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Evolution Time of Stochastic Systems with Multiple Final Sequences of States

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Abstract. A stochastic system with multiple final sequences of states represents a stochastic system that stops its evolution as soon as one of the given final sequences of states is reached. The transition time of the system is unitary and the transition probability depends on source and destination states. We prove that the distribution of the evolution time is a homogeneous linear recurrent sequence and, based on this, a polynomial algorithm for determining the initial state and the generating vector of this recurrence is developed. Using the generating function, the main probabilistic characteristics are determined.

Mathematics subject classification: 65C40, 60J22, 90C39, 90C40.
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1 Introduction and Problem Formulation

The zero-order Markov processes with multiple final sequences of states were introduced in [1] and the linear recurrent homogeneity property of the distribution of their evolution time has been used for determining the main probabilistic characteristics of the evolution time in [3].

In this paper a generalization of these Markov processes is considered. The stochastic systems with multiple final sequences of states are defined in the same way, the difference consisting only in transition probability matrix which does not have all the rows equal. This means that the transition probability from one source state to another destination state depends on both states, not only on destination state.

Here, similarly as in [1], we consider a discrete stochastic system L with finite set of states $V = \{v_1, v_2, \dots, v_\omega\}$, $|V| = \omega$. At every discrete moment of time $t \in \mathbb{N}$ the state of the system is $v(t) \in V$. The system L starts its evolution from the state v with the probability $p^*(v)$, for each $v \in V$, where $\sum\limits_{v \in V} p^*(v) = 1$.

At this point, we introduce the transition probability matrix of the Markov process L. The transition from one state u to another state v is performed according to the probability $p(u, v)$ that depends on the source state u and destination state v, for every $u \in V$ and $v \in V$. Additionally, we assume that r different sequences of states $X^{(\ell)} = (x_1^{(\ell)}, x_2^{(\ell)}, \dots, x_m^{(\ell)}) \in V^m, \ell = \overline{1, r}$, are given and the stochastic

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system stops as soon as the states $x_1^{(\ell)}, x_2^{(\ell)}, \ldots, x_m^{(\ell)}$ are reached consecutively in given order for an arbitrary $\ell \in \{1, 2, \ldots, r\}$. The time T, when the system stops, is called evolution time of the stochastic system L with given final sequences of states $X = \{X^{(1)}, X^{(2)}, \ldots, X^{(r)}\}$.

The system L, described above, represents a stochastic system with final sequences of states $X = \{X^{(1)}, X^{(2)}, \ldots, X^{(r)}\}$. For the particular case $r = 1$, important results were elaborated and presented in [3], [9] and [10]. Also, several interpretations of these Markov processes for the games, compositions and optimization problems were analyzed in [2], [4] and [5]. Based on polynomial algorithms proposed in [3], the main probabilistic characteristics (expectation, variance, mean square deviation, n-order moments) of evolution time and game duration were efficiently determined.

Next, in this paper, the generalization of this problem for any $r \geq 1$ is considered. This generalized problem is a bit different than the parallel compositions, studied in [5], because the dynamics of the systems are performed in a mixed one and they are interdependent.

Our goal is to analyze the evolution time T of the stochastic system L. We prove that the distribution of the evolution time T is a homogeneous linear recurrent sequence, and a polynomial algorithm to determine the initial state and the generating vector of this recurrence is developed. Having the generating vector and the initial state of the recurrence, we can use the related algorithm from [3], which was mentioned above, for determining the main probabilistic characteristics of the evolution time.

2 Determining the Distribution of the Evolution Time

In this section we will determine the distribution of the evolution time T. We assume that $m \geq 2$ and introduce the following notations

$$\begin{aligned} &X_k^{(\ell)} = \{x_k^{(\ell)}\},\ \pi_k^{(\ell)} = p^*(x_k^{(\ell)}),\ \pi_{ik}^{(\ell)} = p(x_i^{(\ell)}, x_k^{(\ell)}),\ w_k^{(\ell)} = \prod_{j=3}^{k} \pi_{j-1,j}^{(\ell)}\ , \\ &Y_k^{(\ell)} = (x_1^{(\ell)}, x_2^{(\ell)}, \ldots, x_k^{(\ell)}),\ Y_k = \{Y_k^{(1)}, Y_k^{(2)}, \ldots, Y_k^{(r)}\}, \end{aligned} \tag{1}$$

for each $i, k = \overline{1, m}$ and $\ell = \overline{1, r}$.

Let $a = (a_n)_{n=0}^{\infty}$ be the distribution of the evolution time T, i.e. $a_n = \mathbb{P}(T = n)$, $n = \overline{0, \infty}$. Since $T \geq m - 1$, we have $a_n = 0$, $n = \overline{0, m-2}$. If $T = m - 1$, then $\exists \ell \in \{1, 2, \ldots, r\}$ such that $v(j) = x_{j+1}^{(\ell)}$, $j = \overline{0, m-1}$, that implies

$$a_{m-1} = \mathbb{P}(T = m-1) = \sum_{\ell=1}^{r} \left(p^*(x_1^{(\ell)}) \prod_{j=2}^{m} p(x_{j-1}^{(\ell)}, x_j^{(\ell)}) \right) =$$

$$= \sum_{\ell=1}^{r} \left(\pi_1^{(\ell)} \pi_{1,2}^{(\ell)} \pi_{2,3}^{(\ell)} \ldots \pi_{m-1,m}^{(\ell)} \right) = \sum_{\ell=1}^{r} \left(\pi_1^{(\ell)} \pi_{1,2}^{(\ell)} w_m^{(\ell)} \right). \tag{2}$$

We consider $\forall n \in \mathbb{Z}$. Let be $S(V) = \{A \mid A \subseteq V\}$. Denote by $P_\Phi(n)$ the probability that $T = n$ and $v(j) \in \Phi_j$, $j = \overline{0, t-1}$, supposing that the initial state of the system is known, for all $\Phi = (\Phi_j)_{j=0}^{t-1} \in (S(V))^t$, $t \in \mathbb{N}$ and $\ell = \overline{1,r}$. We introduce the following functions on $\mathbb{Z}$, $k = \overline{1,m}$, $\ell = \overline{1,r}$:

$$\begin{aligned} \beta_k^{(\ell)}(n) &= P_{(X_1^{(\ell)}, X_2^{(\ell)}, \ldots, X_k^{(\ell)})}(n), \\ \gamma_k^{(\ell)}(n) &= P_{(X_2^{(\ell)}, X_3^{(\ell)}, \ldots, X_k^{(\ell)})}(n). \end{aligned} \tag{3}$$

After that, we extend the definition of the functions $\beta_k^{(\ell)}(n)$ for $\ell = \overline{r+1, r+\omega}$ in the following way:

$$\beta_k^{(r+i)}(n) = P_{(\{v_i\})}(n), \ i = \overline{1,\omega}, \ k = \overline{1,m}, \tag{4}$$

where $v_1, v_2, \ldots, v_\omega$, as defined in Section 1, are all the states of the stochastic system L in a predefined order. If, for each $x \in V$, we denote by $\iota(x)$ the index that satisfies the equality $v_{\iota(x)} = x$, then the relation (4) becomes

$$P_{(\{x\})}(n) = P_{(\{v_{\iota(x)}\})}(n) = \beta_1^{(r+\iota(x))}(n), \ \forall x \in V. \tag{5}$$

On the other hand, for each $x \in \{x_1^{(1)}, x_1^{(2)}, \ldots, x_1^{(r)}\}$, there exists at least one index $\ell(x) \in \{1, 2, \ldots, r\}$ such that $x = x_1^{(\ell(x))}$. So, in this case, we also have

$$\beta_k^{(r+\iota(x))}(n) = P_{(\{x\})}(n) = \beta_1^{(\ell(x))}(n), \ k = \overline{1,m}. \tag{6}$$

Instead, for $x \notin \{x_1^{(1)}, x_1^{(2)}, \ldots, x_1^{(r)}\}$, we can write the following relation:

$$\begin{aligned} \beta_k^{(r+\iota(x))}(n) = P_{(\{x\})}(n) &= \sum_{y \in V} p(x,y) P_{(\{y\})}(n-1) = \\ &= \sum_{y \in V} p(x,y) \beta_1^{(r+\iota(y))}(n-1), \ k = \overline{1,m}. \end{aligned} \tag{7}$$

Let firstly analyze the functions $\beta_k^{(\ell)}(n)$ for $\forall n \geq m$, $k = \overline{1,m}$, $\ell = \overline{1,r}$. For every $k > 1$ we have

$$\begin{aligned} \beta_k^{(\ell)}(n) &= P_{(X_1^{(\ell)}, X_2^{(\ell)}, \ldots, X_k^{(\ell)})}(n) = \\ &= \pi_{1,2}^{(\ell)} P_{(X_2^{(\ell)}, \ldots, X_k^{(\ell)})}(n-1) - \sum_{j:\ 1 \leq j \leq r,\ Y_k^{(j)} = Y_k^{(\ell)}} \pi_{1,2}^{(j)} P_{(X_2^{(j)}, \ldots, X_m^{(j)})}(n-1) = \\ &= \pi_{1,2}^{(\ell)} \gamma_k^{(\ell)}(n-1) - \sum_{j=1}^{r} \left(\pi_{1,2}^{(j)} \cdot I_{\{s \mid Y_k^{(s)} = Y_k^{(\ell)}\}}(j) \cdot \gamma_m^{(j)}(n-1) \right). \end{aligned} \tag{8}$$

Similarly, for $k = 1$, the next formula holds

$$\beta_1^{(\ell)}(n) = P_{(X_1^{(\ell)})}(n) =$$

$$= \sum_{y \in V} p(x_1^{(\ell)}, y) P_{(\{y\})}(n-1) - \sum_{j:\ 1 \le j \le r,\ Y_1^{(j)} = Y_1^{(\ell)}} \pi_{1,2}^{(j)} P_{(X_2^{(j)}, \ldots, X_m^{(j)})}(n-1) =$$

$$= \sum_{y \in V} p(x_1^{(\ell)}, y) \beta_1^{(r+\iota(y))}(n-1) - \sum_{j=1}^{r} \left(\pi_{1,2}^{(j)} \cdot I_{\{s | Y_k^{(s)} = Y_k^{(\ell)}\}}(j) \cdot \gamma_m^{(j)}(n-1) \right). \quad (9)$$

By combining the relations (8) – (9) we obtain the following formula for $\forall n \ge m$, $k = \overline{1,m}$, $\ell = \overline{1,r}$:

$$\beta_k^{(\ell)}(n) = \pi_{1,2}^{(\ell)} \cdot I_{\mathbb{N} \setminus \{1\}}(k) \cdot \gamma_k^{(\ell)}(n-1) +$$
$$+ \sum_{y \in V} \left(p(x_1^{(\ell)}, y) \cdot I_{\{1\}}(k) \cdot \beta_1^{(r+\iota(y))}(n-1) \right) -$$
$$- \sum_{j=1}^{r} \left(\pi_{1,2}^{(j)} \cdot I_{\{s | Y_k^{(s)} = Y_k^{(\ell)}\}}(j) \cdot \gamma_m^{(j)}(n-1) \right). \quad (10)$$

We consider the sets

$$T_s^{(\ell)} = \{s+1\} \cup \{t \in \{2, 3, \ldots, s\} \mid (x_t^{(\ell)}, x_{t+1}^{(\ell)}, \ldots, x_s^{(\ell)}) \in Y_{s+1-t}\},$$

for each $s = \overline{1,m}$ and $\ell = \overline{1,r}$. The minimal elements from these sets are

$$t_s^{(\ell)} = \min_{k \in T_s^{(\ell)}} k, \ s = \overline{1,m}, \ \ell = \overline{1,r}. \quad (11)$$

The value $t_s^{(\ell)}$ represents the position in the sequence $(x_1^{(\ell)}, x_2^{(\ell)}, \ldots, x_s^{(\ell)})$ starting with which, if we overlap a final sequence of states $X^{(\tau_s^{(\ell)})} \in X$, the superposed elements are equal. Here by $\tau_s^{(\ell)}$ we denote the minimal index from the set $\{1, 2, \ldots, r\}$ that satisfies given condition.

Next, we analyze the expression $\gamma_s^{(\ell)}(n-1)$ for $s = \overline{1,m}$ and $\ell = \overline{1,r}$. For $s = 1$ we have

$$\gamma_1^{(\ell)}(n-1) = P_{()}(n-1) = \sum_{x \in V} P_{(\{x\})}(n-1) = \sum_{x \in V} \beta_1^{(r+\iota(x))}(n-1). \quad (12)$$

Instead, for $s = \overline{2,m}$, we have two cases. In the case when $t_s^{(\ell)} \le s$ we obtain

$$\gamma_s^{(\ell)}(n-1) = P_{(X_2^{(\ell)}, X_3^{(\ell)}, \ldots, X_s^{(\ell)})}(n-1) =$$
$$= \pi_{2,3}^{(\ell)} \pi_{3,4}^{(\ell)} \ldots \pi_{t_s^{(\ell)}-1, t_s^{(\ell)}}^{(\ell)} P_{(X_{t_s^{(\ell)}}, X_{t_s^{(\ell)}+1}, \ldots, X_s)}(n - t_s^{(\ell)} + 1) =$$
$$= w_{t_s^{(\ell)}}^{(\ell)} P_{(X_1^{(\tau_s^{(\ell)})}, X_2^{(\tau_s^{(\ell)})}, \ldots, X_{s+1-t_s^{(\ell)}}^{(\tau_s^{(\ell)})})}(n - t_s^{(\ell)} + 1) =$$
$$= w_{t_s^{(\ell)}}^{(\ell)} \beta_{s+1-t_s^{(\ell)}}^{(\tau_s^{(\ell)})}(n - t_s^{(\ell)} + 1). \quad (13)$$

and for $t_s^{(\ell)} = s+1$ we have

$$\gamma_s^{(\ell)}(n-1) = P_{(X_2^{(\ell)}, X_3^{(\ell)}, \dots, X_s^{(\ell)})}(n-1) =$$
$$= \pi_{2,3}^{(\ell)} \pi_{3,4}^{(\ell)} \dots \pi_{s-1,s}^{(\ell)} P_{(x_s^{(\ell)})}(n-s+1) =$$
$$= w_s^{(\ell)} \beta_1^{(r+\iota(x_s^{(\ell)}))}(n-s+1). \tag{14}$$

By combining the relations (12) – (14) we obtain the next formula for $\forall n \geq m$, $s = \overline{1,m}$, $\ell = \overline{1,r}$:

$$\gamma_s^{(\ell)}(n-1) = \sum_{x \in V} I_{\{1\}}(s) \cdot \beta_1^{(r+\iota(x))}(n-1) +$$
$$+ w_{t_s^{(\ell)}}^{(\ell)} \cdot I_{\{k|2 \leq t_k^{(\ell)} \leq k\}}(s) \cdot \beta_{s+1-t_s^{(\ell)}}^{(\tau_s^{(\ell)})}(n - t_s^{(\ell)} + 1) +$$
$$+ w_s^{(\ell)} \cdot I_{\{k|3 \leq t_k^{(\ell)} = k+1\}}(s) \cdot \beta_1^{(r+\iota(x_s^{(\ell)}))}(n-s+1). \tag{15}$$

Substituting the relation (15) in (10) and taking into account that $I_{\{1\}}(m) = 0$ for $m \geq 2$ and $I_{\mathbb{N}\setminus\{1\}}(k) \cdot I_{\{1\}}(k) = 0$, we obtain the following recurrence for $k = \overline{1,m}$, $\ell = \overline{1,r}$ and $\forall n \in \mathbb{Z}$:

$$\beta_k^{(\ell)}(n) = \pi_{1,2}^{(\ell)} \cdot I_{\mathbb{N}\setminus\{1\}}(k) \cdot \gamma_k^{(\ell)}(n-1) +$$
$$+ \sum_{y \in V} \left(p(x_1^{(\ell)}, y) \cdot I_{\{1\}}(k) \cdot \beta_1^{(r+\iota(y))}(n-1) \right) -$$
$$- \sum_{j=1}^{r} \left(\pi_{1,2}^{(j)} \cdot I_{\{s|Y_k^{(s)} = Y_k^{(\ell)}\}}(j) \cdot \gamma_m^{(j)}(n-1) \right) =$$
$$= \sum_{y \in V} \left(p(x_1^{(\ell)}, y) \cdot I_{\{1\}}(k) \cdot \beta_1^{(r+\iota(y))}(n-1) \right) +$$
$$+ \Bigg[\pi_{1,2}^{(\ell)} \cdot I_{\mathbb{N}\setminus\{1\}}(k) \cdot$$
$$\Bigg(\sum_{y \in V} I_{\{1\}}(k) \cdot \beta_1^{(r+\iota(y))}(n-1) +$$
$$+ w_{t_k^{(\ell)}}^{(\ell)} \cdot I_{\{s|2 \leq t_s^{(\ell)} \leq s\}}(k) \cdot \beta_{k+1-t_k^{(\ell)}}^{(\tau_k^{(\ell)})}(n - t_k^{(\ell)} + 1) +$$
$$+ w_k^{(\ell)} \cdot I_{\{s|3 \leq t_s^{(\ell)} = s+1\}}(k) \cdot \beta_1^{(r+\iota(x_k^{(\ell)}))}(n-k+1) \Bigg) \Bigg]$$
$$- \sum_{j=1}^{r} \Bigg[\pi_{1,2}^{(j)} \cdot I_{\{s|Y_k^{(s)} = Y_k^{(\ell)}\}}(j) \cdot$$

$$\Big(\sum_{y\in V} I_{\{1\}}(m)\cdot\beta_1^{(r+\iota(y))}(n-1)+$$

$$+w_{t_m^{(j)}}^{(j)}\cdot I_{\{s|2\le t_s^{(j)}\le s\}}(m)\cdot\beta_{m+1-t_m^{(j)}}^{(\tau_m^{(j)})}(n-t_m^{(j)}+1)+$$

$$+w_m^{(j)}\cdot I_{\{s|3\le t_s^{(j)}=s+1\}}(m)\cdot\beta_1^{(r+\iota(x_m^{(j)}))}(n-m+1)\Big)\Big]=$$

$$=\sum_{y\in V}\Big(p(x_1^{(\ell)},y)\cdot I_{\{1\}}(k)\Big)\cdot\beta_1^{(r+\iota(y))}(n-1)+$$

$$+\pi_{1,2}^{(\ell)}\cdot w_{t_k^{(\ell)}}^{(\ell)}\cdot I_{\{s|2\le t_s^{(\ell)}\le s\}}(k)\cdot\beta_{k+1-t_k^{(\ell)}}^{(\tau_k^{(\ell)})}(n-t_k^{(\ell)}+1)+$$

$$+\pi_{1,2}^{(\ell)}\cdot w_k^{(\ell)}\cdot I_{\{s|3\le t_s^{(\ell)}=s+1\}}(k)\cdot\beta_1^{(r+\iota(x_k^{(\ell)}))}(n-k+1)-$$

$$-\sum_{j=1}^{r}\Big(\pi_{1,2}^{(j)}\cdot w_{t_m^{(j)}}^{(j)}\cdot I_{\{s|Y_k^{(s)}=Y_k^{(\ell)},\ 2\le t_m^{(j)}\le m\}}(j)\cdot\beta_{m+1-t_m^{(j)}}^{(\tau_m^{(j)})}(n-t_m^{(j)}+1)\Big)-$$

$$-\sum_{j=1}^{r}\Big(\pi_{1,2}^{(j)}\cdot w_m^{(j)}\cdot I_{\{s|Y_k^{(s)}=Y_k^{(\ell)},\ 3\le t_m^{(j)}=m+1\}}(j)\cdot\beta_1^{(r+\iota(x_m^{(j)}))}(n-m+1)\Big). \qquad (16)$$

According to recurrent relations (6), (7) and (16), there exist some real coefficients $v_{jks\ell}^{(i)}$, $j=\overline{0,m-1}$, $k,s=\overline{1,m}$, $i,\ell=\overline{1,r+\omega}$, such that

$$\beta_k^{(\ell)}(n)=\sum_{i=1}^{r+\omega}\sum_{j=0}^{m-1}\sum_{s=1}^{m}v_{jks\ell}^{(i)}\ \beta_s^{(i)}(n-1-j),\ k=\overline{1,m},\ \ell=\overline{1,r+\omega},\ \forall n\ge m. \qquad (17)$$

So, we have

$$\beta_k(n)=\sum_{j=0}^{m-1}\sum_{s=1}^{m}V_{jks}\ \beta_s(n-1-j),\ k=\overline{1,m},\ \forall n\ge m,$$

where $V_{jks}=(v_{jks\ell}^{(i)})_{\ell,\ i=\overline{1,r+\omega}}$, $\beta_k(n)=(\beta_k^{(\ell)}(n))_{\ell=\overline{1,r+\omega}}$, $k,s=\overline{1,m}$, $j=\overline{0,m-1}$. This recurrence relation can be written in the form

$$\beta(n)=\sum_{j=0}^{m-1}V_j\ \beta(n-1-j),\ \forall n\ge m,$$

where $V_j=(V_{jks})_{k,s=\overline{1,m}}$ and $\beta(n)=((\beta_k(n))_{k=1}^m)^T$, $j=\overline{0,m-1}$, $\forall n\in\mathbb{Z}$. From this relation, we obtain that $\beta=(\beta(n))_{n=0}^{\infty}\in Rol^*[\mathcal{M}_m(\mathcal{M}_{r+\omega}(\mathbb{R}))][m]$ with generating vector $V=(V_j)_{j=0}^{m-1}\in G^*[\mathcal{M}_m(\mathcal{M}_{r+\omega}(\mathbb{R}))][m](\beta)$. Using the results from [2], we have $\beta\in Rol^*[\mathbb{R}][m^2(r+\omega)]$, which implies that also

$$(\beta_k^{(\ell)}(n))_{n=0}^{\infty}\in Rol^*[\mathbb{R}][m^2(r+\omega)],\ k=\overline{1,m},\ \ell=\overline{1,r+\omega},$$

with the same generating vector. Since

$$a_n = \sum_{x\in V} p^*(x)P_{(x)}(n) = \sum_{x\in V} p^*(x)\beta_1^{(r+\iota(x))}(n),\ \forall n\in\mathbb{Z}, \tag{18}$$

we have

$$a = (a_n)_{n=0}^{\infty} \in Rol^*[\mathbb{R}][m^2(r+\omega)].$$

Next, we will use only the relation $a\in Rol^*[\mathbb{C}][m^2(r+\omega)]$, the minimal generating vector being determined using the minimization method based on the matrix rank, described in [3]. So, according to this method, we have that the minimal generating vector $q=(q_0,q_1,\ldots,q_{R-1})\in G^*[\mathbb{C}][R](a)$ is obtained from the unique solution $x=(q_{R-1},q_{R-2},\ldots,q_0)$ of the system

$$A_R^{[a]}x^T = (f_R^{[a]})^T, \tag{19}$$

where

$$f_R^{[a]} = (a_R, a_{R+1},\ldots,a_{2R-1}),\ A_n^{[a]} = (a_{i+j})_{i,j=\overline{0,n-1}},\ \forall n\in\mathbb{N}^* \tag{20}$$

and R is the rank of the matrix $A_{m^2(r+\omega)}^{[a]}$.

In order to apply this minimization method, we need to have only the values a_k, $k=\overline{0,2m^2(r+\omega)-1}$. These values can be determined using the recurrences (2), (6), (7), (16) and (18).

For the case $m=1$ we have similar recurrent formula as (17). Indeed, it is easy to observe the following relations for $P_{(\{x\})}(n)$, $\forall x\in V$, $\forall n\in\mathbb{N}$:

$$\beta_1^{(\ell)}(0) = \beta_1^{(r+\iota(x_1^{(\ell)}))}(0) = P_{(X_1^{(\ell)})}(0) = 1,$$

$$\beta_1^{(\ell)}(n) = \beta_1^{(r+\iota(x_1^{(\ell)}))}(n) = P_{(X_1^{(\ell)})}(n) = 0,\ \forall n\geq 1,\ \ell=\overline{1,r},$$

$$\beta_1^{(r+\iota(x))}(0) = P_{(\{x\})}(0) = 0,$$

$$\beta_1^{(r+\iota(x))}(n) = P_{(\{x\})}(n) =$$

$$= \sum_{y\in V} p(x,y)P_{(\{y\})}(n-1) = \sum_{y\in V} p(x,y)\beta_1^{(r+\iota(y))}(n-1),$$

$$\forall n\geq 1,\ \forall x\in V\backslash\{x_1^{(1)},x_1^{(2)},\ldots,x_1^{(r)}\}. \tag{21}$$

As result, these relations can be written in the same form as (17):

$$\beta_1^{(\ell)}(n) = \sum_{i=1}^{r+\omega}\sum_{j=0}^{0}\sum_{s=1}^{1} v_{jks\ell}^{(i)}\ \beta_s^{(i)}(n-1-j),\ \ell=\overline{1,r+\omega},\ \forall n\geq 1,$$

this meaning that the homogeneous linear recurrence of the sequence $a=(a_n)_{n=0}^{\infty}$, proved above, is applicable for $m=1$ too.

3 Describing the developed algorithm

In previous section we theoretically grounded the following algorithm for determining the main probabilistic characteristics of the evolution time T: the distribution $(\mathbb{P}(T=n))_{n=0}^{\infty}$, the expectation $\mathbb{E}(T)$, the variance $\mathbb{V}(T)$, the mean square deviation $\sigma(T)$ and the k-order moments $\nu_k(T)$, $k=1,2,\ldots$.

Algorithm 1.

Input: $X^{(\ell)}=(x_1^{(\ell)},x_2^{(\ell)},\ldots,x_m^{(\ell)})\in V^m$, $\ell=\overline{1,r}$, $p^*(x)$ and $p(x,y)$, $\forall x,y\in V$;
Output: $\mathbb{E}(T),\ \mathbb{V}(T),\ \sigma(T),\ \nu_k(T),\ k=\overline{1,t},\ t\geq 2$.

1. Determine the values a_k, $k=\overline{0,2m^2(r+\omega)-1}$, using the recurrence (16) and the relations (2), (6), (7), (18) and (21);

2. Find the minimal generating vector $q=(q_0,q_1,\ldots,q_{R-1})\in G^*[\mathbb{R}][R](a)$ by solving the system (19), taking into account the relation (20);

3. Consider the distribution $a=(a_n)_{n=0}^{\infty}=(\mathbb{P}(T=n))_{n=0}^{\infty}$ of the evolution time T as a homogeneous linear recurrence with the initial state $I_R^{[a]}=(a_n)_{n=0}^{R-1}$ and the minimal generating vector $q=(q_k)_{k=0}^{R-1}$, determined at the steps 1 and 2;

4. Determine the expectation $\mathbb{E}(T)$, the variance $\mathbb{V}(T)$, the mean square deviation $\sigma(T)$ and the k-order moments $\nu_k(T)$, $k=\overline{1,t}$, of the evolution time T by using the corresponding algorithm from [3].

4 Conclusions

A generalization of zero-order Markov processes with multiple final sequences of states is formulated and studied. It is shown that the evolution time of such stochastic system is a discrete random variable with homogeneous linear recurrent distribution. Based on this, algorithms for determining the main probabilistic characteristics of the evolution time are proposed.

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