtain the inequalities:

$$\alpha x_1 + \beta x_4 \neq \alpha x_4 + \beta x_1,$$

$$\alpha(\alpha x_1 + I x_2 + I x_3 + \beta x_4 + c) + I x_2 + I x_3 + \beta x_4 + c \neq x_1,$$

$$\alpha x_1 + I x_2 + I x_3 + \beta(\alpha x_1 + I x_2 + I x_3 + \beta x_4 + c) + c \neq x_4,$$

which imply, respectively, $\alpha \neq \beta$, $\alpha \neq I$ and $\beta \neq I$.

9) $H = H_{19} = \langle (245), (24) \rangle = \{\varepsilon, (245), (254), (24), (25), (25), (45)\}$.

From $(24), (25) \in H_{19}$ it follows, respectively,

$$\alpha_2 x_2 + \alpha_4 x_4 = \alpha_2 x_4 + \alpha_4 x_2,$$

$$\alpha_1 x_1 + \alpha_2(\alpha_1 x_1 + \alpha_2 x_2 + \alpha_3 x_3 + \alpha_4 x_4 + c) + \alpha_3 x_3 + \alpha_4 x_4 + c = x_2, \tag{14}$$

which imply, $\alpha_2 = \alpha_4$ and $\alpha_2 c + c = 0$. Using the equality $\alpha_2 c + c = 0$ and taking $x_1 = x_2 = x_4 = 0$ in (14), we get $\alpha_2(\alpha_3 x_3) = I(\alpha_3 x_3)$, so $\alpha_2 = I$, i.e. $\alpha_2 = \alpha_4 = I$. Denoting $\alpha_1 = \alpha$ and $\alpha_3 = \beta$, we obtain the T–form $T_9 = ((Q,+), \alpha, I, I, \beta, I, c)$. Now, using T_9 and the fact that $(13) \notin H_{19}, (15) \notin H_{19}$ and $(45) \notin H_{19}$ we obtain the inequalities:

$$\alpha x_1 + \beta x_3 \neq \alpha x_3 + \beta x_1,$$

$$\alpha(\alpha x - 1 + I x_2 + \beta x_3 + I x_4 + c) + I x_2 + \beta x_3 + I x_4 + c \neq x_1,$$

$$\alpha x_1 + I x_2 + \beta(\alpha x_1 + I x_2 + \beta x_3 + I x_4 + c) + I x_3 + c \neq x_3,$$

which imply, respectively, $\alpha \neq \beta$, $\beta \neq I$, $\alpha \neq I$.

10) $H_{20} = \langle (345), (34) \rangle = \{\varepsilon, (345), (354), (54), (35), (45)\}$.

As $(34), (35) \in H_{20}$, we have $A = {}^{(34)}A$ and $A = {}^{(35)}A$, i.e.

$$\alpha_3 x_3 + \alpha_4 x_4 = \alpha_3 x_4 + \alpha_4 x_3,$$

$$\alpha_1 x_1 + \alpha_2 x_2 + \alpha_3(\alpha_1 x_1 + \alpha_2 x_2 + \alpha_3 x_3 + \alpha_4 x_4 + c) + \alpha_4 x_4 + c = x_3, \tag{15}$$

which imply, respectively, $\alpha_3 = \alpha_4$ and $\alpha_3 c + c = 0$. Using the equality $\alpha_3 c + c = 0$ and taking $x_1 = x_2 = x_3 = 0$ in (15), we obtain $\alpha(\alpha_4 x_4) = I(\alpha_4 x_4)$, so $a\alpha_3 = I$, i.e. $\alpha_3 = \alpha_4 = I$. Denoting $\alpha_1 = \alpha$ and $\alpha_2 = \beta$ we get the T–form $T_{20} = ((Q,+), \alpha, \beta, I, I, c)$. The inequalities $\alpha \neq \beta$, $\alpha \neq I$ and $\beta \neq I$ follow from the fact that $(12) \notin H_{20}$, $(15) \notin H_{20}$ and $(25) \notin H_{20}$ (using the T–form T_{10}), i.e. from the following inequalities, respectively:

$$\alpha x_1 + \beta x_2 \neq \alpha x_2 + \beta x_1,$$

$$\alpha(\alpha x_1 + \beta x_2 + I x_3 + I x_4 + c) + \beta x_2 + I x_3 + I x_4 + c \neq x_1,$$

$$\alpha x_1 + \beta(\alpha x_1 + \beta x_2 + I x_3 + I x_4 + c) + I x_3 + I x_4 + c \neq x_2.$$

□

Proposition 1. *The* $4-T-$*quasigroup* (Q,A) *with the* $T-$*form*

$$T_1 = ((Q,+),\alpha,\alpha,\alpha,\beta,c),$$

where $\alpha\neq\beta$, $\alpha\neq I$, $\beta\neq I$, $I(x)=-x,\forall x\in Q$, *has exactly 20 distinct parastrophes.*

Proof. It is sufficient to show that $H=\{\sigma\in S_5|{}^{\sigma}\!A=A\}$ is a subgroup of order six. If T_1 is a $T-$form of the $n-T-$quasigroup (Q,A), then

$$A(x_1^4)=\alpha x_1+\alpha x_2+\alpha x_3+\beta x_4+c,\forall x_1,x_2,x_3,x_4\in Q,$$

so ${}^{\sigma}\!A=A,\forall\sigma\in S_5'$, where $S_5'=\{\sigma\in S_5|\sigma(4)=4,\sigma(5)=5\}$, i.e. $S_5'\subseteq H$.

Let consider the following set of representativies of the cosets $\{S_5'\tau|\tau\in S_5\}$:

$$U=\{\varepsilon,(14),(24),(34),(15),(25),(35),(45),(14)(25),(14)(35),(15)(24),$$

$$(15)(34),(24)(35),(25)(34),(145),(154),(245),(254),(345),(354)\}.$$

If $\tau\in U$, then $\beta\in S_5'\tau\Leftrightarrow\exists\sigma\in S_5':\beta=\sigma\tau\Leftrightarrow{}^{\beta}A={}^{\sigma\tau}A={}^{\tau}({}^{\sigma}\!A)={}^{\tau}A$. We will prove below that $\tau\notin H,\forall\tau\in U\setminus\{\varepsilon\}$, i.e. that $H=S_5'$.

If $(14)\in H$ then: $A={}^{(14)}A\Leftrightarrow\alpha x_1+\alpha x_2+\alpha x_3+\beta x_4+c=\alpha x_4+\alpha x_2+\alpha x_3+\beta x_1+c\Rightarrow\alpha=\beta$, a contradiction, so $(14)\notin H$.

Analogously, $\alpha\neq\beta$ implies $(24)\notin H$ and $(34)\notin H$.

If $(15)\in H$ then: $A={}^{(15)}A\Leftrightarrow A(A(x_1^4),x_2,x_3,x_4)=x_2\Leftrightarrow\alpha(\alpha x_1+\alpha x_2+\alpha x_3+\beta x_4+c)+\alpha x_2+\alpha x_3+\beta x_4+c=x_1$. Denoting $\alpha x_2+\alpha x_3+\alpha x_4+c$ by y, we get $\alpha(\alpha x_1+y)+y=x_1$, which (for $x_1=0$) implies $\alpha=I-$ a contradiction, so $(15)\notin H$.

Analogously, we get $(25)\notin H$ and $(35)\notin H$.

Also:

$$A={}^{(45)}A\Leftrightarrow A(x_1^3,A(x_1^4))=x_4\Leftrightarrow$$

$$\alpha x_1+\alpha x_2+\alpha x_3+\beta(\alpha x_1+\alpha x_2+\alpha x_3+\beta x_4+c)+c=x_4.$$

Denoting $\alpha x_1+\alpha x_2+\alpha x_3+x=y$ in the last equality we get: $\beta(\beta x_4+y)+y=x_4$, which (for $x_4=0$) implies $\beta=I-$ a contradiction, hence $(45)\notin H$.

If $(14)(35)\in H$, then: $A={}^{(14)(35)}A\Leftrightarrow A(x_1^4)={}^{(34)}A(x_4,x_2,x_3,x_1)\Leftrightarrow A(x_4,x_2,A(x_1^4),x_1)=x_3\Leftrightarrow$

$$\alpha x_4+\alpha x_2+\alpha(\alpha x_1+\alpha x_2+\alpha x_3+\beta x_4+c)+\beta x_1+c=x_3. \tag{16}$$

Taking $x_1=x_2=x_3=x_4$ in the last equality we get $\alpha c+c=0$. Using $\alpha c+c=0$ and taking $x_1=x_3=x_4=0$ in (16) we obtain $\alpha(\alpha x_2)=I(\alpha x_2)$, i.e. $\alpha=I$, a contradiction, so $(14)(35)\notin H$. Analogously, we obtain $(14)(25)\notin H$, $(15)(24)\notin H$, $(15)(34)\notin H$, $(24)(35)\notin H$ and $(25)(34)\notin H$.

If $(145)\in H$, then we have:

$$A={}^{(145)}A\Leftrightarrow A(x_4,x_2,x_3,A(x_1^4))=x_1\Leftrightarrow$$

$$\alpha x_4+\alpha x_2+\alpha x_3+\beta(\alpha x_1+\alpha x_2+\alpha x_3+\beta x_4+c)+c=x_1. \tag{17}$$

Taking $x_1 = x_2 = x_3 = x_4 = 0$ in (17) we get $\beta c + c = 0$. Using the equality $\beta c + c = 0$ and taking $x_1 = x_3 = x_4 = 0$ in (17), we obtain $\beta(\alpha x_2) = I(\alpha x_2)$, which implies $\beta = I$, a contradiction, so $(145) \notin H$.

Analogously, we get that each of the substitutions: $(154), (245), (254), (345), (354)$ does not belong to H. Hence, $H = S_5' \cong S_3$, i.e. (Q, A) has exactly 20 distinct parastrophes. □

Corollary 1. *A $4 - T-$quasigroup (Q, A), where $\{\sigma \in S_5 | A =^{\sigma} A\} = \langle(123), (23)\rangle$, has exactly 20 distinct parastrophes if and only if it has a $T-$form $((Q, +), \alpha, \alpha, \alpha, \beta, c)$, where $\alpha \neq \beta$, $\alpha \neq I$, $\beta \neq I, I(x) = -x, \forall x \in Q$.*

Theorem 2. *There do not exist $4 - T-$quasigroups (Q, A) such that the group $H = \{\sigma \in S_5 | A =^{\sigma} A\}$ is isomorphic to $\mathbb{Z}_6$.*

Proof. Let (Q, A) be a $4 - T-$quasigroup with $T-$form $((Q, +), \alpha_1, \alpha_2, \alpha_3, \alpha_4, c)$, and let $H = \{\sigma \in S_5 | A = {}^{\sigma}A\}$. The group S_5 has 10 subgroups isomorphic to $\mathbb{Z}_6$: $H_i, \overline{1, 10}$. It will be shown below that $H \neq H_i$, for every $i = \overline{1, 10}$.

1) Let $H = H_1 = \langle(123)(45)\rangle = \{\varepsilon, (123)(45), (132), (45), (123), (132)(45)\}$. As $(123) \in H_1$, we have $A =^{(123)} A \Leftrightarrow A(x_1^4) = A(x_3, x_1, x_2, x_4)$, which implies

$$\alpha_1 x_1 + \alpha_2 x_2 + \alpha_3 x_3 = \alpha_1 x_3 + \alpha_2 x_1 + \alpha_3 x_2. \tag{18}$$

Taking $x_1 = x_3 = 0$, and respectively $x_1 = x_2 = 0$, in (18) we get $\alpha_2 = \alpha_3$ and $\alpha_1 = \alpha_3$, i.e. $\alpha_1 = \alpha_2 = \alpha_3$.

Denoting $\alpha_1 = \alpha_2 = \alpha_3 = \alpha$, we obtain $T = ((Q, +), \alpha, \alpha, \alpha, \alpha_4, c)$, i.e. $A(x_1^4) = \alpha x_1 + \alpha x_2 + \alpha x_3 + \alpha_4 x_4 + c$. Also, $(45) \in H_1$, so $A =^{(45)} A$, i.e. $x_4 = A(x_1, x_2, x_3, A(x_1^4)) \Leftrightarrow$

$$\alpha x_1 + \alpha x_2 + \alpha x_3 + \alpha_4(\alpha x_1 + \alpha x_2 + \alpha x_3 + \alpha_4 x_4 + c) + c = x_4, \tag{19}$$

which (for $x_1 = x_2 = x_3 = x_4 = 0$) implies $\alpha_4 c + c = 0$. Using the last equality in (19) and taking $x_1 = x_2 = x_4 = 0$, we have $\alpha_4(\alpha(x_3)) = I(\alpha x_3)$, i.e. $a\alpha_4 = I$, hence $T = ((Q, +), \alpha, \alpha, \alpha, I, c)$. But an $4 - T-$quasigroups (Q, A) with such a $T-$form satisfies, for example, the equality $A =^{(12)} A$ and $(12) \notin H_1$, hence $H \neq H_1$.

2) Let $H = H_2 = \langle(124)(35)\rangle = \{\varepsilon, (124)(35), (1142), (35), (124), (142)(35)\}$. As $(124) \in H$, we have $A =^{(124)} A \Leftrightarrow A(x_1^4) = A(x_4, x_1, x_3, x_2)$, which implies

$$\alpha_1 x_1 + \alpha_2 x_2 + \alpha_4 x_4 = \alpha_1 x_4 + \alpha_2 x_1 + \alpha_4 x_2. \tag{20}$$

Taking $x_1 = x_2 = 0$ and respectively $x_1 = x_4 = 0$, in (20), we get $\alpha_1 = \alpha_2 = \alpha_4$.

Denoting $\alpha_1 = \alpha_2 = \alpha_4 = \alpha$, we get $T = ((Q, +), \alpha, \alpha, \alpha_3, \alpha, c)$, i.e. $A(x_1^4) = \alpha x_1 + \alpha x_2 + \alpha_3 x_3 + \alpha x_4 + c$, which implies in particular $A =^{(12)} A$, a contradiction, as $(12) \notin H_2$, so $H \neq H_2$.

3) Let $H = H_3 = \langle(125)(34)\rangle = \{\varepsilon, (125)(34), (152), (34), (125), (152)(34)\}$.

As $(34) \in H_3$ we have $A =^{(34)} A$, which implies $\alpha_3 x_3 + \alpha_4 x_4 = \alpha_3 x_4 + \alpha_4 x_3$, so (taking $x_4 = 0$) $\alpha_3 = \alpha_4$.

Let us denote $\alpha_3 = \alpha_4 = \alpha$ and let us consider $(125) \in H_3$. As $A = ^{(125)}A$, we get $A(x_2, A(x_1^4), x_3, x_4) = x_1$, i.e.

$$\alpha_1 x_2 + \alpha_2(\alpha_1 x_1 + \alpha_2 x_2 + \alpha x_3 + \alpha x_4 + c) + \alpha x_3 + \alpha x_4 + c = x_1, \tag{21}$$

which (for $x_1 = x_2 = x_3 = x_4 = 0$) implies $\alpha_2 c + c = 0$. Using the last equality in (21) and taking $x_1 = x_2 = x_4 = 0$, we get $\alpha_2(\alpha x_3) = I(\alpha x_3)$, i.e. $\alpha_2 = I$. Also, for $x_1 = x_3 = x_4 = 0$, the equality (21) implies

$$\alpha_2^2 x_2 = I(\alpha_1 x_2) \Leftrightarrow x_2 = I(\alpha_1 x_2),$$

i.e. $\alpha_1 = I$, hence $T = ((Q,+), I, I, \alpha, \alpha, c)$. In particular, we obtained $A = ^{(12)}A$, impossible, as $(12) \notin H_3$.

4) Let $H = H_4 = \langle(134)(25)\rangle = \{\varepsilon, (134)(25), (143), (25), (134), (143)(25)\}$. From $(134) \in H_4$ we have $A = ^{(134)}A \Leftrightarrow A(x_1^4) = A(x_4, x_2, x_1, x_3)$, hence

$$\alpha_1 x_1 + \alpha_3 x_3 + \alpha_4 x_4 = \alpha_1 x_4 + \alpha_3 x_1 + \alpha_4 x_3. \tag{22}$$

Taking $x_1 = x_3 = 0$ and, respectively $x_1 = x_4 = 0$, in (22) we get $\alpha_1 = \alpha_4$ and $\alpha_3 = \alpha_4$, so $\alpha_1 = \alpha_3 = \alpha_4$.

Denoting $\alpha_1 = \alpha_3 = \alpha_4 = \alpha$, we obtain the $T-$form $T = ((Q,+), \alpha, \alpha_2, \alpha, \alpha, c)$, so $A(x_1^4) = \alpha x_1 + \alpha_2 x_2 + \alpha x_3 + \alpha x_4 + c$, which implies, for example, $A = ^{(13)}A$, a contradiction as $(13) \notin H_4$, so $H \neq H_4$.

5) Let $H = H_5 = \langle(135)(24)\rangle = \{\varepsilon, (135)(24), (153), (24), (135), (153)(24)\}$. As $(24) \in H_5$ we have $A = ^{(24)}A$, i.e. $\alpha_2 x_2 + \alpha_4 x_4 = \alpha_2 x_4 + \alpha_4 x_2$, which implies $\alpha_2 = \alpha_4$. Let us denote $\alpha_2 = \alpha_4 = \alpha$ and let us consider $A = ^{(135)}A \Leftrightarrow A(A(x_1^4), x_2, x_1, x_4) = x_3 \Leftrightarrow$

$$\alpha_1^2 + (\alpha\alpha_1)x_2 + (\alpha_3\alpha_1)x_3 + (\alpha\alpha_1)x_4 + \alpha_1 c + \alpha x_2 + \alpha_3 x_1 + \alpha x_4 + c = x_3. \tag{23}$$

Taking $x_1 = x_2 = x_3 = 0$ and, respectively $x_2 = x_3 = x_4 = 0$, we obtain $\alpha_1 = \alpha_3 = I$, so $A(x_1^4) = Ix_1 + \alpha x_2 + Ix_3 + \alpha x_4 + c$, which implies $A = ^{(13)}A$, where $(13) \notin H_5$.

6) Let $H = H_6 = \langle(145)(23)\rangle = \{\varepsilon, (145)(23), (154), (23), (145), (154)(23)\}$. From $(23) \in H$, i.e. $A = ^{(23)}A$, we get $\alpha_2 x_2 + \alpha_3 x_3 = \alpha_2 x_3 + \alpha_3 x_2$, so $\alpha_2 = \alpha_3$. Denoting $\alpha_2 = \alpha_3 = \alpha$, it follows $T = ((Q,+), \alpha_1, \alpha, \alpha, \alpha_4, c)$.

Also, $(145) \in H_6$, so $A = ^{(145)}A \Leftrightarrow A(A(x_1^4), x_2, x_3, x_1) = x_4 \Leftrightarrow$

$$\alpha_1^2 + (\alpha\alpha_1)x_2 + (\alpha\alpha_1)x_3 + (\alpha_4\alpha_1)x_4 + \alpha_1 c + \alpha x_2 + \alpha x_3 + \alpha_4 x_4 + c = x_4, \tag{24}$$

which (for $x_1 = x_2 = x_3 = x_4 = 0$) implies $\alpha_1 c + c = 0$.

Using the last equality and taking, respectively, $x_1 = x_2 = x_4 = 0$ and $x_1 = x_2 = x_3 = 0$ in (24), we obtain $\alpha_1 = \alpha_4 = I$, hence $A(x_1^4) = Ix_1 + \alpha x_2 + \alpha x_3 + Ix_4 + c$ which implies, in particular, $A = ^{(14)}A$ – a contradiction, as $(14) \notin H_6$.

7) $H = H_7 = \langle(234)(15)\rangle = \{\varepsilon, (234)(15), (243), (15), (234), (15)(243)\}$.

From $(234) \in H_7$ it follows $A = ^{(234)}A$, hence

$$\alpha_2 x_2 + \alpha_3 x_3 + \alpha_4 x_4 = \alpha_2 x_4 + \alpha_3 x_2 + \alpha_4 x_3. \tag{25}$$

Taking $x_2 = x_3 = 0$ and, respectively $x_2 = x_4 = 0$, in (25) we obtain $\alpha_2 = \alpha_4$ and $\alpha_3 = \alpha_4$, so $\alpha_2 = \alpha_3 = \alpha_4$.

Denoting $\alpha_2 = \alpha_3 = \alpha_4 = \alpha$, we get $A(x_1^4) = \alpha_1 x_1 + \alpha x_2 + \alpha x_3 + \alpha x_4 + c$, which implies, for example, $A =^{(23)}A$ − a contradiction, as $(23) \notin H_7$.

8) $H = H_8 = \langle(14)(235)\rangle = \{\varepsilon, (14)(235), (253), (14), (235), (14)(253)\}$.

As $(14) \in H_8$, i.e. $A =^{(14)}A$, it follows $\alpha_1 x_1 + \alpha_4 x_4 = \alpha_1 x_4 + \alpha_4 x_1$, so $\alpha_1 = \alpha_4$. Also $(235) \in H_8 \Rightarrow A =^{(235)}A \Leftrightarrow A(A(x_1^4), x_2, x_4) = x_3 \Leftrightarrow$

$$\alpha_1 x_1 + \alpha_2(\alpha_1 x_1) + \alpha_2^2(\alpha_3 x_3) + \alpha_2(\alpha_1 x_4) + \alpha_2 c + \alpha_3 x_2 + \alpha_1 x_4 + c = x_3, \qquad (26)$$

which implies $\alpha_2 c + c = 0$. Using the last equality and taking $x_1 = x_2 = x_3 = 0$ and, respectively $x_1 = x_3 = x_4 = 0$, in (26) we get $\alpha_2 = \alpha_3 = I$.

Denoting $\alpha_1 = \alpha_4 = \alpha$ we obtain $A(x_1^4) = \alpha x_1 + I x_2 + I x_3 + \alpha x_4 + c$, which implies, in particular, $A =^{(23)}A$ − a contradiction as $(23) \in H_8$.

9) Let $H = H_9 = \langle(13)(245)\rangle = \{\varepsilon, (13)(245), (254), (13), (245), (13)(254)\}$. As $(13) \in H_9$ we have $A =^{(13)}A$ which implies $\alpha_1 x_1 + \alpha_3 x_3 = \alpha_1 x_3 + \alpha_3 x_1$, hence $\alpha_1 = \alpha_3$. Also, $(245) \in H_9$, so $A =^{(245)}A \Rightarrow$

$$\alpha_1 x_1 + \alpha_2(\alpha_1 x_1) + \alpha_2^2 x_2 + \alpha_2(\alpha_1 x_3) + \alpha_2(\alpha_4 x_4) + \alpha_2 c + \alpha_1 x_3 + \alpha_4 x_2 + c = x_4, \quad (27)$$

which implies $\alpha_2 c + c = 0$. Now, using the last equality and taking $x_1 = x_2 = x_4 = 0$ and, respectively $x_1 = x_3 = x_4 = 0$, in (27) we get $\alpha_2 = \alpha_4 = I$.

Denoting $\alpha_2 = \alpha_4 = I$ we obtain $A(x_1^4) = \alpha x_1 + I x_2 + \alpha x_3 + I x_4 + c$, so $A =^{(24)}A$, which is a contradiction as $(24) \notin H_9$.

10) Let $H = H_{10} = \langle(12)(345)\rangle = \{\varepsilon, (12)(345), (354), (12), (12)(354), (345)\}$. As $(12) \in H_{10}$, i.e. $A =^{(12)}A$, it follows $\alpha_1 x_+ \alpha_2 x_2 = \alpha_1 x_2 + \alpha_2 x_1$, i.e. $\alpha_1 = \alpha_2$.

Also $(345) \in H_{10}$, i.e. $A =^{(345)}A \Leftrightarrow A(x_1, x_2, A(x_1^4), x_3) = x_4 \Leftrightarrow$

$$\alpha_1 x_1 + \alpha_2 x_2 + \alpha_3(\alpha_1 x_1) + \alpha_3(\alpha_2 x_2) + \alpha_3(\alpha_3 x_3) + \alpha_3(\alpha_4 x_4) + \alpha_3 c + \alpha_4 x_3 + c = x_4, \quad (28)$$

which (for $x_1 = x_2 = x_3 = x_4 = 0$) implies $\alpha_3 c + c = 0$. Using the last equality and taking $x_2 = x_3 = x_4 = 0$ and, respectively $x_1 = x_2 = x_3 = 0$, in (28), we get $\alpha_3 = \alpha_4 = I$. Denoting $\alpha_1 = \alpha_2 = \alpha$, we have $A(x_1^4) = \alpha x_1 + \alpha x_2 + I x_3 + I x_4 + c$, which implies in particular $A =^{(34)}A$ − a contradiction, so $H \neq H_{10}$. □

Theorem 3. *There are no* $4-T-$*quasigoups* (Q, A) *such that* $H \in \{H_{21}, H_{22}, \ldots, H_{30}\}$, *where* $H = \{\sigma \in S_5 | A =^{\sigma}A\}$.

Proof. In this case the subgroup H is generated by two substitutions α and β, of order 3 and, respectively 2, where β is a product of two independent transpositions. The proof is analogous to those of Theorem 2. We will prove that each equality $H = K$, where $K \in \{H_{21}, H_{22}, \ldots, H_{30}\}$, leads to a contradiction.

Let (Q, A) be a $4-T-$ quasigroup with the $T-$form $T = ((Q, +), \alpha_1, \alpha_2, \alpha_3, \alpha_4, c)$ and let $H = \{\sigma \in S_5 \mid A = {}^{\sigma}A\}$.

1. If $H = H_{21} = \langle(123), (12)(45)\rangle$ then $(123) \in H$, i.e. $A = {}^{(123)}A$, so

$$\alpha_1 x_1 + \alpha_2 x_2 + \alpha_3 x_3 = \alpha_1 x_3 + \alpha_2 x_1 + \alpha_3 x_2.$$

Taking $x_1 = x_2 = 0$ and respectively, $x_1 = x_3 = 0$, in the previous equality, we get $\alpha_1 = \alpha_2 = \alpha_3$, hence $T = ((Q, +), \alpha_1, \alpha_1, \alpha_1, \alpha_4, c)$ which implies, in particular, $A = {}^{(12)}A$ − a contradiction, as $(12) \notin H$.

2. Let $H = H_{22} = \langle(124), (12)(35)\rangle$, then $(124) \in H$, i.e. $A = {}^{(124)}A$, hence
$$\alpha_1 x_1 + \alpha_2 x_2 + \alpha_4 x_4 = \alpha_1 x_4 + \alpha_2 x_1 + \alpha_4 x_2.$$
From the last equality we get $\alpha_1 = \alpha_2 = \alpha_4$ and $T = ((Q, +), \alpha_1, \alpha_1, \alpha_3, \alpha_1, c)$, which implies $A = {}^{(12)}A$, which is not possible as $(12) \notin H$.

3. If $H = H_{23} = \langle(125), (12)(34)\rangle$ then $(123) \in H$, hence $A = {}^{(12)(34)}A$, which implies $\alpha_1 = \alpha_2$ and $\alpha_3 = \alpha_4$, i.e. $T = ((Q, +), \alpha_1, \alpha_1, \alpha_3, \alpha_3, c)$. In particular, we get $A = {}^{(12)}A$ − a contradiction, as $(12) \notin H$.

4. Let $H = H_{24} = \langle(134), (13)(25)\rangle$. As $(134) \in H$, we have: $A = {}^{(134)}A$, so
$$\alpha_1 x_1 + \alpha_3 x_3 + \alpha_4 x_4 = \alpha_1 x_4 + \alpha_3 x_1 + \alpha_4 x_3.$$
From the previous equality we get $\alpha_1 = \alpha_3 = \alpha_4$, i.e. $T = ((Q, +), \alpha_1, \alpha_2, \alpha_1, \alpha_1, c)$ and, in particular, $A = {}^{(13)}A$ − a contradiction.

5. If $H = H_{25} = \langle(135), (13)(24)\rangle$ then $(13)(24) \in H$, i.e. $A = {}^{(13)(24)}A$, so
$$\alpha_1 x_1 + \alpha_2 x_2 + \alpha_3 x_3 + \alpha_4 x_4 = \alpha_1 x_3 + \alpha_2 x_4 + \alpha_3 x_1 + \alpha_4 x_2,$$
which implies $\alpha_2 = \alpha_4$ and $\alpha_1 = \alpha_3$, hence $T = ((Q, +), \alpha_1, \alpha_2, \alpha_1, \alpha_2, c)$ so, in particular, $A = {}^{(13)}A$, which is not possible as $(13) \notin H$.

6. Let $H = H_{26} = \langle(145), (14)(23)\rangle$. As $(14)(23) \in H$, we have $A = {}^{(13)(24)}A$, so
$$\alpha_1 x_1 + \alpha_2 x_2 + \alpha_3 x_3 + \alpha_4 x_4 = \alpha_1 x_4 + \alpha_2 x_3 + \alpha_3 x_2 + \alpha_4 x_1,$$
which implies $\alpha_1 = \alpha_4$ and $\alpha_2 = \alpha_3$, hence $T = ((Q, +), \alpha_1, \alpha_2, \alpha_2, \alpha_1, c)$ and then $A = {}^{(23)}A$ − a contradiction, as $(23) \notin H$.

7. Let $H = H_{27} = \langle(234), (15)(23)\rangle$. As $(234) \in H$, we have $A = {}^{(234)}A$, so
$$\alpha_2 x_2 + \alpha_3 x_3 + \alpha_4 x_4 = \alpha_2 x_4 + \alpha_3 x_2 + \alpha_4 x_3,$$
which implies $\alpha_2 = \alpha_3 = \alpha_4$, i.e. $T = ((Q, +), \alpha_1, \alpha_2, \alpha_2, \alpha_2, c)$ and, in particular, $A = {}^{(13)}A$, but $(13) \notin H$.

8. If $H = H_{28} = \langle(235), (14)(23)\rangle$ then $(14)(23) \in H$, hence $A = {}^{(14)(23)}A$, which implies $\alpha_1 = \alpha_4$ and $\alpha_2 = \alpha_3$, i.e. $T = ((Q, +), \alpha_1, \alpha_2, \alpha_2, \alpha_1, c)$. In particular, we get $A = {}^{(14)}A$, which is not possible as $(14) \notin H$.

9. If $H = H_{29} = \langle(245), (13)(24)\rangle$ then $(13)(24) \in H$, hence $A = {}^{(13)(24)}A$, which implies $\alpha_1 = \alpha_3$ and $\alpha_2 = \alpha_4$, i.e. $T = ((Q, +), \alpha_1, \alpha_2, \alpha_1, \alpha_2, c)$. In particular, we get $A = {}^{(13)}A$ which is impossible as $(13) \notin H$.

10. Let $H = H_{30} = \langle(345), (12)(34)\rangle$ then $(12)(34) \in H$, hence $A = {}^{(12)(34)}A$, which implies $\alpha_1 = \alpha_2$ and $\alpha_3 = \alpha_4$, i.e. $T = ((Q, +), \alpha_1, \alpha_1, \alpha_2, \alpha_2, c)$. In particular, we get $A = {}^{(12)}A$ − a contradiction, as $(12) \notin H$.

□

An example of a $4-T-$quasigroup with exactly 20 distinct parastrophes is (Z_5, A), where $A(x_1, x_2, x_3, x_4) = x_1 + x_2 + x_3 + 2x_4$. In this case $H = \{\sigma \in S_5 \mid A = {}^{\sigma}A\} = H_{12} = \langle(124), (12)\rangle$. A maximum set of its distinct parastrophes is given by any set of representatives of cosets by H, for example,

1. $A(x_1^4) = x_1 + x_2 + x_3 + 2x_4$;
2. ${}^{(14)}A(x_1^4) = x_1 + x_2 + 2x_3 + x_4$;
3. ${}^{(24)}A(x_1^4) = x_1 + 2x_2 + x_3 + x_4$;
4. ${}^{(34)}A(x_1^4) = 2x_1 + x_2 + x_3 + x_4$;
5. ${}^{(15)}A(x_1^4) = x_1 + 4x_2 + 4x_3 + 3x_4$;
6. ${}^{(25)}A(x_1^4) = x_1 + 4x_2 + 3x_3 + 4x_4$;

7. $^{(35)}A(x_1^4) = x_1 + 3x_2 + 4x_3 + 4x_4$; 8. $^{(45)}A(x_1^4) = 4x_1 + x_2 + 4x_3 + 3x_4$;
9. $^{(145)}A(x_1^4) = 4x_1 + x_2 + 3x_3 + 4x_4$; 10. $^{(14)(25)}A(x_1^4) = 4x_1 + 4x_2 + x_3 + 3x_4$;
11. $^{(15)(24)}A(x_1^4) = 4x_1 + 4x_2 + 3x_3 + x_4$; 12. $^{(15)(34)}A(x_1^4) = 4x_1 + 3x_2 + x_3 + 4x_4$;
13. $^{(14)(35)}A(x_1^4) = 4x_1 + 3x_2 + 4x_3 + x_4$; 14. $^{(154)}A(x_1^4) = 3x_1 + x_2 + 4x_3 + 4x_4$;
15. $^{(245)}A(x_1^4) = 3x_1 + 4x_2 + x_3 + 4x_4$; 16. $^{(24)(35)}A(x_1^4) = 3x_1 + 4x_2 + 4x_3 + x_4$;
17. $^{(254)}A(x_1^4) = 3x_1 + 2x_2 + 2x_3 + 2x_4$; 18. $^{(25)(34)}A(x_1^4) = 2x_1 + 3x_2 + 2x_3 + 2x_4$;
19. $^{(345)}A(x_1^4) = 2x_1 + 2x_2 + 3x_3 + 2x_4$; 20. $^{(354)}A(x_1^4) = 2x_1 + 2x_2 + 2x_3 + 3x_4$.

Corollary 2. *There exist $4-T-$quasigroups with exactly 20 distinct parastrophes of any odd order $q \geq 5$, where $(q,3) = 1$.*

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Tatiana Rotari
Alecu Russo Balti State University,
Chair of Mathematics and Science
E-mail: *tatiana.rotari@usarb.md*

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Parascovia Syrbu
Moldova State University,
Department of Mathematics
E-mail: *parascovia.syrbu@gmail.com*