BULETINUL ACADEMIEI DE ŞTIINŢE
A REPUBLICII MOLDOVA. MATEMATICA
Numbers 3(106), 2024, Pages 71–102
ISSN 1024–7696, E-ISSN 2587–4322

Remotely Almost Periodic Motions of Dynamical Systems

David Cheban

Abstract. The aim of this paper is studying the remotely almost periodic motions of dynamical systems and solutions of differential equations. We establish some properties of remotely almost periodic motions. The notion of remote comparability (comparability at the infinity) by the character of recurrence for remotely almost periodic motions is introduced. This notion plays a very important role in the study of the remotely almost periodic motions.

Mathematics subject classification: 37B20.
Keywords and phrases: almost periodic, asymptotically almost periodic, remotely almost periodic motions.

1 Introduction

The aim of this paper is studying the remotely almost periodic motions of dynamical systems. This study continues the author's series of works devoted to the study of asymptotically almost periodic motions of dynamical systems [2].

The notion of remotely almost periodicity (on the real axis $\mathbb{R} := (-\infty, +\infty)$) for scalar function was introduced and studied by Donald Sarason [9]. Remotely almost periodic functions on the semi-axis $\mathbb{R}_+ := [0, +\infty)$ with values in the Banach space were introduced by Ruess W. M. and Summers W. H. [8] (see also Baskakov A. G. [1]). Remotely almost periodic motions for dynamical systems were introduced by Ruess W. M. and Summers W. H. [8].

In this paper we study systematically the remotely almost periodic motions on the semi-axis (both positive semi-axis $\mathbb{R}_+$ and negative semi-axis $\mathbb{R}_-$) and on the real axis $\mathbb{R}$. We introduce and study the important notion of the comparability by the character of recurrence at the infinity for the remotely almost periodic motions.

The paper is organized as follows. In the second Section, we collect some known notions and facts that we use in this paper.

In the third Section we study the remotely stationary (respectively, remotely τ-periodic and remotely stationary) motions and the relation between them.

The fourth Section is dedicated to the important notion of the remote comparability (comparability at the infinity) by the character of recurrence for remotely almost periodic motions. We show that if the given motion $\pi(t, x)$ is remotely comparable by the character of recurrence with the remotely stationary (respectively,

DOI: https://doi.org/10.56415/basm.y2024.i3.p71

remotely τ-periodic or remotely almost periodic) motion $\sigma(t, y)$, then the motion $\pi(t, x)$ is also remotely stationary (respectively, remotely τ-periodic or remotely almost periodic).

In the fifth Section we introduce and study the notion of equi-almost periodicity for the compact invariant sets of given dynamical system. We establish the relation between the equi-almost periodicity of compact invariant set and almost periodicity of some abstract function associated with given compact invariant set.

The sixth Section is dedicated to the study of the relation between two-sided remotely almost periodic motions and one-sided remotely almost periodic motions. We show that the given motion of the two-sided dynamical system is remotely almost periodic on the real axis $\mathbb{R}$ (it is two-sided remotely almost periodic) if and only if it is one-sided remotely almost periodic on the positive semi-axis $\mathbb{R}_+$ and on the negative semi-axis $\mathbb{R}_-$.

In the next publications we plane to study:

1. different classes of remotely almost periodic functions and their relations with the shift dynamical systems and the remotely almost periodic motions on them;

2. remotely almost periodic solutions of different classes of differential equations.

2 Preliminary

Let X and Y be two complete metric spaces, let $\mathbb{R} := (-\infty, +\infty)$, $\mathbb{Z} := \{0, \pm 1, \pm 2, \dots\}$, $\mathbb{S} = \mathbb{R}$ or $\mathbb{Z}$, $\mathbb{S}_+ = \{t \in \mathbb{S} | \quad t \geq 0\}$ and $\mathbb{S}_- = \{t \in \mathbb{S} | \quad t \leq 0\}$. Let $\mathbb{T} = \mathbb{S}$ or $\mathbb{S}_+$, $(X, \mathbb{S}_+, \pi)$ (respectively, $(Y, \mathbb{S}, \sigma)$) be an autonomous one-sided (respectively, two-sided) dynamical system on X (respectively, on Y).

Let $(X, \mathbb{T}, \pi)$ be a dynamical system and $\pi(t, x) = \pi^t x = xt$.

Definition 1. A point $x \in X$ (respectively, a motion $\pi(t, x)$) is said to be:

– stationary if $\pi(t, x) = x$ for any $t \in \mathbb{T}$;

– τ-periodic ($\tau > 0$ and $\tau \in \mathbb{T}$) if $\pi(\tau, x) = x$;

– asymptotically stationary (respectively, asymptotically τ-periodic) if there exists a stationary (respectively, τ-periodic) point $p \in X$ such that

$$\lim_{t \to \infty} \rho(\pi(t, x), \pi(t, p)) = 0. \tag{1}$$

Remark 1. Note that the almost periodic point p in the relation (1) is defined uniquely (see, for example, [4, Ch.I]).

Theorem 1. *[2, Ch.I] A point $x \in X$ is asymptotically τ-periodic if and only if the sequence $\{\pi(k\tau, x)\}_{k=0}^{\infty}$ converges.*

Definition 2. A point $\tilde{x} \in X$ is said to be ω-limit for $x \in X$ if there exists a sequence $\{t_k\} \subset \mathbb{S}_+$ such that $t_k \to +\infty$ and $\pi(t_k, x) \to \tilde{x}$ as $k \to \infty$.

Denote by ω_x the set of all ω-limit points of $x \in X$.

Definition 3. We will call a point $x \in X$ (respectively, a motion $\pi(t,x)$) remotely τ-periodic ($\tau \in \mathbb{T}$ and $\tau > 0$) if

$$\lim_{t\to+\infty} \rho(\pi(t+\tau,x),\pi(t,x)) = 0. \tag{2}$$

Lemma 1. *Let $x \in X$ be a remotely τ-periodic point then for any $\tau^{'} \in \{k\tau |\ k \in \mathbb{Z}\}\bigcap \mathbb{T}$ we have*

$$\lim_{t\to+\infty} \rho(\pi(t+\tau',x),\pi(t,x)) = 0. \tag{3}$$

Proof. Assume that this statement is not true, then there exist $\varepsilon_0 > 0$, $\tau_0' = k_0\tau \in \mathbb{T}$ and $t_n \to +\infty$ ($t_n \in \mathbb{T}$) as $n \to \infty$ such that

$$\rho(\pi(t_n + k_0\tau, x), \pi(t_n,x)) \ge \varepsilon_0$$

for any $n \in \mathbb{N}$.

On the other hand we have

$$\varepsilon_0 \le \rho(\pi(t_n + k_0\tau,x),\pi(t_n,x)) \le \tag{4}$$
$$\sum_{m=1}^{k_0} \rho(\pi(t_n + m\tau,x),\pi(t_n+(m-1)\tau,x)) = \sum_{m=1}^{k_0} \rho(\pi(t_n^m+\tau,x),\pi(t_n^m,x)),$$

where $t_n^m := t_n + (m-1)\tau$ for any $n \in \mathbb{N}$ and $m = 1,\dots,k_0$. Note that $t_n^m \to +\infty$ as $n \to \infty$ ($m = 1,\dots,k_0$). Passing to the limit in (4) as $n \to \infty$ and taking into account (3) we obtain $\varepsilon_0 \le 0$. The last relation contradicts the choice of ε_0. The obtained contradiction proves our statement. Lemma is proved. □

Remark 2. If the point $x \in X$ is remotely τ-periodic, then it is remotely τ'-periodic for any $\tau' \in \{\tau k |\ k \in \mathbb{Z}\}\bigcap\mathbb{T}$.

Remark 3. The motions of dynamical systems possessing the property (2) were studied in the works of K. Cryszka [5] and A. Pelczar [7].

Definition 4. A point x is called positively Lagrange stable if the semi-trajectory $\Sigma_x^+ := \{\pi(t,x) |\ t \in \mathbb{S}_+\}$ is a precompact subset of X.

Theorem 2. *[4, Ch.I] Let $x \in X$ be positively Lagrange stable and $\tau \in \mathbb{T}$ ($\tau > 0$). Then the following statements are equivalent:*

- *a. the motion $\pi(t,x)$ is remotely τ-periodic;*
- *b. any point $p \in \omega_x$ is τ-periodic.*

Definition 5. A point x (respectively, a motion $\pi(t,x)$) is said to be remotely stationary if it is remotely τ-periodic for any $\tau \in \mathbb{T}$.

Corollary 1. *Let $x \in X$ be positively Lagrange stable. Then the following statements are equivalent:*

a. *the motion* $\pi(t, x)$ *is remotely stationary;*

b. *any point* $p \in \omega_x$ *is stationary.*

Proof. This statement follows directly from the corresponding definition and Theorem 2. □

Corollary 2. *Every asymptotically* τ*-periodic (respectively, asymptotically stationary) point* $x \in X$ *is remotely* τ*-periodic (respectively, remotely stationary).*

Proof. Assume that the point $x \in X$ is asymptotically τ-periodic (respectively, asymptotically stationary) then there exist a (unique) τ-periodic (respectively, stationary) point $p \in X$ such that (1) holds and, consequently, the point x is positively Lagrange stable and $\omega_x = \omega_p$. The last equality means that the set ω_x is minimal and consists of τ-periodic (respectively, stationary) points. By Corollary 1 the point x is remotely τ-periodic (respectively, remotely stationary). □

Definition 6. A subset $A \subseteq \mathbb{T}$ is said to be relatively dense in $\mathbb{T}$ if there exists a positive number $l \in \mathbb{T}$ such that $[a, a+l] \bigcap A \neq \emptyset$ for any $a \in \mathbb{T}$, where $[a, a+l] := \{x \in \mathbb{T} |\ a \leq x \leq a+l\}$.

Lemma 2. *Assume that a subset* $\mathcal{P}^{\pm} \subseteq \mathbb{S}^{\pm}$ *is relatively dense in* $\mathbb{S}^{\pm}$*. Then the set* $\mathcal{P} := \mathcal{P}^{-} \bigcup \mathcal{P}^{+} \subseteq \mathbb{S}$ *is relatively dense in* $\mathbb{S}$*.*

Proof. Since the set $\mathcal{P}^{\pm} \subseteq \mathbb{S}^{\pm}$ is relatively dense in $\mathbb{S}^{\pm}$ then there is a positive number $\ell_{\pm}$ such that in any segment $[a', b'] \subseteq \mathbb{S}^{\pm}$ of length $\ell_{\pm}$ at least one point from $\mathbb{S}_{\pm}$ is contained. Denote by $\ell := \ell_{-} + \ell_{+}$ and we will show that any segment $[a, b] \subset \mathbb{S}$ of length ℓ contains at least one point from $\mathbb{S}$. Let $c \in \mathbb{S}$ be number such that $a < c < b$, $[a, b] = [a, c] \bigcup [c, b]$ and the segment $[a, c]$ (respectively, $[c, b]$) has the length ℓ_{-} (respectively, ℓ_{+}).

If $a \geq 0$ (respectively, $b \leq 0$), then the segment $[a, b]$ contains at least one point from $\mathbb{S}_{+}$ (respectively, $\mathbb{S}_{-}$) and, consequently, $[a, b] \bigcap \mathbb{S} \neq \emptyset$.

If $a < 0 < b$, then logically the following two cases are possible:

1. $c \geq 0$. In this case we have the segment $[c, b] \subset \mathbb{S}_{+}$ of the length ℓ_{+} which contains at least one point from $\mathcal{P}_{+}$ and, consequently, $\mathcal{P} \bigcap [a, b] \neq \emptyset$.

2. $c < 0$. In this case the segment $[a, c] \subset \mathbb{S}_{-}$ of the length ℓ_{-} contains at least one point from $\mathcal{P}_{-}$ and, consequently, $\mathcal{P} \bigcap [a, b] \neq \emptyset$.

Thus for the set $\mathcal{P}$ there exists a positive number ℓ $(= \ell_{-} + \ell_{+})$ such that every segment $[a, b]$ (from $\mathbb{S}$) of the length ℓ is such that $[a, b] \bigcap \mathcal{P} \neq \emptyset$. This means that the set $\mathcal{P} \subset \mathbb{S}$ is relatively dense in $\mathbb{S}$. Lemma is proved. □

Definition 7. A point $x \in X$ of the dynamical system $(X, \mathbb{T}, \pi)$ is said to be:

1. almost periodic if for any $\varepsilon > 0$ the set

$$\mathcal{P}(\varepsilon, p) := \{\tau \in \mathbb{T} |\ \rho(\pi(t+\tau, p), \pi(t, p)) < \varepsilon \ \text{ for any } t \in \mathbb{T}\}$$

is relatively dense in $\mathbb{T}$;

2. positively Poisson stable if $x \in \omega_x$;
3. asymptotically stationary (respectively, asymptotically τ-periodic, asymptotically almost periodic or positively asymptotically Poisson stable) if there exists a stationary (respectively, τ-periodic, almost periodic or positively Poisson stable) point $p \in X$ such that

$$\lim_{t\to\infty} \rho(\pi(t,x),\pi(t,p)) = 0. \tag{5}$$

Definition 8. A subset $M \subseteq X$ is said to be positively invariant (respectively, negatively invariant or invariant) if $\pi(t,M) \subseteq M$ (respectively, $M \subseteq \pi(t,M)$ or $\pi(t,M) = M$) for any $t \in \mathbb{T}$.

Theorem 3. *[4,11] Assume that the point $x \in X$ is positively Lagrange stable, then the following statement hold:*

1. *$\omega_x \neq \emptyset$;*
2. *ω_x is a compact subset of X;*
3. *the set ω_x is invariant, that is, $\pi(t,\omega_x) = \omega_x$ for any $t \in \mathbb{S}_+$.*

3 Remotely Almost Periodic Motions of Dynamical Systems on the Semi-Axis

Remark 4. For any $\tau > 0$ ($\tau \in \mathbb{T}$) the set $A := \{k\tau |\ k \in \mathbb{Z}\} \bigcap \mathbb{T}$ is relatively dense in $\mathbb{T}$.

Definition 9. A point $x \in X$ (respectively, a motion $\pi(t,x)$) is said to be remotely almost periodic [8] if for an arbitrary positive number ε there exists a relatively dense subset $\mathcal{P}(\varepsilon,x) \subseteq \mathbb{T}$ such that for any $\tau \in \mathcal{P}(\varepsilon,x)$ there exists a number $L(\varepsilon,x,\tau) > 0$ for which we have

$$\rho(\pi(t+\tau,x),\pi(t,x)) < \varepsilon$$

for any $t \geq L(\varepsilon,x,\tau)$.

Remark 5. Every almost periodic point $x \in X$ is remotely almost periodic.

Lemma 3. *Every remotely τ-periodic (respectively, remotely stationary) point x of the dynamical system $(X,\mathbb{T},\pi)$ is remotely almost periodic.*

Proof. Let ε be an arbitrary positive number. Denote by $\mathcal{P}(\varepsilon) := \{k\tau |\ k \in \mathbb{Z}\} \bigcap \mathbb{T}$. If $\tau' \in \mathcal{P}(\varepsilon)$, then there exists $k_0 \in \mathbb{Z}$ such that $\tau' = k_0\tau \in \mathbb{T}$. By Lemma 1 for given ε and $\tau' = k_0\tau$ there exists a positive number $L(\varepsilon,\tau)$ such that

$$\rho(\pi(t+\tau',x),\pi(t,x)) < \varepsilon$$

for any $t \geq L(\varepsilon,\tau')$. □

Lemma 4. *A point x is remotely τ-periodic (respectively, remotely stationary) if and only if for any $\varepsilon > 0$ there exists a relatively dense in $\mathbb{T}$ subset $\mathcal{P}(x,\varepsilon)$ such that*

1. *for any $\tau \in \mathcal{P}(x,\varepsilon)$ there exists a number $L(x,\varepsilon,\tau) > 0$ for which we have*
$$\rho(\pi(t+\tau,x),\pi(t,x)) < \varepsilon$$
for any $t \geq L(x,\varepsilon,\tau)$ and

2. *$\{\tau\mathbb{Z}\} \subset \mathcal{P}(x,\varepsilon)$ (respectively, $\mathbb{T} \subseteq \mathcal{P}(x,\varepsilon)$).*

Proof. This statement follows directly from the corresponding definitions. □

Lemma 5. *If the point $x \in X$ is asymptotically almost periodic, then it is remotely almost periodic.*

Proof. Assume that the point $x \in X$ is asymptotically almost periodic, then there exists a unique almost periodic point $p_x \in X$ such that

$$\lim_{t\to+\infty} \rho(\pi(t,x),\pi(t,p_x)) = 0. \tag{6}$$

Let ε be an arbitrary positive number, then from (6) there exists a positive number $L(\varepsilon,x)$ such that
$$\rho(\pi(t,x),\pi(t,p_x)) < \varepsilon$$
for any $t \geq L(\varepsilon,x)$. On the other hand by almost periodicity of $p_x \in X$ for given $\varepsilon > 0$ the set

$$\mathcal{P}(\varepsilon,p_x) := \{\tau \in \mathbb{T}|\ \rho(\pi(t+\tau,p_x),\pi(t,p_x)) < \varepsilon \text{ for any } t \in \mathbb{T}\}$$

is relatively dense in $\mathbb{T}$. Denote by $\mathcal{P}(\varepsilon,x) := \mathcal{P}(\varepsilon/3,p_x)$. Let $\tau \in \mathcal{P}(\varepsilon,x)$ and $L(\varepsilon,x,\tau) := L(\varepsilon/3,x) + |\tau|$. Note that if $t \geq L(\varepsilon,x,\tau)$, then $t+\tau \geq L(\varepsilon/3,x)$ and

$$\rho(\pi(t+\tau,x),\pi(t,x)) \leq \rho(\pi(t+\tau,x),\pi(t+\tau,x)) + \rho(\pi(t+\tau,p_x),\pi(t,p_x)) +$$
$$\rho(\pi(t,p_x),\pi(t,x)) < \varepsilon/3 + \varepsilon/3 + \varepsilon/3 = \varepsilon,$$

i.e., the point x is remotely almost periodic. Lemma is proved. □

Lemma 6. *Let $(X,\mathbb{T},\pi)$ and $(Y,\mathbb{T},\sigma)$ be two dynamical systems and $x \in X$ (respectively, $y \in Y$). Assume that the following conditions hold:*

1. *the point $y \in Y$ is remotely almost periodic (respectively, remotely τ-periodic, remotely stationary);*

2. *there exists a uniformly continuous mapping $h : \Sigma_y \to \Sigma_x$ satisfying the conditions*
$$h(\sigma(t,y)) = \pi(t,x)$$
for any $t \in \mathbb{T}$.

Then the point $x \in X$ is also remotely almost periodic (respectively, remotely τ-periodic, remotely stationary).

Proof. Let ε be an arbitrary positive number and $\delta = \delta(\varepsilon) > 0$ be the positive number from the uniform continuity of the map h, i.e.,

$$\rho(\sigma(t_1, y), \sigma(t_2, y)) < \delta \ \text{ implies } \ \rho(\pi(t_1, x), \pi(t_2, x)) < \varepsilon \ (t_1, t_2 \in \mathbb{T}).$$

Since the point y is remotely almost periodic then for the number $\delta > 0$ there exists a relatively dense in $\mathbb{T}$ subset $\mathcal{P}(y, \delta)$ such that for any $\tau \in \mathcal{P}(y, \delta)$ there exists a number $L(y, \delta, \tau) > 0$ for which we have

$$\rho(\sigma(t + \tau, y), \sigma(t, y)) < \delta$$

and, consequently,

$$\rho(\pi(t + \tau, x), \pi(t, x)) < \varepsilon \tag{7}$$

for any $t \geq L(y, \delta, \tau)$. Denote by $\mathcal{P}(x, \varepsilon) := \mathcal{P}(y, \delta(\varepsilon))$ and for $\tau \in \mathcal{P}(x, \varepsilon)$ we put $L(x, \varepsilon, \tau) := L(y, \delta(\varepsilon), \tau)$, then (7) means that the point x is remotely almost periodic.

Let now the point $y \in Y$ be remotely τ-periodic, ε be an arbitrary positive number and $\delta = \delta(\varepsilon) > 0$ as above. We have

$$\lim_{t \to \infty} \rho(\sigma(t + \tau, y), \sigma(t, y)) = 0$$

and, consequently, there exists a positive number $L(y, \delta)$ such that

$$\rho(\sigma(t + \tau, y), \sigma(t, y)) < \delta$$

for any $t \geq L(y, \delta)$ and, consequently,

$$\rho(\pi(t + \tau, x), \pi(t, x)) < \varepsilon \tag{8}$$

for any $t \geq L(x, \varepsilon) := L(y, \delta(\varepsilon))$. On the other hand by Lemma 4 we have

$$\{\tau\mathbb{Z}\} \subset \mathcal{P}(y, \delta(\varepsilon)) = \mathcal{P}(x, \varepsilon). \tag{9}$$

The relations (8)-(9) (see Lemma 4) mean that the point x is remotely τ-periodic.

Let now the point $y \in Y$ be remotely stationary, then it is remotely τ-periodic for every $\tau \in \mathbb{T}$. Now we prove that under the conditions of Lemma 6 the point x is remotely stationary. Applying Lemma 4 (item (ii)) we conclude that the point x is remotely stationary. □

Lemma 7. *Let M (respectively, N) be a positively invariant subset of $(X, \mathbb{T}, \pi)$ (respectively, $(Y, \mathbb{T}, \sigma)$. Assume that the following conditions hold:*

1. *there exists a uniformly continuous mapping $h : N \to M$;*
2. *$h(\sigma(t, q)) = \pi(t, h(p))$ for any $(t, q) \in \mathbb{T} \times N$.*

Then the mapping h admits a unique continuous extension $h : \overline{N} \to \overline{M}$ with the property

$$h(\sigma(t,q)) = \pi(t, h(p))$$

for any $(t,q) \in \mathbb{T} \times \overline{N}$.

Proof. Let $q \in \overline{N}$ be an arbitrary point then there exists a sequence $\{q_n\} \subset M$ such that $q_n \to q$ as $n \to \infty$. Denote by $p_n := h(q_n)$ and note that by uniform continuity of the map h the sequence $\{p_n\}$ is fundamental. Since the metric space X is complete then the sequence $\{p_n\}$ converges. Denote by p its limit and we put

$$h(q) := \lim_{n\to\infty} h(q_n). \tag{10}$$

It is known (see, for example, Theorem 26 [6, Ch.VI, p.195]) that the mapping $h : \overline{N} \to \overline{M}$ defined above is uniquely defined and continuous.

To finish the proof of lemma we note that by (10) we have

$$h(\sigma(t,q)) = h(\sigma(t, \lim_{n\to\infty} q_n)) = \lim_{n\to\infty} h(\sigma(t,q_n)) =$$
$$\lim_{n\to\infty} \pi(t, h(q_n)) = \pi(t, \lim_{n\to\infty} h(q_n)) = \pi(t, h(q))$$

for any $(t,q) \in \mathbb{T} \times \overline{N}$. Lemma is completely proved. $\square$

Corollary 3. *Assume that the following conditions hold:*

1. *the point $y \in Y$ is Lagrange stable and remotely almost periodic (respectively, remotely τ-periodic, remotely stationary);*
2. *there exists a continuous mapping $h : H(y) \to H(x)$ satisfying the conditions*

$$h(y) = x \quad and \quad h(\sigma(t,q)) = \pi(t, h(q)) \tag{11}$$

for any $(t,q) \in \mathbb{T} \times H(y)$.

Then the point $x \in X$ is also remotely almost periodic (respectively, remotely τ-periodic, remotely stationary).

Proof. Since the set $H(y)$ is compact then the mapping $h : H(y) \to H(x)$ is uniformly continuous and by (11) we have $h(\sigma(t,y)) = \pi(t,x)$ for any $t \in \mathbb{T}$. Now to finish the proof of Lemma it suffices to apply Lemmas 4 and 7. $\square$

Lemma 8. *Assume that the following conditions hold:*

1. *the point $y \in Y$ is asymptotically stationary (respectively, asymptotically τ-periodic, asymptotically almost periodic);*
2. *there exists a continuous mapping $h : H(y) \to H(x)$ satisfying the conditions*

$$h(y) = x \quad and \quad h(\sigma(t,q)) = \pi(t, h(q)) \tag{12}$$

for any $(t,q) \in \mathbb{T} \times H(y)$.

Then the point $x \in X$ is also asymptotically stationary (respectively, asymptotically τ-periodic, asymptotically almost periodic).

Proof. By condition (i) of lemma there exists a stationary (respectively, τ-periodic, almost periodic) $q \in Y$ such that

$$\lim_{t \to +\infty} \rho(\sigma(t, y), \sigma(t, q)) = 0 \tag{13}$$

and, consequently, $q \in \omega_q \subseteq H^+(y)$. By (13) the set $H^+(y)$ is compact and, hence, the mapping $h : H^+(y) \to H^+(x)$ is uniformly continuous. According to (12) the point $p := h(q)$ is stationary (respectively, τ-periodic, almost periodic). Since the mapping $h : H^+(y) \to H^+(x)$ is uniformly continuous then from (13) we obtain

$$\lim_{t \to +\infty} \rho(\pi(t, x), \pi(t, p)) = 0.$$

Lemma is proved. □

Lemma 9. *Let (X, ρ) (respectively, (Y, d)) be a metric space and $(X, \mathbb{T}, \pi)$ (respectively, $(Y, \mathbb{T}, \sigma)$) be a dynamical system on X (respectively, on Y). Assume that the following conditions are fulfilled:*

1. *there exist a homomorphism h from $(Y, \mathbb{T}, \sigma)$ into $(X, \mathbb{T}, \pi)$;*
2. *the mapping $h : Y \to X$ is uniformly continuous.*

Then the following statements hold:

1. *if the point y is asymptotically stationary (respectively, asymptotically τ-periodic or asymptotically almost periodic), then the point $x := h(y)$ is so;*
2. *if the point y is remotely stationary (respectively, remotely τ-periodic or remotely almost periodic), then the point $x = h(y)$ is so;*
3. *if the point $y \in Y$ is stationary (respectively, τ-periodic or almost periodic), then the point $x = h(y)$ is so.*

Proof. The first (respectively, the second) statement directly follows from Lemma 8 (respectively, from Lemma 6).

Let now the point $y \in Y$ be almost periodic (respectively, τ-periodic or stationary) and ε be an arbitrary positive number. Denote by $\delta = \delta(\varepsilon) > 0$ a positive number from the uniform continuity of the map h. Since the point y is almost periodic (respectively, τ-periodic or stationary) then there exist a relatively dense in $\mathbb{T}$ subset $\mathcal{P}(y, \delta)$ (respectively, relatively dense subset $\mathcal{P}(y, \delta)$ with $\{\tau\mathbb{Z}\} \bigcap \mathbb{T} \subseteq \mathcal{P}(y, \delta)$ or $\mathcal{P}(y, \delta)$ with $\{\tau\mathbb{Z}\} \bigcap \mathbb{T} \subseteq \mathcal{P}(y, \delta)$ for any $\tau \in \mathbb{T}$). Denote by $\mathcal{P}(x, \varepsilon) := \mathcal{P}(y, \delta(\varepsilon))$ then we have

$$\rho(\pi(t + \tau, x), \pi(t, x)) = \rho(h(\sigma(t + \tau, y)), h(\sigma(t, y)) < \varepsilon \tag{14}$$

for any $(t, \tau) \in \mathbb{T} \times \mathcal{P}(x, \varepsilon)$ and $\{\tau\mathbb{Z}\} \bigcap \mathbb{T} \subseteq \mathcal{P}(y, \delta)$ and $\{\tau\mathbb{Z}\} \bigcap \mathbb{T} \subseteq \mathcal{P}(y, \delta)$ (respectively, $\{\tau\mathbb{Z}\} \bigcap \mathbb{T} \subseteq \mathcal{P}(y, \delta)$ for any $\tau \in \mathbb{T}$). Lemma is proved. □

Corollary 4. *Let (X,ρ) (respectively, (Y,d)) be a metric space and $(X,\mathbb{T},\pi)$ (respectively, $(Y,\mathbb{T},\sigma)$) be a dynamical system on X (respectively, on Y). Assume that the following conditions are fulfilled:*

1. *there exists a homeomorphism h from $(Y,\mathbb{T},\sigma)$ onto $(X,\mathbb{T},\pi)$;*
2. *the mappings $h : Y \to X$ and $h^{-1} : X \to Y$ are uniformly continuous.*

Then the following statements hold:

1. *the point y is asymptotically stationary (respectively, asymptotically τ-periodic or asymptotically almost periodic) if and only if the point $x := h(y)$ is so;*
2. *the point y is remotely stationary (respectively, remotely τ-periodic or remotely almost periodic) if and only the point $x = h(y)$ is so;*
3. *the point $y \in Y$ is stationary (respectively, τ-periodic or almost periodic) if and only the point $x = h(y)$ is so.*

Definition 10. A subset M is said to be equi-almost periodic if for any $\varepsilon > 0$ there exists a relatively dense subset $\mathcal{P}(\varepsilon, M)$ such that

$$\rho(\pi(t+\tau,p),\pi(t,p)) < \varepsilon \tag{15}$$

for any $t \in \mathbb{T}$, $\tau \in \mathcal{P}(\varepsilon, M)$ and $p \in M$.

Lemma 10. *Let M be a compact invariant minimal set consisting of almost periodic motions. Then the set M is equi-almost periodic.*

Proof. Let ε be an arbitrary positive number. We fixe a point $p_0 \in M$. By almost periodicity of p_0 the set

$$\mathcal{P}(\varepsilon,p_0) := \{\tau \in \mathcal{T} |\ \rho(\pi(t+\tau,p_0),\pi(t,p_0)) < \varepsilon \text{ for any } t \in \mathbb{T}\}$$

is relatively dense in $\mathbb{T}$. Denote by $\mathcal{P}(\varepsilon, M) := \mathcal{P}(\varepsilon/2, p_0)$. Let $p \in M$ be an arbitrary point, then by the minimality of M there exists a sequence $\{t_n\} \subset \mathbb{T}$ such that $\pi(t_n,p_0) \to p$ as $n \to \infty$. From above and (15) we obtain

$$\rho(\pi(\tau,\pi(t_n,p_0)),\pi(t_n,p_0)) = \rho(\pi(t_n+\tau,p_0),\pi(t_n,p_0)) < \varepsilon/2 \tag{16}$$

for any $\tau \in \mathcal{P}(\varepsilon, M)$ and $n \in \mathcal{N}$. Passing to the limit in (16) as $n \to \infty$ we obtain

$$\rho(\pi(\tau,p),p) \le \varepsilon/2 < \varepsilon \tag{17}$$

for any $p \in M$ and $\tau \in \mathcal{P}(\varepsilon, M)$. Since the set M is invariant, then $\pi(t,p) \in M$ for any $t \in M$ and, consequently, from (17) we obtain

$$\rho(\pi(t+\tau,p),\pi(t,p)) < \varepsilon$$

for any $(t,p) \in \mathbb{T} \times M$. Lemma is proved. □

Lemma 11. *Let $x \in X$ be a positively Lagrange stable point of the dynamical system $(X, \mathbb{T}, \pi)$.*

Then the following statements are equivalent:

1. *motion $\pi(t,x)$ is remotely almost periodic;*

2. *the ω-limit set ω_x of the motion $\pi(t,x)$ is equi-almost periodic.*

Proof. Let $\pi(t,x)$ be a remotely almost motion, then for an arbitrary positive number ε there exists a relatively dense in $\mathbb{T}$ subset $\mathcal{P}(\varepsilon,x)$ with the property that for any $\tau \in \mathcal{P}(\varepsilon/2,x)$ there exists a positive number $L(\varepsilon,x,\tau)$ such that

$$\rho(\pi(t+\tau,x),\pi(t,x)) < \varepsilon/2 \tag{18}$$

for any $t \geq L$. Denote by $\mathcal{P}(\varepsilon,\omega_x) := \mathcal{P}(\varepsilon/2,x)$ and we will show that

$$\rho(\pi(t+\tau,p),\pi(t,p)) < \varepsilon \tag{19}$$

for any $t \in \mathbb{T}$, $\tau \in \mathcal{P}(\varepsilon,\omega_x)$ and $p \in \omega_x$. Note that under the conditions of Lemma 11 the set ω_x is invariant (see Theorem 3) and, consequently, to show (19) it suffices to prove that

$$\rho(\pi(\tau,p),p) < \varepsilon$$

for any $\tau \in \mathcal{P}(\varepsilon,\omega_x)$ and $p \in \omega_x$.

Let now p be an arbitrary element from ω_x, then there exists a sequence $t_n \to +\infty$ as $n \to \infty$ such that

$$p = \lim_{n\to\infty} \pi(t_n,x). \tag{20}$$

Denote by $n_0 := n(\varepsilon,x,\tau)$ a number from $\mathbb{N}$ such that $t_n \geq L(\varepsilon,x,\tau)$ for any $n \geq n_0$, then from (18) we obtain

$$\rho(\pi(t_n+\tau,x),\pi(t_n,x)) < \varepsilon/2 \tag{21}$$

for any $n \geq n_0$. Passing to the limit in (21) and taking into account (20) we obtain

$$\rho(\pi(\tau,p),p) \leq \varepsilon/2 < \varepsilon$$

for any $\tau \in \mathcal{P}(\varepsilon,\omega_x)$ and $p \in \omega_x$.

Now we will prove the converse statement. Assume that ω_x is equi-a.p. For $\varepsilon > 0$, choose a relatively dense in $\mathbb{T}$ subset $\mathcal{P}(s/2,\omega_x)$ of $\varepsilon/2$-almost periods for ω_x. Let $\mathcal{P}(\varepsilon,x) := \mathcal{P}(\varepsilon/2,\omega_x)$ and $\tau \in \mathcal{P}(\varepsilon,x)$. We will show that there exists a positive number $L(\varepsilon,x,\tau)$ such that

$$\rho(\pi(t+\tau,x),\pi(t,x)) < \varepsilon \tag{22}$$

for any $t \geq L$. If we suppose that (22) is not true, then there exist a positive number $\varepsilon_0 > 0$, $\tau_0 \in \mathcal{P}(\varepsilon_0,x) = \mathcal{P}(\varepsilon_0/2,\omega_x)$ and a sequence $\{t_n\}$ such that $t_n \to +\infty$ as $n \to \infty$ and

$$\rho(\pi(t_n+\tau_0,x),\pi(t_n,x)) \geq \varepsilon_0 \tag{23}$$

for any $n \in \mathbb{N}$. Since the point x is Lagrange stable then without loss of generality we can suppose that the sequence $\{\pi(t_n, x)\}$ is convergent and denote its limit by

$$\bar{p} := \lim_{n\to\infty} \pi(t_n, x). \tag{24}$$

Note that $\bar{p} \in \omega_x$. Passing to the limit in (23) and taking into account (24) we obtain

$$\rho(\pi(\bar{p}, \tau_0), \bar{p}) \geq \varepsilon_0. \tag{25}$$

On the other hand since $\tau_0 \in \mathcal{P}(\varepsilon_0/2, \omega_x)$ then we have

$$\rho(\pi(p, \tau), p) < \varepsilon_0/2 \tag{26}$$

for any $p \in \omega_x$. The relation (25) contradicts to (26). The obtained contradiction proves our statement. Lemma is completely proved. □

Theorem 4. *Assume that the following conditions are fulfilled:*

1. *the point x is remotely stationary (respectively, remotely τ-periodic or remotely almost periodic);*
2. *$x \in X$ is positively asymptotically Poisson stable.*

Then the point x is asymptotically stationary (respectively, asymptotically τ-periodic or asymptotically almost periodic).

Proof. Since the point x is positively asymptotically Poisson stable, then there exists a positively Poisson stable point p such that (5) holds. From (5) we obtain $\omega_x = \omega_p$. On the other hand $p \in \omega_p$ because the point p is positively Poisson stable. Thus we have $p \in \omega_p = \omega_x$. By Corollary 1 (respectively, by Theorem 2 or by Lemma 11) the point $p \in \omega_x$ is stationary (respectively, τ-periodic or almost periodic) and taking into account (5) we conclude that the point x is asymptotically stationary (respectively, asymptotically τ-periodic or asymptotically almost periodic). Theorem is proved. □

4 Comparability of remotely almost periodic motions by their character of recurrence at the infinity

Denote by

- $\mathfrak{L}_x^{+\infty} := \{\{t_n\} \subset \mathbb{T} |\ \{\pi(t_n, x)\}$ converges and $t_n \to +\infty$ as $n \to \infty$ $\}$;
- $\mathfrak{L}_{x,p}^{+\infty} := \{\{t_n\} \in \mathfrak{L}_x^{+\infty} |$ such that $\pi(t_n, x) \to p$ as $n \to \infty\}$.

Definition 11. Let $(X, \mathbb{T}, \pi)$ and $(Y, \mathbb{T}, \sigma)$ be two dynamical systems. A point $x \in X$ is said to be (positively) remotely comparable by the character of recurrence with the point $y \in Y$ if $\mathfrak{L}_y^{+\infty} \subseteq \mathfrak{L}_x^{+\infty}$.

Let $\mathbb{T}_i$ $(i = 1, 2)$ be a subsemigroup of the group $\mathbb{S}$, $\mathbb{S}_+ \subseteq T_1 \subseteq T_2$ and $(X, \mathbb{T}_1, \pi)$ and $(Y, \mathbb{T}_2, \sigma)$ be two dynamical systems.

Lemma 12. *Let $x \in X$ (respectively, $y \in Y$) be the point of the dynamical system $(X, \mathbb{T}_1, \pi)$ (respectively, $(Y \mathbb{T}_2, \sigma)$). Assume that the following conditions are fulfilled:*

1. $\mathfrak{L}_y^{+\infty} \subseteq \mathfrak{L}_x^{+\infty}$;
2. $\mathfrak{L}_{y,q}^{+\infty} \bigcap \mathfrak{L}_{x,p}^{+\infty} \neq \emptyset$ *($q \in \omega_y$ and $p \in \omega_x$).*

Then

$$\mathfrak{L}_{y,q}^{+\infty} \subseteq \mathfrak{L}_{x,p}^{+\infty}. \tag{27}$$

Proof. If we suppose that the relation (27) is false, then there exists a sequence

$$\{\tau_n\} \in \mathfrak{L}_{y,q}^{+\infty} \setminus \mathfrak{L}_{x,p}^{+\infty} \neq \emptyset.$$

Under the conditions of lemma we have $\sigma(\tau_n, y) \to q$, $\pi(\tau_n, x) \to p'$ as $n \to \infty$ and $p' \neq p$.

Note that $\mathfrak{L}_{y,q}^{+\infty} \bigcap \mathfrak{L}_{x,p}^{+\infty} \neq \emptyset$ and, consequently, there exists a sequence $\{s_n\} \in \mathfrak{L}_{y,q}^{+\infty} \bigcap \mathfrak{L}_{x,p}^{+\infty}$. This means that $\sigma(s_n, y) \to q$ and $\pi(s_n, x) \to p$ as $n \to \infty$. Let $t_n = \tau_k$ if $n = 2k - 1$ and $t_n = s_k$ if $n = 2k$ for any $k \in \mathbb{N}$. It is clear that $\{t_n\} \in \mathfrak{L}_{y,q}^{+\infty} \subseteq \mathfrak{L}_y^{+\infty} \subseteq \mathfrak{L}_x^{+\infty}$ and, consequently, there exists a point $p'' \in \omega_x$ such that $\pi(t_n, x) \to p''$ as $n \to \infty$.

On the other hand we have

$$p'' = \lim_{k \to \infty} \pi(s_k, x) = p$$

and

$$p'' = \lim_{k \to \infty} \pi(\tau_k, x) = p'$$

and, consequently, $p = p'$. The last relation contradicts the choice of the point p'. The obtained contradiction prove our statement. Lemma is proved. □

Lemma 13. *Assume that $\mathfrak{L}_y^{+\infty} \subseteq \mathfrak{L}_x^{+\infty}$ and $\mathfrak{L}_{y,q}^{+\infty} \subseteq \mathfrak{L}_{x,p}^{+\infty}$ ($q \in \omega_y$ and $p \in \omega_x$).*

Then for any $t \in \mathbb{T}_1$ we have

$$\mathfrak{L}_{y,\sigma(t,q)}^{+\infty} \subseteq \mathfrak{L}_{x,\pi(t,p)}^{+\infty}. \tag{28}$$

Proof. Let $t \in \mathbb{T}_1$ and $\{t_n\} \in \mathfrak{L}_{y,q}^{+\infty} \subseteq \mathfrak{L}_{x,p}^{+\infty}$, then $\sigma(t_n + t, y) = \sigma(t, \sigma(t_n, y)) \to \sigma(t, q)$ (respectively, $\pi(t_n + t, x) = \pi(t, \pi(t_n, x)) \to \pi(t, p)$) as $n \to \infty$ and consequently,

$$\{t_n + t\} \in \mathfrak{L}_{y,\sigma(t,q)}^{+\infty} \bigcap \mathfrak{L}_{x,\pi(t,p)}^{+\infty} \neq \emptyset.$$

By Lemma 12 we have the relation (28). Lemma is proved. □

Theorem 5. *[2, Ch.II,p.31] Let $y \in Y$ be an asymptotically stationary (respectively, asymptotically τ-periodic, asymptotically almost periodic) point. If the point $x \in X$ is remotely comparable by the character of recurrence with the point $y \in Y$, then the point x is also asymptotically stationary (respectively, asymptotically τ-periodic, asymptotically almost periodic).*

Theorem 6. *Let $y \in Y$ be a Lagrange stable point of $(Y, \mathbb{T}_2, \sigma)$. If a point x is remotely comparable with the point y by the character of recurrence, then there exists a continuous mapping $h : \omega_y \to \omega_x$ satisfying the condition*

$$h(\sigma(t, q)) = \pi(t, h(q))$$

for any $t \in \mathbb{T}_1$ and $q \in \omega_y$.

Proof. Since the point $y \in Y$ is Lagrange stable then ω_y is a nonempty, compact and invariant subset of $(Y, \mathbb{T}_2, \sigma)$. Let $q \in \omega_y$ be an arbitrary point, then there exists a sequence $t_n \to +\infty$ such that $\sigma(t_n, y) \to q$ as $n \to \infty$. It is clear that $\{t_n\} \in \mathfrak{L}_{y,q}^{+\infty} \subseteq \mathfrak{L}_y^{+\infty} \subseteq \mathfrak{L}_x^{+\infty}$. This means that there exists a point $p \in \omega_x$ such that $\{t_n\} \in \mathfrak{L}_{x,p}^{+\infty}$ and, consequently, $\mathfrak{L}_{y,q}^{+\infty} \bigcap \mathfrak{L}_{x,p}^{+\infty} \neq \emptyset$. By Lemma 12 we have $\mathfrak{L}_{y,q}^{+\infty} \subseteq \mathfrak{L}_{x,p}^{+\infty}$. It is easy to see that the point p is defined uniquely. If we assume that it is not true, then there are the points $p_1, p_2 \in \omega_x$ $(p_1 \neq p_2)$ such that $\mathfrak{L}_{y,q}^{+\infty} \subseteq \mathfrak{L}_{x,p_i}^{+\infty}$ $(i = 1, 2)$ and, consequently.

$$\mathfrak{L}_{y,q}^{+\infty} \subseteq \mathfrak{L}_{x,p_1}^{+\infty} \bigcap \mathfrak{L}_{x,p_2}^{+\infty} \neq \emptyset. \tag{29}$$

On the other hand since $p_1 \neq p_2$ then it is easy to check that

$$\mathfrak{L}_{x,p_1}^{+\infty} \bigcap \mathfrak{L}_{x,p_2}^{+\infty} = \emptyset. \tag{30}$$

The relation (30) contradicts (29). The obtained contradiction proves our statement.

Taking into account the facts established above we can define the mapping $h : \omega_y \to \omega_x$ as follows:

$$h(q) = p \;\text{ if and only if }\; \mathfrak{L}_{y,q}^{+\infty} \subseteq \mathfrak{L}_{x,p}^{+\infty}. \tag{31}$$

It is clear that by (31) a mapping $h : \omega_y \to \omega_x$ is well defined and by Lemma 13 we have $h(\sigma(t, q)) = \pi(t, h(q))$ for any $t \in \mathbb{T}_2$ and $q \in \omega_x$.

To finish the proof of Theorem it suffices to show that the mapping $h : \omega_y \to \omega_x$ is continuous. Let $\{q_k\} \to q$ $(q_k, q \in \omega_y)$. Show that $\{p_k\} = \{h(q_k)\}$ converges to $p = h(q)$. For every $k \in \mathbb{N}$ we choose $\{t_n^{(k)}\} \in \mathfrak{L}_{y,q_k}^{+\infty}$, then $p_k = h(q_k) = \lim_{n \to +\infty} x t_n^{(k)}$. Let $\varepsilon_k \downarrow 0$. For every $k \in \mathbb{N}$ we will choose $n_k \in \mathbb{N}$ such that the following inequalities would be fulfilled simultaneously

$$t_{n_k}^{(k)} \geq k, \quad \rho(x t_{n_k}^{(k)}, p_k) < \varepsilon_k \quad \text{and} \quad d(y t_{n_k}^{(k)}, q_k) < \varepsilon_k$$

(it is clear that such n_k exists). Assume $t_k' := t_{n_k}^{(k)}$ and let us show that the sequence $\{t_k'\}$ belongs to $\mathfrak{L}_{y,q}^{+\infty}$. For this aim we will note that

$$d(y t_k', q) \leq d(y t_k', q_k) + d(q_k, q) < \varepsilon_k + d(q_k, q). \tag{32}$$

Passing to the limit in (32) as $k \to +\infty$ we will obtain $\{t'_k\} \in \mathfrak{L}^{+\infty}_{y,q}$. Since $\mathfrak{L}^{+\infty}_{y,q} \subseteq \mathfrak{L}^{+\infty}_{x,p}$, then $\{t'_k\} \in \mathfrak{L}^{+\infty}_{x,p}$. As

$$\rho(p_k, p) \le \rho(p_k, xt'_k) + \rho(xt'_k, p) < \varepsilon_k + \rho(xt'_k, p), \tag{33}$$

then passing to the limit in (33) and taking into account that $\{t'_k\} \in \mathfrak{L}^{+\infty}_{x,p}$ we will obtain $p_k \to p$. Theorem is completely proved. □

Remark 6. Under the conditions of Theorem 6 we have $h(\omega_y) \subseteq \omega_x$.

Theorem 7. *Assume that the point $y \in Y$ is positively Lagrange stable. If a point x is remotely comparable with y by the character of recurrence, then the following statements hold:*

1. *the point x is positively Lagrange stable;*
2. *there exists a continuous mapping $h : \omega_y \to \omega_x$ satisfying the condition*

$$h(\sigma(t, q)) = \pi(t, h(q))$$

   for any $t \in \mathbb{T}$ and $q \in \omega_y$;
3. *$h(\omega_y) = \omega_x$.*

Proof. Let $\{t_n\} \subset \mathbb{S}_+$ be an arbitrary sequence. If the sequence $\{t_n\}$ is bounded, then the sequence $\{\pi(t_n, x)\}$ is evidently precompact. If the sequence $\{t_n\}$ is unbounded, then we can extract from $\{t_n\}$ a subsequence $\{t_{n_k}\}$ which converges to $+\infty$ as $k \to \infty$. Since the point y is positively Lagrange stable, then without loss of generality we can assume that the sequence $\{\sigma(t_{n_k}, y)\}$ converges. Thus we have $\{t_{n_k}\} \in \mathfrak{L}^{+\infty}_y \subseteq \mathfrak{L}^{+\infty}_x$, i.e., the sequence $\{\pi(t_{n_k}, x)\}$ is convergent. This means that x is positively Lagrange stable.

The seconde statement follows directly from Theorem 6.

Let now p be an arbitrary point from ω_x, then there exists a sequence $\{t_n\} \in \mathfrak{L}^{+\infty}_x$ such that $\pi(t_n, x) \to p$ as $n \to \infty$. Since the point y is positively Lagrange stable we can assume that the sequence $\{\sigma(t_n, y)\}$ converges. Denote its limit by q then $h(q) = p$ (see the proof of Theorem 6), i.e., $h(\omega_y) = \omega_x$. Theorem is completely proved. □

Theorem 8. *[4, Ch.I] Let $y \in Y$ be asymptotically stationary (respectively, asymptotically τ-periodic or asymptotically almost periodic) point. If the point $x \in X$ is remotely comparable by the character of recurrence with the point y, then the point x is also asymptotically stationary (respectively, asymptotically τ-periodic or asymptotically almost periodic).*

Theorem 9. *Let $y \in Y$ be Lagrange stable and remotely stationary (respectively, remotely τ-periodic or remotely almost periodic) point. If the point $x \in X$ is remotely comparable by the character of recurrence with the point y, then the point x is also remotely stationary (respectively, remotely τ-periodic or remotely almost periodic).*

Proof. Let $y \in Y$ be a Lagrange stable and remotely stationary (respectively, remotely τ-periodic or remotely almost periodic) point. Since the point y is positively Lagrange stable, then by Theorem 7 the point $x \in X$ is positively Lagrange stable and there exists a continuous map $h : \omega_y \to \omega_x$ satisfying the following conditions:

1. $$h(\omega_y) = \omega_x; \tag{34}$$

2. $$h(\sigma(t, q)) = \pi(t, h(q)) \tag{35}$$

   for any $t \in \mathbb{T}$ and $q \in \omega_y$.

By Lemma 11 and Theorem 2 the set ω_y consists of stationary points (respectively, τ-periodic points or the set ω_y is equi-almost periodic). From the equalities (34) and (35) we obtain that the set ω_x also consists of stationary points (respectively, τ-periodic motions).

If the point y is remotely stationary (respectively, remotely τ-periodic), then by Corollary 1 (respectively, by Theorem 2) the point x is remotely stationary (respectively, remotely τ-periodic).

Let now the point y be remotely almost periodic. We will show that since the set ω_y is equi-almost periodic (see Lemma 11) then ω_x is also equi-almost periodic. Let ε be an arbitrary positive number then by compactness of ω_y there exists a positive number $\delta = \delta(\varepsilon)$ such that

$$\rho(y_1, y_2) < \delta$$

implies

$$\rho(h(y_1), h(y_2)) < \varepsilon.$$

Since the set ω_y is equi-almost periodic, then for $\delta(\varepsilon) > 0$ there exists a relatively dense subset $\mathcal{P}(\delta(\varepsilon), \omega_y) = \mathcal{P}(\varepsilon, \omega_y)$ such that

$$\rho(\sigma(t + \tau, \tilde{q}), \sigma(t, \tilde{q})) < \delta(\varepsilon) \tag{36}$$

for any $t \in \mathbb{T}$, $\tau \in \mathcal{P}(\omega_y, \varepsilon)$ and $\tilde{q} \in \omega_y$. For arbitrary $p \in \omega_x$ there exists a point $q \in \omega_y$ (see Theorem 7, item (iii)) such that $h(q) = p$ and by (36) we obtain

$$\rho(\pi(t + \tau, p), \pi(t, p)) = \rho(\pi(t + \tau, h(q)), \pi(t, h(q)) = \rho(h(\sigma(t + \tau, q)), h(\sigma(t, q)) < \varepsilon$$

for any $(t, \tau, p) \in \mathbb{T} \times \mathcal{P}(\varepsilon, \omega_y) \times \omega_x$. This means that the set ω_x is equi-almost periodic. By Lemma 11 the point x is remotely almost periodic. Theorem is completely proved. □

Consider a dynamical system $(X, \mathbb{S}_-, \pi)$. Denote by $(X, \mathbb{S}_+, \hat{\pi})$ the dynamical system, where $\hat{\pi}$ is the mapping from $\mathbb{S}_+ \times X$ into X defined by the equality $\hat{\pi}(t, x) := \pi(-t, x)$ for any $(t, x) \in \mathbb{S}_+ \times X$.

Remark 7. Taking into account the construction above we can define the notion of remote almost periodicity in the negative direction and to establish the analogues of all results of Sections 2-4 for negatively remotely almost periodic motions.

Lemma 14. *Assume that M is a nonempty compact positively invariant equi-almost periodic subset of $(X,\mathbb{S}_+,\pi)$. Then the relation $\pi(t_0,x_1)=\pi(t_0,x_2)$ ($t_0\in\mathbb{S}_+,\ t_0>0$ and $x_1,x_2\in M$) implies $x_1=x_2$.*

Proof. If we suppose that this statement is not true, then there are $x_1^0,x_2^0\in M$ ($x_1^0\neq x_2^0$) and $t_0>0$ such that

$$\pi(t_0,x_1^0)=\pi(t_0,x_2^0).$$

Denote by $d:=\rho(x_1^0,x_2^0)>0$. Let $\varepsilon\in(0,d)$ be an arbitrary number. Since the set M is equi-almost periodic then for given ε there exists a relatively dense subset $\mathcal{P}(\varepsilon,M)\subset\mathbb{S}_+$ such that

$$\rho(\pi(\tau,x_i^0),x_i^0)<\varepsilon/3$$

for any $\tau\in\mathcal{P}(\varepsilon/3,M)$ and $i=1,2$. Let $\tau\in\mathcal{P}(\varepsilon/3,M)$ and $\tau>t_0$ then

$$\pi(\tau,x_1^0)=\pi(\tau,x_2^0)$$

and, consequently,

$$d=\rho(x_1^0,x_2^0)\leq\rho(x_1^0,\pi(\tau,x_1^0))+\rho(\pi(\tau,x_1^0),\pi(\tau,x_2^0))+$$
$$\rho(\pi(\tau,x_2^0),x_2^0)<\varepsilon/3+0+\varepsilon/3<\varepsilon.$$

The last inequality contradicts the choice of the number ε. The obtained contradiction proves our statement. Lemma is proved. □

Definition 12. A continuous mapping $\gamma:\mathbb{S}\to X$ is said to be an entire (full) trajectory of $(X,\mathbb{S}_+,\pi)$ if $\pi(t,\gamma(s))=\gamma(t+s)$ for any $t\in\mathbb{S}_+$ and $s\in\mathbb{S}$.

Denote by $\mathcal{F}_x$ the family of all full trajectories of $(X,\mathbb{S}_+,\pi)$ passing through the point x at the initial moment $t=0$, i.e., $\gamma(0)=x$.

Theorem 10. *[3, Ch.I] Let M be a nonempty, compact invariant subset of $(X,\mathbb{T},\pi)$. Then for any $x\in M$ the set $\mathcal{F}_x\neq\emptyset$, i.e., there exists at least one full trajectory γ of $(X,\mathbb{T},\pi)$ passing through the point x at the initial moment $t=0$.*

Corollary 5. *Assume that M is a nonempty compact invariant equi-almost periodic subset of $(X,\mathbb{S}_+,\pi)$. Then for any $x\in M$ there exists a unique full trajectory γ_x of $(X,\mathbb{S}_+,\pi)$ such that $\gamma_x(0)=x$ and $\gamma_x(s)\in M$ for any $s\in\mathbb{S}$.*

Proof. Since the set M is invariant then by Theorem 10 for any $x\in M$ the set $\mathcal{F}_x$ / consists of a single full trajectory of $(X,\mathbb{T},\pi)$. If we assume that this statement is not true, then there are a point $x_0\in M$ and full trajectories $\gamma_{x_0}^1$ and $\gamma_{x_0}^2$ such that $\gamma_{x_0}^i(s)\in M$ ($i=1,2$) for any $s\in\mathbb{S}$ and $\gamma_{x_0}^1(s_0)\neq\gamma_{x_0}^2(s_0)$ for certain $s_0<0$ ($s_0\in\mathbb{S}$). On the other hand since $\pi(-s_0,\gamma_{x_0}^i(s_0))=x_0$ ($i=1,2$) then by Lemma 14 we have $\gamma_{x_0}^1(s_0)=\gamma_{x_0}^2(s_0)$. The obtained contradiction completes the proof. □

Definition 13. An entire trajectory $\gamma \in \mathcal{F}_x$ is called almost periodic if the function $\gamma : \mathbb{S} \to X$ is almost periodic, i.e., for every $\varepsilon > 0$ there exists a relatively dense in $\mathbb{S}$ subset $\mathcal{P}(\varepsilon, \gamma) \subset \mathbb{S}$ such that $\rho(\gamma(t+\tau), \gamma(t)) < \varepsilon$ for any $t \in \mathbb{S}$ and $\tau \in \mathcal{P}(\varepsilon, \gamma)$.

Theorem 11. *[4, Ch.I] Let $(X, \mathbb{S}_+, \pi)$ be a semigroup dynamical system and suppose that the point $x \in X$ is almost periodic. Then the motion $\pi(t, x)$ admits a unique almost periodic extension, i.e., there exists a unique almost periodic entire trajectory $\gamma \in \mathcal{F}_x$ such that $\gamma_x(s) \in M := H(x)$ for any $s \in \mathbb{S}$ and $\gamma(t) = \pi(t, x)$ for any $t \in \mathbb{S}_+$.*

Corollary 6. *Assume that M is a nonempty compact invariant equi-almost periodic subset of $(X, \mathbb{S}_+, \pi)$. Then for any $x \in M$ there exists a unique full trajectory γ_x of $(X, \mathbb{S}_+, \pi)$ such that $\gamma_x(0) = x$, $\gamma_x(s) \in M$ for any $s \in \mathbb{S}$ and γ_x is almost periodic.*

Proof. This statement follows from Theorem 11 and Corollary 5. □

5 Equi-Almost Periodicity

Let M be a nonempty compact invariant and equi-almost periodic subset of the semi-group dynamical system $(X, \mathbb{S}_+, \pi)$. Denote by $\hat{\pi}$ the mapping from $\mathbb{S} \times M$ into M defined by equality

$$\hat{\pi}(t, x) := \gamma_x(t) \quad (\gamma_x \in \mathcal{F}_x) \tag{37}$$

for any $x \in M$ and $t \in \mathbb{S}$, where $\gamma_x \in \mathcal{F}_x$ is a unique full trajectory of $(X, \mathbb{S}_+, \pi)$ such that $\gamma_x(s) \in M$ for any $s \in \mathbb{S}$ and $\gamma(0) = x$ (see Corollary 6).

Lemma 15. *The mapping $\hat{\pi}$ defined by equality (37) possesses the following properties:*

1. $\hat{\pi}(0, x) = x$ *for any* $x \in M$*;*
2. $$\hat{\pi}(t_1 + t_2, x) = \hat{\pi}(t_2, \hat{\pi}(t_1, x)) \tag{38}$$
   for any $x_1, x_2 \in M$ *and* $t \in \mathbb{S}$*;*
3. *the mapping* $\hat{\pi}$ *is continuous;*
4. $\hat{\pi}(t, x) = \pi(t, x)$ *for any* $(t, x) \in \mathbb{S}_+ \times M$*.*

Proof. The first and fourth statements are evident. Let $t_1, t_2 \in \mathbb{S}$ be two arbitrary numbers. Note that

$$\hat{\pi}(t_1 + t_2, x) = \gamma_x(t_1 + t_2) \quad \text{and} \quad \hat{\pi}(t_2, \hat{\pi}(t_1, x)) = \hat{\pi}(t_2, \gamma_x(t_1)) = \gamma_{\gamma_x(t_1)}(t_2) \tag{39}$$

Without loss of generality we can suppose that $t_1 \le t_2$. Consider the following three logically possible cases:

(a) $t_1 \le t_2 \le 0$;

(b) $t_1 \le 0 \le t_2$;

(c) $0 \le t_1 \le t_3$.

(a) If $t_1 \le t_2 \le 0$, then

$$\pi(-t_1 - t_2, \gamma_x(t_1 + t_2)) = x \text{ and } \pi(-t_1 - t_2, \gamma_{\gamma_x(t_1)}(t_2)) = \pi(-t_1, \gamma_x(t_1)) = x.$$

According to Lemma 14 $\gamma_x(t_1 + t_2) = \gamma_{\gamma_x(t_1)}(t_2)$ and by (39) we obtain (38).

(b) If $t_1 \le 0 \le t_2$, then $\gamma_x(t_1 + t_2) = \pi(t_2, \gamma_x(t_1))$ and $\gamma_{\gamma_x(t_1)}(t_2) = \pi(t_2, \gamma_x(t_1))$ and, consequently, $\gamma_x(t_1 + t_2) = \gamma_{\gamma_x(t_1)}(t_2)$ and by (39) we obtain (38).

(c) If $0 \le t_1 \le t_2$, then $\hat{\pi}(t_1 + t_2, x) = \pi(t_1 + t_2, x) = \pi(t_2, \pi(t_1, x)) = \hat{\pi}(t_2, \hat{\pi}(t_1, x))$. Thus the second statement is proved.

To finish the proof of lemma it suffices to show that the mapping $\hat{\pi} : \mathbb{S} \times M \to M$ is continuous. Let $x \in M, \quad t \in \mathbb{S}_-, x_n \to x$ and $t_n \to t$, then there is a number $l_0 > 0$ such that $t_n \in [-l_0, l_0]$ and, consequently,

$$\rho(\tilde{\pi}(t_n, x_n), \tilde{\pi}(t, x)) = \rho(\pi^{t_n + l_0}\gamma_{x_n}(-l_0), \pi^{t + l_0}\gamma_x(-l_0)) \le$$
$$\rho(\pi^{t_n + l_0}\gamma_{x_n}(-l_0), \pi^{t_n + l_0}\gamma_x(-l_0)) + \rho(\pi^{t_n + l_0}\gamma_x(-l_0), \pi^{t + l_0}\gamma_x(-l_0)).$$

We will establish that the sequence $\{\gamma_{x_n}\}$ is relatively compact in $C(\mathbb{S}, M)$ and that every limiting point γ of this sequence belongs to $\mathcal{F}_x$. Indeed, to prove that the sequence $\{\gamma_{x_n}\}$ is precompact in $C(\mathbb{S}, M)$ it is sufficient to show that for any $l > 0$ the sequence $\{\gamma_{x_n}\}$ is equi-continuous on $[-l.l]$, because $\gamma_{x_n}(s) \in M$ for any $(n, s) \in \mathbb{N} \times \mathbb{S}$ and M is a compact subset of X. Let l be an arbitrary positive number. We will prove that the sequence $\{\gamma_{x_n}\}$ is equi-continuous on segment $[-l, l] \subset \mathbb{T}$. If we suppose that it is not true, then there exist $\varepsilon_0, l_0 > 0$, $t_n^i \in [-l_0, l_0]$ $(i = 1, 2)$ and $\delta_n \to 0 \quad (\delta_n > 0)$ such that

$$|t_n^1 - t_n^2| \le \delta_n \quad \text{and} \quad \rho(\gamma_{x_n}(t_n^1), \gamma_{x_n}(t_n^2)) \ge \varepsilon_0. \tag{40}$$

We may suppose that $t_n^i \to t_0$ $(i = 1, 2)$. From (40) we obtain

$$\varepsilon_0 \le \rho(\gamma_n(t_n^1), \gamma_n(t_n^2)) =$$
$$\rho(\pi(t_n^1 + l_0, \gamma_{x_n}(-l_0), \pi(t_n^2 + l_0, \gamma_{x_n}(-l_0))) \tag{41}$$

for any $n \in \mathbb{N}$. Since $\gamma_{x_n}(s) \in M$ for any $s \in \mathbb{S}$ and M is a compact subset of X, then the sequence $\{\gamma_{x_n}(-l_0)\}$ is precompact. Without loss of generality we can suppose that $\{\gamma_{x_n}(-l_0)\}$ converges and denote by $\bar{x}$ its limit. Passing to the limit in (41) as $n \to \infty$ and taking into account above we obtain

$$\varepsilon_0 \le \rho(\pi(t_0 + l_0, \bar{x}), \pi(t_0 + l_0, \bar{x})) = 0.$$

The last relation contradicts to the choice of ε_0. The obtained contradiction proves our statement. Let γ be a limiting point of the sequence $\{\gamma_n\}$, then there exists a subsequence $\{\gamma_{k_n}\}$ such that $\gamma(t) = \lim_{n\to\infty} \gamma_{k_n}(t)$ uniformly on every segment $[-l, l] \subset \mathbb{T}$. In particular $\gamma \in C(\mathbb{S}, X)$ and $\gamma(s) \in M$ for any $s \in \mathbb{S}$ because $\gamma(s) = \lim_{n\to\infty} \gamma_{x_{k_n}}(s)$. We note that

$$\pi^t\gamma(s) = \lim_{n\to\infty} \pi^t\gamma_{k_n}(s) = \lim_{n\to\infty} \gamma_{k_n}(s + t) = \gamma(s + t)$$

for any $t \in \mathbb{T}$ and $s \in \mathbb{S}$. Finally, we see that $\gamma(0) = \lim_{n\to\infty} \gamma_{k_n}(0) = \lim_{n\to\infty} \gamma_{x_{k_n}}(0) = \lim_{n\to\infty} x_{k_n} = x$, i.e., γ is an entire trajectory of $(X, \mathbb{T}, \pi)$ passing through the point x. Lemma is completely proved. $\square$

Definition 14. A dynamical system $(X, \mathbb{T}, \pi)$ is said to be *distal* on the positively invariant subset $M \subseteq X$ if

$$\inf_{t\in\mathbb{T}} (\pi(t,p), \pi(t,q)) > 0$$

for any $p, q \in M$ $(p \neq q)$.

Lemma 16. *Let $M \subseteq X$ be a nonempty, compact and positively invariant equi-almost periodic subset, then the following statements hold:*

1. *the set M is uniformly Lyapunov stable, i.e., for any $\varepsilon > 0$ there exists a positive number $\delta = \delta(\varepsilon)$ such that $\rho(p, q) < \delta$ $(p, q \in M)$ implies $\rho(\pi(t, p), \pi(t, q)) < \varepsilon$ for any $t \geq 0$;*
2. *the dynamical system $(M, \mathbb{T}, \pi)$ is distal.*

Proof. Let us prove the first statement of Lemma. Let $\varepsilon > 0$, then by equi-almost periodicity of M there exists a relatively dense in $\mathbb{T}$ subset $\mathcal{P}(\varepsilon, M)$ such that

$$\rho(\pi(t + \tau, p), \pi(t, p)) < \varepsilon$$

for any $(t, \tau, p) \in \mathbb{T} \times \mathcal{P}(\varepsilon, M) \times M$. By relative density of $P(\varepsilon, M)$ there exists a positive number $l = l(\varepsilon, M) > 0$ such that

$$\mathcal{P}(\varepsilon, M) \bigcap [a, a + l] \neq \emptyset$$

for any $a \in \mathbb{T}$, where $[a, a + l] := \{t \in \mathbb{T} |\ a \leq t \leq a + l\}$.

Since the set M is compact, then on M the integral continuity is uniform. This means that for $\varepsilon/3$ and $l(\varepsilon/3)$ there exists $\delta = \delta(\varepsilon) > 0$ such that

$$\rho(pt, qt) < \frac{\varepsilon}{3} \tag{42}$$

for any $t \in [0, l]$ as soon as $\rho(p, q) < \delta$ $(p, q \in M)$. Let now $q \in M$ and $\rho(p, q) < \delta$. Then on the segment $[t - l, t] \subset \mathbb{T}$ there is a number τ such that

$$\rho(r(t + \tau), rt) < \varepsilon/3. \tag{43}$$

Present the number t as $s + \tau$, where $s \in [0, l]$. Then for $t = s + \tau$ we have

$$\begin{aligned} &\rho(pt, qt) = \rho(p(s + \tau), q(s + \tau)) \\ &\leq \rho(p(s + \tau), ps) + \rho(ps, qs) + \rho(qs, q(s + \tau)). \end{aligned}$$

From the last inequality and the inequalities (42) and (43) it follows that

$$\rho(pt, qt) < \varepsilon \tag{44}$$

for any $t \geq l$. From (42) and (44) we obtain the first statement of lemma.

Assume that the second statement of lemma does not take place. Then there exist $p, q \in M$ $(p \neq q)$ and $t_n \in \mathbb{T}$ such that

$$\rho(pt_n, qt_n) \to 0 \tag{45}$$

as $n \to +\infty$. According to the first statement of Lemma the set M is positively uniformly stable and, consequently, for the number $0 < \varepsilon < \rho(p, q)$ there is $\delta = \delta(\varepsilon/3)$ such that

$$\rho(pt, qt) < \frac{\varepsilon}{3}$$

for any $t \in \mathbb{T}$ as soon as $\rho(p, q) < \delta$ $(p, q \in M)$. From (45) it follows that for n large enough $\rho(pt_n, qt_n) < \delta$ and, consequently,

$$\rho(p(t_n + t), q(t_n + t)) < \varepsilon/3 \tag{46}$$

for any $t \in \mathbb{T}$. By the numbers $\varepsilon/3$ and $t_n \in \mathbb{T}$ we chose $\tau \geq t_n$ such that

$$\rho(r\tau, r) < \varepsilon/3 \tag{47}$$

for any $r \in M$ (according to the equi-almost periodicity of M such τ exists). Then

$$\rho(p, q) \leq \rho(p\tau, p) + \rho(p\tau, q\tau) + \rho(q\tau, q)$$

and according to (46) and (47) $\rho(p, q) < \varepsilon$. This fact contradicts the choice of ε. Lemma is proved. □

Let $V : X \times X \mapsto \mathbb{R}_+$ be a continuous and positive definite function, i.e., $V(x_1, x_2) = 0$ if and only if $x_1 = x_2$.

Definition 15. A dynamical system $(X, \mathbb{T}, \pi)$ is said to be V-*monotone* on the positively invariant subset $M \subseteq X$ if $V(\pi(t, x_1), \pi(t, x_2)) \leq V(x_1, x_2)$ for any $(x_1, x_2) \in M \times M$ and $t \geq 0$.

Lemma 17. *Let M be a nonempty, compact and positively invariant equi-almost periodic subset of X, then the following statements hold:*

1. *there exists a sequence $\{t_n\} \subseteq \mathbb{T}$ such that $t_n \to +\infty$ and $\pi(t_n, p) \to p$ as $n \to \infty$ uniformly with respect to $p \in M$;*
2. *the set M is invariant, i.e., $\pi(t, M) = M$ for any $t \in \mathbb{T}$;*
3. *there exists a group dynamical system $(M, \mathbb{S}, \tilde{\pi})$ such that $\tilde{\pi}(t, p) = \pi(t, p)$ for any $t \in \mathbb{T}$ and $p \in M$, i.e., the semigroup dynamical system $(M, \mathbb{T}, \pi)$ admits a group extension on M;*
4. *the dynamical system $(M, \mathbb{T}, \pi)$ is V-monotone, where*

$$V(p, q) := \sup\{\rho(\pi(t, p), \pi(t, q)) : \ t \in \mathbb{T}\}; \tag{48}$$

5. *the group dynamical system* $(M, \mathbb{S}, \tilde{\pi})$ *is bilaterally Lyapunov stable, i.e., for any* $\varepsilon > 0$ *there exists a* $\delta = \delta(\varepsilon) > 0$ *such that* $\rho(p, q) < \delta$ $(p, q \in M)$ *implies* $\rho(\tilde{\pi}(t, p), \tilde{\pi}(t, q)) < \varepsilon$ *for any* $t \in \mathbb{S}$;

6. *the point* $x \in M$ *is almost periodic with respect to group dynamical system* $(M, \mathbb{S}, \tilde{\pi})$.

Proof. By equi-almost periodicity of M for $\varepsilon_n := 1/n$ there exists a number $t_n \geq n$ $(t_n \in \mathbb{T})$ such that $\rho(\pi(t_n, p), p)) < 1/n$ $(\forall\ p \in M)$ and, consequently, $\pi(t_n, p) \to p$ as $n \to +\infty$ uniformly with respect to $p \in M$.

To prove the second statement it suffices to show that $M \subseteq \pi(t, M)$ for any $t \in \mathbb{T}$. Let $p \in M$, $t \in \mathbb{T}$ and $t_n \to +\infty$ such that $\pi(t_n, p) \to p$ as $n \to \infty$ uniformly with respect to $p \in M$ (see the first statement). Note that for sufficiently large n we have $t_n \geq t$ and

$$\pi(t_n, p) = \pi(t_n - t + t, p) = \pi(t, \pi(t_n - t, p)). \tag{49}$$

Since $\pi(t_n - t, p) \in M$ for sufficiently large n and the set M is compact then without loss of generality we can suppose that the sequence $\{\pi(t_n - t, p)\}$ converges. Denote its limit by p_t, then $p_t \in \omega_p \subseteq M$. Passing to the limit in (49) as $n \to \infty$ we obtain $p = \pi(t, p_t)$, i.e., $p \in \pi(t, M)$.

The third statement follows from Lemma 15.

Denote by $V : M \times M \mapsto \mathbb{R}_+$ the mapping defined by the equality (48). It is easy to check that V is a new metric on M topologically equivalent to ρ. Note that

$$|V(u, v) - V(p, q)| \leq V(u, p) + V(v, y) \tag{50}$$

for any $u, v, p, q \in M$. Since the dynamical system $(M, \mathbb{T}, \pi)$ is uniformly Lyapunov stable, then $V(u, p) + V(v, y) \to 0$ as $u \to p$ and $v \to y$, hence from (50) the continuity of V follows. Finally, notice that by definition of V we have $V(\pi(t, p), \pi(t, q)) \leq V(p, q)$ for any $t \in \mathbb{T}$ and $p, q \in M$. Thus the fourth statement is proved.

Let $p, q \in M$, consider the function $\psi(t) := V(\tilde{\pi}(t, p), \tilde{\pi}(t, p))$ (for any $t \in \mathbb{S}$). Note that $\psi : \mathbb{S} \mapsto \mathbb{R}_+$ is a continuous mapping and

$$\psi(t_2) = V(\tilde{\pi}(t_2, p), \tilde{\pi}(t_2, p)) = V(\tilde{\pi}(t_2 - t_1, \tilde{\pi}(t_1, p)), \tilde{\pi}(t_2 - t_1,$$
$$\tilde{\pi}(t_1, p))) \leq V(\tilde{\pi}(t_1, p), \tilde{\pi}(t_1, p)) = \psi(t_1)$$

for any $t_1 \leq t_2$ $(t_1, t_2 \in \mathbb{S})$. Thus ψ is a monotone decreasing function and, consequently, there exists the limit $\lim_{t \to +\infty} \psi(t) = C$, where C is a nonnegative constant. By the first statement of Lemma there exists a sequence $t_n \to +\infty$ such that $\pi(t_n, p) \to p$ and $\pi(t_n, q) \to q$ as $n \to \infty$. Since the function $V : M \times M \mapsto \mathbb{R}_+$ is continuous, we have

$$V(\tilde{\pi}(s, p), \tilde{\pi}(s, q)) = \lim_{n \to \infty} \psi(s + t_n) = C \tag{51}$$

for any $s \in \mathbb{S}$. Using the identity (51) it is not difficult to finish the proof of fourth statement. Indeed, if we suppose that it is not true, then there are positive number

$\varepsilon_0 > 0$, sequences $\{s_n\} \subseteq \mathbb{S}$, $\{\delta_n\}$ and $\{p_n\}, \{q_n\} \subseteq M$ such that $\delta_n > 0$, $\delta_n \to 0$ as $n \to \infty$,

$$\rho(p_n, q_n) < \delta_n \text{ and } \rho(\tilde{\pi}(s_n, p_n), \tilde{\pi}(s_n, q_n)) \geq \varepsilon_0. \tag{52}$$

Under the conditions of Lemma without loss of generality we may suppose that $s_n \to -\infty$. Since M is compact then we may suppose that the sequence $\{\tilde{\pi}(s_n, p_n)\}$ (respectively, $\{\tilde{\pi}(s_n, q_n)\}$) is convergent. Denote by $\bar{p}$ (respectively, $\bar{q}$) its limit. Note that

$$V(\tilde{\pi}(s_n, , p_n), \tilde{\pi}(s_n, q_n)) = V(\tilde{\pi}(-s_n + s_n, p_n), \tilde{\pi}(-s_n + s_n, q_n)) =$$
$$V(\tilde{\pi}(-s_n, \tilde{\pi}(s_n, p_n)), \tilde{\pi}(-s_n, \tilde{\pi}(s_n, q_n)) = V(p_n, q_n) \to 0$$

as $n \to \infty$ and, consequently, $\bar{p} = \bar{q}$. On the other hand passing to the limit in (52) as $n \to \infty$ we obtain $\rho(\bar{p}, \bar{q}) \geq \varepsilon_0$. The obtained contradiction proves the fifth statement of Lemma.

The sixth statement follows from Theorem 11. □

Theorem 12. *Let M be a nonempty, compact and positively invariant equi-almost periodic subset of X, then the following statements hold:*

1. *the set M is invariant;*
2. *the semigroup dynamical system $(M, \mathbb{T}, \pi)$ admits a group extension on M, i.e., on the set M a group dynamical system $(M, \mathbb{S}, \tilde{\pi})$ such that $\pi = \tilde{\pi}\big|_{\mathbb{S}_+ \times M}$ is defined;*
3. *the point $x \in X$ is almost periodic with respect to group dynamical system $(M, \mathbb{S}, \tilde{\pi})$;*
4. *the dynamical system $(M, \mathbb{S}, \tilde{\pi})$ is equi-almost periodic, i.e., for any $\varepsilon > 0$ the set*

$$\mathcal{P}(\varepsilon) := \{\tau \in \mathbb{S} |\ \rho(\tilde{\pi}(t + \tau, p), \tilde{\pi}(t, p)) < \varepsilon \ \ \textit{for any } (t, p) \in \mathbb{S} \times M\}$$

is relatively dense in $\mathbb{S}$.

Proof. The first three statements of theorem directly follows from Lemma 17 (see items (ii), (iii) and (vi) respectively). To finish the proof of theorem it suffices to establish the fourth statement. Since the dynamical system $(M, \mathbb{S}_+, \pi)$ is equi-almost periodic then for arbitrary positive number ε the set

$$A^+(\varepsilon) := \{\tau \in \mathbb{S}_+ |\ \rho(\pi(t + \tau, p), \pi(t, p)) < \varepsilon \text{ for any } (t, p) \in \mathbb{S}_+ \times M\}$$

is relatively dense in $\mathbb{S}_+$. Denote by

$$\tilde{\mathcal{P}}^+(\varepsilon) := \{\tau \in \mathbb{S}_+ |\ \rho(\tilde{\pi}(t + \tau, p), \tilde{\pi}(t, p)) < \varepsilon \text{ for any } (t, p) \in \mathbb{S} \times M\}$$

and note that $A^+(\varepsilon) \subseteq \tilde{\mathcal{P}}^+(\varepsilon)$. Indeed. Consider the group dynamical system $(M, \mathbb{S}, \tilde{\pi})$. Let $p \in M$ be an arbitrary point then by the fifth statement of theorem this point is almost periodic and in particular we have $p \in \tilde{\alpha}_p = \tilde{\omega}_p$, where $\tilde{\alpha}_p$ (respectively, $\tilde{\omega}_p$) is the alpha (respectively, omega) limit set of the point p in the dynamical system $(M, \mathbb{S}, \tilde{\pi})$. Let $t \in \mathbb{S}_-$, $\tau \in A^+(\varepsilon)$ and $p \in M$ then

$$\rho(\tilde{\pi}(t+\tau, p), \tilde{\pi}(t,p)) = \rho(\pi(\tau, p_t), p_t) < \varepsilon$$

because $p_t := \widetilde{\pi}(t,p) \in M$ for any $t \in \mathbb{S}$. Taking into account that the set $A^+(\varepsilon)$ is relatively dense in S_+ we conclude that the set $\widetilde{\mathcal{P}}^+(\varepsilon)$ is also relatively dense in $\mathbb{S}_+$. Denote by $\widetilde{\mathcal{P}}^-(\varepsilon) := -\widetilde{\mathcal{P}}^+(\varepsilon)$. I is clear that the set $\widetilde{\mathcal{P}}^-(\varepsilon)$ is relatively dense in $\mathbb{S}_-$ and by Lemma 2 the set $\widetilde{\mathcal{P}}(\varepsilon) := \widetilde{\mathcal{P}}^-(\varepsilon) \bigcup \widetilde{\mathcal{P}}^+(\varepsilon)$ is relatively dense in $\mathbb{S}$. This means that the dynamical system $(M, \mathbb{S}, \widetilde{\pi})$ is equi-almost periodic. Theorem is completely proved. □

Theorem 13. *Let $x \in X$ be a Lagrange stable and remotely almost periodic point of semigroup dynamical system $(X, \mathbb{S}_+, \pi)$. Then the following statements hold:*

1. *ω_x is a nonempty, compact and invariant set;*
2. *the semigroup dynamical system $(\omega_x, \mathbb{S}_+, \pi)$ admits a group extension $(\omega_x, \mathbb{S}, \tilde{\pi})$ on ω_x;*
3. *every point $p \in \omega_x$ is almost periodic with respect to group dynamical system $(\omega_x, \mathbb{S}, \tilde{\pi})$;*
4. *the dynamical system $(\omega_x, \mathbb{S}, \tilde{\pi})$ is equi-almost periodic;*
5. *for arbitrary $\varepsilon > 0$ there exists a relatively dense in $\mathbb{S}$ subset $\mathcal{P}(\varepsilon, x)$ with the property that for any $\tau \in \mathcal{P}(\varepsilon, x)$ there exists a positive number $L(\varepsilon, x, \tau)$ such that*
$$\rho(\pi(t+\tau, x), \pi(t,x)) < \varepsilon \tag{53}$$
for any $t \geq L(\varepsilon, x, \tau)$ and $t + \tau \geq L(\varepsilon, x, \tau)$.

Proof. Let $x \in X$ be Lagrange stable and remotely almost periodic. Then by Lemma 11 omega limit set ω_x of the point x is a compact, invariant and equi-almost periodic set of $(X, \mathbb{T}, \pi)$. The first four statements of Theorem follows directly from Theorem 12. Thus to finish the proof it suffices to prove the fifth statement. Let $\varepsilon > 0$ be an arbitrary positive number. Since the set ω_x is equi-almost periodic with respect to the group dynamical system $(\omega_x, \mathbb{S}, \widetilde{\pi})$ then there exists a relatively dense in $\mathbb{S}$ subset $\widetilde{\mathcal{P}}(\varepsilon, \omega_x)$ such that

$$\rho(\widetilde{\pi}(t+\tau, p), \widetilde{\pi}(t,p)) < \varepsilon$$

for any $t \in \mathbb{S}$ and $\tau \in \widetilde{\mathcal{P}}(\varepsilon, \omega_x)$. Denote by $\mathcal{P}(\varepsilon, x) := \widetilde{\mathcal{P}}(\varepsilon/2, \omega_x)$ the relatively dense in $\mathbb{S}$ subset of $\mathbb{S}$ and let τ be and arbitrary number from $\mathcal{P}(\varepsilon, x)$. We will show that there exists a positive number $L(\varepsilon, x, \tau)$ with the property that (53) holds. If we assume that it is not true, then there are $\varepsilon_0 > 0$, $\tau_0 \in \mathcal{P}(\varepsilon_0, x)$ and $t_n \to +\infty$ as $n \to \infty$ such that

$$\rho(\pi(t_n + \tau_0, x), \pi(t_n, x)) \geq \varepsilon_0 \tag{54}$$

for any $n \in \mathbb{N}$. Since the point x is Lagrange stable then we can assume that the sequence $\{\pi(t_n, x)\}$ converges. Denote by

$$\widetilde{p} := \lim_{n\to\infty} \pi(t_n, x). \tag{55}$$

Passing to the limit in (54) and taking into account (55) we obtain

$$\rho(\widetilde{\pi}(\tau_0, \widetilde{p}), \widetilde{p}) \geq \varepsilon_0. \tag{56}$$

On the other hand $\tau_0 \in \widetilde{\mathcal{P}}(\varepsilon_0/2, \omega_x) = \mathcal{P}(\varepsilon_0, x)$ and, consequently, we have

$$\rho(\widetilde{\pi}(\tau_0, p), p) < \varepsilon_0/2 < \varepsilon_0 \tag{57}$$

for any $p \in \omega_x$. Note that (56) contradicts to (57). This contradiction proves the required statement. Theorem is completely proved. $\square$

Theorem 14. *Let $(X, \mathbb{T}, \pi)$ be a semigroup of dynamical system and $x \in X$ be a Lagrange stable point. Then the following statements are equivalent:*

1. *for arbitrary $\varepsilon > 0$ there exists a relatively dense in $\mathbb{T}$ subset $\mathcal{P}(\varepsilon, x)$ with the property that for any $\tau \in \mathcal{P}(\varepsilon, x)$ there exists a positive number $L(\varepsilon, x, \tau)$ such that*
$$\rho(\pi(t+\tau, x), \pi(t, x)) < \varepsilon \tag{58}$$
for any $t \geq L(\varepsilon, x, \tau)$;

2. *for arbitrary $\varepsilon > 0$ there exists a relatively dense in $\mathbb{S}$ subset $\mathcal{P}(\varepsilon, x)$ with the property that for any $\tau \in \mathcal{P}(\varepsilon, x)$ there exists a positive number $L(\varepsilon, x, \tau)$ such that (58) holds for any $t \geq L(\varepsilon, x, \tau)$ and $t + \tau \geq L(\varepsilon, x, \tau)$.*

Proof. It is easy to see that (ii) implies (i). The converse implication follows directly from Theorem 13 (item (v)). Theorem is proved. $\square$

Corollary 7. *Let $(X, \mathbb{S}, \pi)$ be a group of dynamical system and $x \in X$ be a positively Lagrange stable point. Then the following statements are equivalent:*

1. *the point x is remotely almost periodic;*

2. *the omega limit set ω_x of the point x is equi-almost periodic.*

Proof. This statement follows from Theorem 14 and Lemma 11 $\square$

Remark 8. Note if $\mathbb{S} = \mathbb{R}$ (i.e., the dynamical system $(X, \mathbb{T}, \pi)$ is two-sided with the continuous time), then Corollary 7 generalizes a result from [8] (Proposition 2.8).

Let $(X, \mathbb{T}, \pi)$ be a dynamical system on the complete metric space (X, ρ) and K be a nonempty compact invariant subset.

Denote by $C(K, X)$ the family of all continuous functions $f : K \to X$ equipped with the distance

$$d(f, g) := \max_{x\in K} \rho(f(x), g(x)). \tag{59}$$

Remark 9. 1. By the formula (59) a complete metric on the space $C(K,X)$ is defined.

2. The distance d generates on the space $C(K,X)$ the topology of the uniform convergence on K, i.e., $d(f_n,f)\to 0$ as $n\to\infty$ if and only if $\lim_{n\to\infty} f_n(x)=f(x)$ uniformly with respect to $x\in K$.

Let $F\in C(\mathbb{T}\times K,X)$. Denote by F_K the mapping $F_K:\mathbb{T}\to C(K,X)$ defined by the equality

$$F_K(t):=F(t,\cdot)$$

for any $t\in\mathbb{T}$.

Lemma 18. *[10] The mapping $F_K:\mathbb{T}\to C(K,X)$ is continuous.*

Lemma 19. *The compact invariant set $K\subseteq X$ is equi-almost periodic if and only if the function $\Pi_K\in C(\mathbb{T},C(K,X)))$ is almost periodic, where $\Pi_K(t):=\pi^t\big|_K$ for any $t\in\mathbb{T}$ and $\pi^t:=\pi(t,\cdot)$.*

Proof. Let K be an equi-almost periodic compact invariant subset of $(X,\mathbb{T},\pi)$ and ε be an arbitrary positive number. Then there exists a relatively dense in $\mathbb{T}$ subset $\mathcal{P}(\varepsilon,K)$ of $\mathbb{T}$ such that (15) holds. Note that

$$d(\Pi_K(t+\tau),\Pi_K(t))=\max_{x\in K}\rho(\pi(t+\tau,x),\pi(t,x)) \tag{60}$$

for any $(t,\tau)\in\mathbb{T}\times\mathcal{P}(\varepsilon,K)$. From (15) and (60) we obtain

$$d(\Pi_K(t+\tau),\Pi_K(t))<\varepsilon \tag{61}$$

for any $(t,\tau)\in\mathbb{T}\times\mathcal{P}(\varepsilon,K)$. This means that the mapping $\Pi_K\in C(\mathbb{T},C(K,X))$ is almost periodic.

Assume now that the mapping $\Pi_K\in C(\mathbb{T},C(K,X))$ is almost periodic, i.e., for any $\varepsilon>0$ there exists a relatively dense in $\mathbb{T}$ subset $\mathcal{P}(\varepsilon,\Pi_K)$ of $\mathbb{T}$ such that (61) holds. Let now p be an arbitrary element from K and $(t,\tau)\in\mathbb{T}\times\mathcal{P}(\varepsilon,K)$ ($\mathcal{P}(\varepsilon,K):=\mathcal{P}(\varepsilon,\Pi_K)$), then taking into account (60) and (61) we obtain

$$\rho(\pi(t+\tau,p),\pi(t,p))\le\max_{x\in K}\rho(\pi(t+\tau,x),\pi(t,x))=d(\Pi_K(t+\tau),\Pi_K(t))<\varepsilon.$$

Lemma is completely proved. □

Lemma 20. *Assume that $K_1,K_2,\dots,K_m$ are the equi-almost periodic compact invariant subsets of $(X,\mathbb{T},\pi)$. Then the following statements hold:*

1. *the compact invariant subset $K:=K_1\times K_2\times\dots K_m$ of the product dynamical system $(X^m,\mathbb{T},[\pi])$ ($X^m:=X\times X\times\dots\times X$ and $[\pi](t,x):=(\pi(t,x_1),\pi(t,x_2),\dots,\pi(t,x_m))$ for any $t\in\mathbb{T}$ and $x:=(x_1,x_2,\dots,x_m)\in X^m$) is equi-almost periodic;*

2. *the compact invariant subset $K:=\bigcup_{i=1}^m K_i$ of $(X,\mathbb{T},\pi)$ is equi-almost periodic;*

3. *the compact invariant subset* $K := \bigcap_{i=1}^m K_i$ *of* $(X, \mathbb{T}, \pi)$ *is equi-almost periodic.*

Proof. If the compact invariant subset K_i $(i = 1, \dots, m)$ of $(X, \mathbb{T}, \pi)$ is equi-almost periodic, then by Lemma 19 the mapping $\Pi_{K_i} \in C(\mathbb{T}, C(K_i, X))$ $(i = 1, \dots, m)$ is almost periodic and, consequently, the mapping $(\Pi_{K_1}, \dots, \Pi_{K_m})$ is almost periodic (the mappings $\Pi_{K_1}, \dots, \Pi_{K_m}$ are jointly almost periodic). Note that

$$\Pi_K = (\Pi_{K_1}, \dots, \Pi_{K_m}).$$

By Lemma 19 the compact invariant set $K = K_1 \times \dots \times K_m$ of dynamical system $(X^m, \mathbb{T}, [\pi])$ is equi-almost periodic.

Let K_i $(i = 1, \dots, m)$ be a compact invariant subset of $(X, \mathbb{T}, \pi)$, then by the first statement of Lemma the compact invariant subsets $K_1, \dots, K_m$ are jointly equi-almost periodic, i.e., for any $\varepsilon > 0$ there exists a relatively dense subset $\mathcal{P}(\varepsilon, K_1, \dots, K_m)$ of $\mathbb{T}$ such that

$$\rho(\pi(t + \tau, x_i), \pi(t, x)) < \varepsilon/m \tag{62}$$

for any $t \in \mathbb{T}$, $\tau \in \mathcal{P}(\varepsilon, K_1, \dots, K_m)$ and $i = 1, \dots, m$.

Let x be an arbitrary point from $K := \bigcup_{i=1}^m K_i$, then there exists an index $i_0 \in \{1, \dots, m\}$ such that $x \in K_{i_0}$ and, consequently, from (62) we obtain

$$\rho(\pi(t + \tau, x), \pi(t, x)) = \rho(\pi(t + \tau, x_{i_0}), \pi(t, x_{i_0})) < \varepsilon \tag{63}$$

for any $(t, \tau, x) \in \mathbb{T} \times \mathcal{P}(\varepsilon, K) \times K$, where $\mathcal{P}(\varepsilon, K) := \mathcal{P}(\varepsilon, K_1, \dots, K_m)$.

Finally, if $x \in K = \bigcap_{i=1}^m K_i \neq \emptyset$, then $x \in K_i$ for any $i = 1, \dots, m$ and, consequently, we have the relation (63) for the point x. Lema is completely proved. □

Remark 10. Note that Lemma 20 remains true if the subset K_i is an equi-almost periodic compact invariant subset of the dynamical systems $(X_i, \mathbb{T}, \pi_i$ $(i = 1, \dots, m)$.

6 Two-Sided Remotely Almost Periodic Motions

Consider a two-sided dynamical system $(X, \mathbb{S}, \pi)$.

Definition 16. A point $x \in X$ (respectively, a motion $\pi(t, x)$) is said to be two-sided remotely almost periodic if the following two conditions are fulfilled:

1. the point x is Lagrange stable, i.e., the set $\Sigma_x := \{\pi(t, x) |\ t \in \mathbb{S}\}$ is precompact;
2. for any $\varepsilon > 0$ there exists a relatively dense in $\mathbb{S}$ subset $\mathcal{P}(\varepsilon, x)$ such that

   $$\lim_{|t| \to +\infty} \sup \rho(\pi(t + \tau, x), \pi(t, x)) < \varepsilon$$

   or equivalently for any $\varepsilon > 0$ and $\tau \in \mathcal{P}(\varepsilon, x)$ there exists a positive number $L(\varepsilon, x, \tau)$ such that

   $$\rho(\pi(t + \tau, x), \pi(t, x)) < \varepsilon$$

   for any $|t| \geq L(\varepsilon, x, \tau)$.

Theorem 15. *A motion $\pi(t,x)$ of dynamical system $(X,\mathbb{S},\pi)$ is two-sided remotely almost periodic if and only if its dynamical limit set $\Delta_x := \alpha_x \bigcup \omega_x$ is equi-almost periodic.*

Proof. Let the motion $\pi(t,x)$ be two-sided remotely almost periodic. Then for any $\varepsilon > 0$ there exists a relatively dense subset $\mathcal{P}(\varepsilon/2,x)$ such that for $\tau \in \mathcal{P}(\varepsilon/2,x)$ there exists a number $L(\varepsilon/2,x,\tau)$ for which we have

$$\rho(\pi(t+\tau,x),\pi(t,x)) < \varepsilon/2 \tag{64}$$

for any $|t| \geq L(\varepsilon/2,x,\tau)$.

On the other hand since the motion $\pi(t,x)$ is Lagrange stable then its dynamical limit set Δ_x is nonempty, compact and invariant. Let now $p \in \Delta_x$ be an arbitrary point, then there exists a sequence $\{t_n\} \subset \mathbb{S}$ such that $|t_n| \to +\infty$ and $\pi(t_n,x) \to p$ as $n \to \infty$. Let $n_\varepsilon \in \mathbb{N}$ be a natural number such that $|t_n| \geq L(\varepsilon/2,x,\tau)$ for any $n \geq n_\varepsilon$, then from (64) we have

$$\rho(\pi(t_n+\tau,x),\pi(t_n,x)) = \rho(\pi(\tau,\pi(t_n,x)),\pi(t_n,x)) < \varepsilon/2 \tag{65}$$

for any $n \geq n_\varepsilon$. Passing to the limit in (65) as $n \to \infty$ we receive

$$\rho(\pi(\tau,p),p) \leq \varepsilon/2 < \varepsilon$$

for any $\tau \in \mathcal{P}(\varepsilon/2,x)$ and $p \in \Delta_x$. Since the set Δ_x is invariant then $\pi(t,p) \in \Delta_x$ for any $(t,p) \in \mathbb{S} \times \Delta_x$ and, consequently, we have

$$\rho(\pi(t+\tau,\pi(t,p)) = \rho(\pi(\tau,\pi(t,p)),\pi(t,p)) \leq \varepsilon/2 < \varepsilon$$

for any $\tau \in \mathcal{P}(\varepsilon,\Delta_x)$ ($\mathcal{P}(\varepsilon,\Delta_x) := \mathcal{P}(\varepsilon/2,x)$) and $p \in \Delta_x$.

Assume that the motion $\pi(t,x)$ is Lagrange stable and its limit set Δ_x is equi-almost periodic. Then for any $\varepsilon > 0$ there exists a relatively dense in S subset $\mathcal{P}(\varepsilon,\Delta_x)$ of $\mathbb{S}$ such that

$$\rho(\pi(t+\tau,p),\pi(t,p)) < \varepsilon \tag{66}$$

for any $(t,\tau,p) \in \mathbb{S} \times \mathcal{P}(\varepsilon,\Delta_x) \times \Delta_x$. We will show that for any $\varepsilon > 0$ and $\tau \in \mathcal{P}(\varepsilon,x) := \mathcal{P}(\varepsilon,\Delta_x)$ there exists a positive number $L(\varepsilon,x,\tau)$ so that

$$\rho(\pi(t+\tau,x),\pi(t,x)) < \varepsilon$$

for any $|t| \geq L(\varepsilon,x,\tau)$. If we suppose that this is not true, then there exist positive number ε_0, $\tau_0 \in \mathcal{P}(\varepsilon_0,x)$ and a sequence $\{t_n\} \subset \mathbb{S}$ such that

$$\rho(\pi(t_n+\tau_0,x),\pi(t_n,x)) = \rho(\pi(\tau_0,\pi(t_n,x)),\pi(t_n,x)) \geq \varepsilon_0 \tag{67}$$

for any $n \in \mathbb{N}$ and $|t_n| \to +\infty$ as $n \to \infty$. Since the motion $\pi(t,x)$ is Lagrange stable then without loss of generality we can suppose that the sequence $\pi(t_n,x)$ converges. Denote by p_0 its limit. Taking into account that $|t_n| \to +\infty$ as $n \to \infty$ we have $p_0 \in \Delta_x$. Passing to the limit in the inequality (67) as $n \to \infty$ we obtain

$$\rho(\pi(\tau_0,p_0),p_0) \geq \varepsilon_0. \tag{68}$$

The relations (66) and (68) are contradictory. The obtained contradiction proves our statement. Theorem is completely proved. $\square$

Theorem 16. *Let $x \in X$ be a Lagrange stable point of the two-sided dynamical system $(X, \mathbb{S}, \pi)$. The point x is two-sided remotely almost periodic if and only if it is remotely almost periodic in both positive and negative directions.*

Proof. Let x be a two-sided remotely almost periodic point, then it is easy to check that the point x will be remotely almost periodic in both directions.

Let x be remotely almost periodic in the positive and negative directions. Then by Lemma 11 (see also Remark 7) the motion $\pi(t, x)$ is two-sided Lagrange stable and the limit sets ω_x and α_x are equi-almost periodic. According to Lemma 20 (item (ii)) the set $\Delta_x := \omega_x \bigcup \alpha_x$ is equi-almost periodic and by Theorem 15 the motion $\pi(t, x)$ is two-sided remotely almost periodic. Theorem is proved. □

Let $(Y, \mathbb{S}, \sigma)$ be a two-sided dynamical system. Denote by $\mathfrak{L}_y := \{\{t_n\} \subset \mathbb{S} |\ |t_n| \to +\infty$ and $\{\sigma(t_n, y)\}$ converges $\}$.

Theorem 17. *Let $y \in Y$ be a Lagrange stable and two-sided remotely almost periodic point of the dynamical system $(Y, \mathbb{S}, \sigma)$. If $\mathfrak{L}_y \subseteq \mathfrak{L}_x$, then the point x is Lagrange stable and x is two-sided remotely almost periodic.*

Proof. Firstly, under the conditions of theorem the point x is Lagrange stable. Let $\{t_n\}$ be an arbitrary sequence from $\mathbb{S}$. We will show that the sequence $\{\pi(t_n, x)\}$ is precompact in X. Indeed, since the point y is Lagrange stable then without loss of generality we can suppose that the sequence $\{\sigma(t_n, y)\}$ converges.

a. If from the sequence $\{t_n\}$ it is possible to extract a convergent subsequence $\{t_{k_n}\}$, then evidently the sequence $\{\pi(t_{k_n}, x)\}$ converges. Thus in this case the sequence $\{\pi(t_n, x)\}$ is precompact.

b. If from the sequence $\{t_n\}$ it is impossible to extract a convergent subsequence and, consequently, $\{t_n\} \in \mathfrak{L}_y \subseteq \mathfrak{L}_x$. Thus the sequence $\{\pi(t_n, x)\}$ is convergent.

Now we will show that the motion $\pi(t, x)$ is two-sided remotely almost periodic. To this end we note that the inclusion $\mathfrak{L}_y \subseteq \mathfrak{L}_x$ implies $\mathfrak{L}_y^{+\infty} \subseteq \mathfrak{L}_x^{+\infty}$ and $\mathfrak{L}_y^{-\infty} \subseteq \mathfrak{L}_x^{-\infty}$. According to Theorem 7 (see also Remark 7) from the inclusion $\mathfrak{L}_y^{+\infty} \subseteq \mathfrak{L}_x^{+\infty}$ (respectively, $\mathfrak{L}_y^{-\infty} \subseteq \mathfrak{L}_x^{-\infty}$) we obtain positive (respectively, negative) remote almost periodicity of the motion $\pi(t, x)$. Now to finish the proof of Theorem it suffices to apply Theorem 16. Theorem is completely proved. □

Definition 17. Let $(X, \mathbb{S}, \pi)$ be a two-sided dynamical system. A Lagrange stable point $x \in X$ is said to be

1. two-sided remotely τ-periodic if
$$\limsup_{|t| \to +\infty} \rho(\pi(t + \tau, x), \pi(t, x)) = 0; \tag{69}$$
2. two-sided remotely stationary if the point x is two-sided remotely τ-periodic for any $\tau \in \mathbb{S}$.

Theorem 18. *A Lagrange stable point $x \in X$*

1. *is two-sided remotely τ-periodic if and only if its dynamical limit set Δ_x consists of τ-periodic points;*

2. *is two-sided remotely stationary if and only if its dynamical limit set Δ_x consists of stationary points.*

Proof. Let $x \in X$ be a Lagrange stable, two-sided remotely τ-periodic point and ε be an arbitrary positive number. By (69) there exists a positive number $L = L(\varepsilon)$ such that

$$\rho(\pi(t+\tau, x), \pi(t, x)) < \varepsilon \tag{70}$$

for any $|t| \geq L(\varepsilon)$. Let now p be an arbitrary dynamical limit point of x. Then there exists a sequence $\{t_n\} \subset \mathbb{S}$ such that $|t_n| \to +\infty$ and

$$\lim_{n\to\infty} \pi(t_n, x) = p$$

as $n \to \infty$ and, consequently, we have

$$\begin{aligned} \rho(\pi(\tau, p), p) \leq \rho(\pi(\tau, p), \pi(\tau, \pi(t_n, x)) + \\ \rho(\pi(\tau, \pi(t_n, x)), \pi(t_n, x)) + \rho(\pi(t_n, x), p) \end{aligned} \tag{71}$$

for any $n \in \mathbb{N}$. From (70) and (71) we obtain

$$\rho(\pi(\tau, p), p) \leq \rho(\pi(\tau, p), \pi(\tau, \pi(t_n, x)) + \varepsilon + \rho(\pi(t_n, x), p) \tag{72}$$

for any $n \in \mathbb{N}$. Passing to the limit in (72) and taking into account the relation (71) we obtain

$$\rho(\pi(\tau, p), p) \leq \varepsilon$$

for any $\varepsilon > 0$ and, consequently, $\pi(\tau, p) = p$.

Converse. Let $x \in X$ be a Lagrange stable point and its dynamical limit set Δ_x consists of τ-periodic points. If we assume that the point x is not two-sided remotely τ-periodic, then there are $\varepsilon_0 > 0$ and a sequence $\{t_n\} \subset \mathbb{S}$ such that

$$\rho(\pi(t_n + \tau, x), \pi(t_n, x)) \geq \varepsilon_0 \tag{73}$$

for any $n \to \infty$ and $|t_n| \to +\infty$ as $n \to \infty$. Since the point x is Lagrange stable then without loss of generality we can suppose that the sequence $\{\pi(t_n, x)\}$ is convergent. Denote by $p_0 := \lim\limits_{n\to\infty} \pi(t_n, x)$ then $p_0 \in \Delta_x$ and passing to the limit in (73) we obtain

$$\rho(\pi(\tau, p_0), p_0) \geq \varepsilon_0.$$

The last relation contradicts to our assumption that Δ_x consists of τ-periodic points. This contradiction proves required statement.

The second statement of Theorem follows directly from the first one. In fact, if the Lagrange stable point $x \in X$ is two-sided remotely stationary, then the point x is τ-periodic and according to the first statement of Theorem the dynamically limit set Δ_x consists of τ-periodic points for any $\tau \in \mathbb{S}$, i.e., for every point $p \in \Delta_x$ we have $\pi(\tau, p) = p$ for any $\tau \in \mathbb{S}$.

Now assume the point $x \in X$ is Lagrange stable and its dynamically limit set Δ_x consists of stationary points of $(X, \mathbb{S}, \pi)$. We will show that

$$\limsup_{|t| \to +\infty} \rho(\pi(t+\tau, x), \pi(t, x)) = 0 \tag{74}$$

for any $\tau \in \mathbb{S}$. If we suppose that (74) is false, then there are $\tau_0 \in \mathbb{S}$ (with $\tau_0 \neq 0$), $\varepsilon_0 > 0$ and $\{t_n\} \subset \mathbb{S}$ such that $|t_n| \to \infty$ as $n \to \infty$ and

$$\rho(\pi(t_n + \tau_0, x), \pi(t_n, x)) \geq \varepsilon_0 \tag{75}$$

for any $n \in \mathbb{N}$. Since the point x is Lagrange stable then without loss of generality we can assume that the sequence $\{\pi(t_n, x)\}$ converges. Let p_0 be the limit of the sequence $\{\pi(t_n, x)\}$ then passing to the limit in (75) as $n \to \infty$ we obtain

$$\rho(\pi(\tau_0, p_0), p_0) \geq \varepsilon_0,$$

i.e., $p_0 \in \Delta_x$ is not a stationary point. The last fact contradict our assumption that the dynamical limit set Δ_x consists of stationary points. The obtained contradiction completes the proof of the second statement of theorem. □

7 Acknowledgments.

This research was supported by the State Program of the Republic of Moldova "Monotone Nonautonomous Dynamical Systems (24.80012.5007.20SE)" and partially was supported by the Institutional Research Program 011303 "SATGED", Moldova State University.

References

[1] Baskakov A. G., *Analysis of linear differential equations by methods of the spectral theory of difference operators and linear relations.* Uspekhi Matematicheskikh Nauk, 2013, Vol. 68, No.1 (409), pp.77-128 (in Russian) [English translation: Russian Mathematical Surveys, 2013, Vol.68, No.1, pp.69-116]

[2] Cheban D. N., Asymptotically Almost Periodic Solutions of Differential Equations. Hindawi Publishing Corporation, New York, 2009, ix+186 pp.

[3] Cheban D. N., *Global Attractors of Set-Valued Dynamical and Control Systems.* Nova Science Publishers Inc, New York, 2010, xvii+269 pp.

[4] David N. Cheban, *Nonautonomous Dynamics: Nonlinear Oscillations and Global Attractors.* Springer Nature Switzerland AG, 2020, xxii+ 434 pp.

[5] Karol Gryszka, *On Asymptotically Periodic-Like Motions in Flows.* Ann. Univ. Paedagog. Crac. Stud. Math., 17 (2018), pp.45-57.

[6] Kelley J., *General Topology.* D. Van Norstand Company Inc., Toronto–New York–London, 1957.

[7] Andrzej Pelczar, *Asymptotically Periodic Motions.* Bull. Polish Acad. Sci. Math., Vol.33, No. 5–6, 1985, pp.313–319.

[8] Ruess W. M. and Summers W. H., *Minimal Sets of Almost Periodic Motions.* Math. Ann., 276, 1986, pp.145-158.

[9] Sarason D., *Remotely almost periodic functions. Contemporary Mathematics,* Vol.32, 1984, pp.237-242.

[10] Shcherbakov B. A., *Recurrent Solutions of Differential Equations and the General Theory of Dynamical Systems.* Differentsial'nye Uravneniya, Vol.3, No.9, 1967, pp.1450-1460. (in Russian). [English translation in Differential Equations, Vol.3, No.9, 1967, pp.758-763]

[11] Sibirsky K. S., *Introduction to Topological Dynamics.* Kishinev, RIA AN MSSR, 1970, 144 p. (in Russian). [English translationn: Introduction to Topological Dynamics. Noordhoff, Leyden, 1975]

Moldova State University
Vladimir Andrunachievici Institute of
Mathematics and Computer Science
Laboratory of Differential Equations
str. Academiei 5
MD–2028 Chişinău, Moldova
E-mail: *david.ceban@usm.md,*
davidcheban@yahoo.com

Received September 8, 2024