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Existence of positive periodic solution of second-order neutral differential equations with distributed deviating arguments

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Abstract. In this work, sufficient conditions are established for the existence of positive ω-periodic solutions of the second-order neutral differential equations with distributed deviating arguments. The proof of our results is based on the Krasnoselskii fixed point theorem. An example is given to illustrate the application of the results.

Mathematics subject classification: 34C25, 34K13..
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1 Introduction

In this article, we study the existence of positive T-periodic solutions for the following second-order neutral differential equations:

$$\left[x(t) - c(t)\int_a^b x(t-\sigma(t,\mu))d\mu\right]'' = a(t)x(t) - \int_a^b f(t, x(t-\sigma(t,\mu)))d\mu \tag{1}$$

and

$$\left[x(t) - c(t)\int_a^b x(t-\sigma(t,\mu))d\mu\right]'' = -a(t)x(t) + \int_a^b f(t, x(t-\sigma(t,\mu)))d\mu, \tag{2}$$

where $c \in C(\mathbb{R},\mathbb{R})$, $a \in C(\mathbb{R},(0,\infty))$ are T- periodic functions, $\sigma \in C(\mathbb{R}\times[a,b],\mathbb{R})$ $(b > a \geqslant 0)$ is a T- periodic function with respect to t and $f \in C(\mathbb{R}\times\mathbb{R},\mathbb{R})$ is a T-periodic in its first variable.

These equations appear in a number of fields, such as mechanics, physics, and biology, as discussed in [10, 11, 15]. Recently, several authors have investigated the existence of positive periodic solutions for first- and second-order neutral differential equations, as seen in [2–9, 12–14, 16] and related references.

This paper is motivated by recent work [8], in which the authors investigate the existence of positive periodic solutions for this problem.

$$[x(t) - cx(t-\tau(t))]'' = a(t)x(t) - f(t, x(t-\tau(t))) \tag{3}$$

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and

$$[x(t) - cx(t - \tau(t))]'' = -a(t)x(t) + f(t, x(t - \tau(t))), \tag{4}$$

where $\tau \in C(\mathbb{R}, \mathbb{R})$, $a \in C(\mathbb{R}, (0, \infty))$, $f \in C(\mathbb{R} \times [0, \infty), [0, \infty))$, and a, τ are T-periodic functions, with f being T-periodic with respect to the first variable. In this paper, we extend the results in [8] to the case of distributed deviating arguments and variable coefficient $c(t)$ in (1) and (2), as opposed to the constant coefficient c in (3) and (4).

The rest of this paper is organized as follows. In Section 2, we introduce some notation and state some lemmas from [8]. In Section 3, we present our existence results for equations (1) and (2), respectively, along with an example.

2 Preliminaries

Let $\Phi_T = \{\phi(t) : \phi(t) \in C(\mathbb{R}, \mathbb{R}), \quad \phi(t + T) = \phi(t), \quad t \in \mathbb{R}\}$ be with the sup norm $\|\phi\| = \sup\limits_{t \in [0,T]} |\phi(t)|$. It is clear that Φ_T is a Banach space. Define

$$C_T^+ = \{\phi(t) : \phi(t) \in C(\mathbb{R}, (0, \infty)), \quad \phi(t + T) = \phi(t)\},$$
$$C_T^- = \{\phi(t) : \phi(t) \in C(\mathbb{R}, (-\infty, 0)), \quad \phi(t + T) = \phi(t)\}.$$

Let $M = \max\{a(t) : t \in [0, T]\}, \quad m = \min\{a(t) : t \in [0, T]\} \quad$ and $\quad \beta = \sqrt{M}$.

Lemma 1. *([8]) The equation*

$$y''(t) - My(t) = h(t), \quad h \in C_T^-$$

has a unique T-periodic solution

$$y(t) = \int_t^{t+T} G_1(t, s)(-h(s))ds,$$

where

$$G_1(t, s) = \frac{\exp(-\beta(s - t)) + \exp(\beta(s - t - T))}{2\beta(1 - \exp(-\beta T))}, \quad s \in [t, t + T].$$

Lemma 2. *([8]) $G_1(t, s) > 0$ and $\int_t^{t+T} G_1(t, s)ds = \frac{1}{M}$ for all $t \in [0, T]$ and $s \in [t, t + T]$.*

Lemma 3. *([8]) Consider the equation*

$$y''(t) - a(t)y(t) = h(t), \quad h \in C_T^-. \tag{5}$$

Define $T_1, B_1 : \Phi_T \to \Phi_T$ by

$$(T_1 h)(t) = \int_t^{t+T} G_1(t, s)(-h(s))ds, \quad (B_1 y)(t) = [-M + a(t)]y(t).$$

Obviously T_1 *and* B_1 *are completely continuous,* $(T_1h)(t) > 0$ *for* $h(t) < 0$ *and* $\|B_1\| \leq (M-m)$*. By Lemma 1, the solution of (5) can be written as* $y(t) = (T_1h)(t) + (T_1B_1y)(t)$*. Since* $\|T_1B_1\| \leq 1 - \frac{m}{M} < 1$,

$$y(t) = (I - T_1B_1)^{-1}(T_1h)(t).$$

Define $P_1 : \Phi_T \to \Phi_T$ *by*

$$(P_1h)(t) = (I - T_1B_1)^{-1}(T_1h)(t),$$

since T_1 *and* B_1 *are completely continuous,* P_1 *is completely continuous. Moreover, we have*

$$0 < (T_1h)(t) \leqslant (P_1h)(t) \leqslant \frac{M}{m}\|T_1h\|, \quad h \in C_T^-.$$

Lemma 4. *([8]) The equation*

$$y''(t) + My(t) = h(t), \quad h \in C_T^+$$

has a unique T*-periodic solution*

$$y(t) = \int_t^{t+T} G_2(t,s)h(s)ds,$$

where

$$G_2(t,s) = \frac{\cos(\beta(\frac{T}{2} + t - s))}{2\beta \sin(\frac{\beta T}{2})}, \quad s \in [t, t+T].$$

Lemma 5. *([8])* $\int_t^{t+T} G_2(t,s)ds = \frac{1}{M}$ *for all* $t \in [0,T]$*. Furthermore, if* $M < (\frac{\pi}{T})^2$*, then* $G_2(t,s) > 0$ *for all* $t \in [0,T]$ *and* $s \in [t, t+T]$*.*

Lemma 6. *([8]) Let* $M < (\frac{\pi}{T})^2$*. Consider the equation*

$$y''(t) + a(t)y(t) = h(t), \quad h \in C_T^+. \tag{6}$$

Define $T_2, B_2 : \Phi_T \to \Phi_T$ *by*

$$(T_2h)(t) = \int_t^{t+T} G_2(t,s)(h(s))ds, \quad (B_2y)(t) = [M - a(t)]y(t).$$

Clearly, T_2 *and* B_2 *are completely continuous,* $(T_2h)(t) > 0$ *for* $h(t) > 0$ *and* $\|B_2\| \leq (M - m)$*. By Lemmma 1, the solution of (5) can be written as* $y(t) = (T_2h)(t) + (T_2B_2y)(t)$*. Since* $\|T_2B_2\| \leq 1 - \frac{m}{M}$,

$$y(t) = (I - T_1B_1)^{-1}(T_1h)(t).$$

Define $P_2 : \Phi_T \to \Phi_T$ *by*

$$(P_2 h)(t) = (I - T_2 B_2)^{-1}(T_2 h)(t),$$

since T_2 *and* B_2 *are completely continuous,* P_2 *is completely continuous. Moreover, we have*

$$0 < (T_2 h)(t) \leqslant (P_2 h)(t) \leqslant \frac{M}{m}\|T_1 h\|, \quad h \in C_T^+.$$

Lemma 7. *(Krasnoselskii's Fixed Point Theorem [1]). Let* X *be a Banach space. Assume that* Ω *is a bounded closed and convex subset of* X*. If* $Q, S : \Omega \to X$ *satisfy*

1. $Qx + Sy \in \Omega, \quad \forall x, y \in \Omega$
2. Q *is a contractive operator and* S *is completely continuous operator,*

then $Q + S$ *has a fixed point in* Ω*.*

3 Main Results

Theorem 1. *Suppose that* $0 \leqslant c(t)(b-a) \leqslant c_1 < 1$ *and there exist constants* N_1 *and* N_2 *with* $0 < N_1 < N_2$ *such that*

$$\frac{N_1 M}{(b-a)} \leqslant f(t,x) - c(t)a(t)x \leqslant \frac{mN_2(1-c_1)}{(b-a)}, \quad \forall (t,x) \in [0,T] \times [N_1, N_2]. \tag{7}$$

Then (1) has at least one positive T*-periodic solution* $x(t)$ *such that* $N_1 \leqslant x(t) \leqslant N_2$*.*

Proof. We show that

$$\begin{aligned} x(t) &= P_1\left(a(t)c(t)\int_a^b x(t-\sigma(t,\mu))d\mu - \int_a^b f(t, x(t-\sigma(t,\mu)))d\mu\right) \\ &+ c(t)\int_a^b x(t-\sigma(t,\mu))d\mu \end{aligned} \tag{8}$$

is a solution of (1). The equation

$$\begin{aligned} &\left[x(t) - c(t)\int_a^b x(t-\sigma(t,\mu))d\mu\right]'' - a(t)\left[x(t) - c(t)\int_a^b x(t-\sigma(t,\mu))d\mu\right] \\ &= a(t)c(t)\int_a^b x(t-\sigma(t,\mu))d\mu - \int_a^b f(t, x(t-\sigma(t,\mu)))d\mu \end{aligned} \tag{9}$$

is equivalent to (1). Let $y(t) = x(t) - c(t)\int_a^b x(t-\sigma(t,\mu))d\mu$ in the equation (9), then we have

$$y''(t) - a(t)y(t) = a(t)c(t)\int_a^b x(t-\sigma(t,\mu))d\mu - \int_a^b f(t, x(t-\sigma(t,\mu)))d\mu.$$

Applying Lemma 3 to this equation yields

$$y(t) = P_1\left(a(t)c(t)\int_a^b x(t-\sigma(t,\mu))d\mu - \int_a^b f(t,x(t-\sigma(t,\mu)))d\mu\right)$$

which is equivalent to (8). Let $\Omega = \{x \in \Phi_T : N_1 \leqslant x(t) \leqslant N_2,\ t \in [0,T]\}$. One can see that Ω is a bounded, closed and convex subset of Φ_T. We define two mappings $Q, S : \Omega \to \Phi_T$ as follows

$$\begin{aligned}(Qx)(t) &= P_1\left(a(t)c(t)\int_a^b x(t-\sigma(t,\mu))d\mu - \int_a^b f(t,x(t-\sigma(t,\mu)))d\mu\right) \quad \text{and}\\ (Sx)(t) &= c(t)\int_a^b x(t-\sigma(t,\mu))d\mu.\end{aligned}$$

It is easy to verify that Qx and Sx are continuous and T-periodic, i.e we have $Q(\Omega) \subset \Phi_T$ and $S(\Omega) \subset \Phi_T$. For all $x_1, x_2 \in \Omega$ and $t \in \mathbb{R}$, from (7), Lemma 2 and Lemma 3, we get

$$\begin{aligned}(Qx_1)(t) + (Sx_2)(t) = P_1\Bigg(&a(t)c(t)\int_a^b x_1(t-\sigma(t,\mu))d\mu\\ &- \int_a^b f(t,x_1(t-\sigma(t,\mu)))d\mu\Bigg) + c(t)\int_a^b x_2(t-\sigma(t,\mu))d\mu\\ &\leqslant \frac{M}{m}\left\|T_1\left(a(t)c(t)\int_a^b x_1(t-\sigma(t,\mu))d\mu - \int_a^b f(t,x_1(t-\sigma(t,\mu)))d\mu\right)\right\|\\ &+ c_1N_2 = \frac{M}{m}\sup_{t\in[0,T]}\Bigg|\int_t^{t+T} G_1(t,s)\int_a^b \Big[f(s,x_1(s-\sigma(s,\mu)))\\ &- a(s)q(s)x_1(s-\sigma(s,\mu))\Big]d\mu ds\Bigg| + c_1N_2\\ &\leqslant \frac{M}{m}\int_t^{t+T} G_1(t,s)mN_2(1-c_1)ds + c_1N_2 = N_2\end{aligned}$$

and

$$\begin{aligned}(Qx_1)(t) + (Sx_2)(t) = P_1\Bigg(&a(t)c(t)\int_a^b x_1(t-\sigma(t,\mu))d\mu\\ &- \int_a^b f(t,x_1(t-\sigma(t,\mu)))d\mu\Bigg) + c(t)\int_a^b x_2(t-\sigma(t,\mu))d\mu\\ &\geqslant T_1\left(a(t)c(t)\int_a^b x_1(t-\sigma(t,\mu))d\mu - \int_a^b f(t,x_1(t-\sigma(t,\mu)))d\mu\right)\\ &= \int_t^{t+T} G_1(t,s)\int_a^b \Big[f(s,x_1(s-\sigma(s,\mu))) - a(s)q(s)x_1(s-\sigma(s,\mu))\Big]d\mu ds\\ &\geqslant \int_t^{t+T} G_1(t,s)MN_1ds = N_1\end{aligned}$$

from which we conclude that $N_1 \leqslant (Qx_1)(t) + (Sx_2)(t) \leqslant N_2$ for all $x_1, x_2 \in \Omega$ and $t \in \mathbb{R}$, i.e. we have $Qx_1 + Sx_2 \in \Omega$. For $x_1, x_2 \in \Omega$, we obtain

$$\begin{aligned} |(Sx_1)(t) - (Sx_2)(t))| &= \left| c(t) \int_a^b x_1(t - \sigma(t,\mu))d\mu - c(t) \int_a^b x_2(t - \sigma(t,\mu))d\mu \right| \\ &\leqslant c(t) \int_a^b |x_1(t - \sigma(t,\mu)) - x_2(t - \sigma(t,\mu))| d\mu. \end{aligned}$$

By taking the supremum norm, we conclude that

$$\|Sx_1 - Sx_2\| \leqslant c_1 \|x_1 - x_2\|,$$

which implies that S is a contraction mapping on Ω. It is known that P_1 is completely continuous by Lemma 3, so is Q. Then, from Lemma 7, (1) has at least one T-periodic solution $x(t)$. □

Theorem 2. *Suppose that $-1 < c_0 \leqslant c(t)(b-a) \leqslant 0$, $-c_0 M < m$ and there exist constants N_1 and N_2 with $0 < N_1 < N_2$ such that*

$$\frac{(N_1 - c_0 N_2)M}{(b-a)} \leqslant f(t,x) - c(t)a(t)x \leqslant \frac{mN_2}{(b-a)}, \quad \forall (t,x) \in [0,T] \times [N_1, N_2]. \quad (10)$$

Then (1) has at least one positive T-periodic solution $x(t)$ such that $N_1 \leqslant x(t) \leqslant N_2$.

Proof. Define Q and S as in the proof of Theorem 1 and let $\Omega = \{x \in \Phi_T : N_1 \leqslant x(t) \leqslant N_2,\ t \in [0,T]\}$. It is clear that Ω is a bounded, closed and convex subset of Φ_T and $Q(\Omega) \subset \Phi_T$ and $S(\Omega) \subset \Phi_T$. We show that $Qx_1 + Sx_2 \in \Omega$ for all $x_1, x_2 \in \Omega$. For $x_1, x_2 \in \Omega$ and $t \in \mathbb{R}$, we have from (10), Lemma 2 and Lemma 3 that

$$\begin{aligned} (Qx_1)(t) + (Sx_2)(t) = P_1 \bigg(& a(t)c(t) \int_a^b x_1(t - \sigma(t,\mu))d\mu \\ & - \int_a^b f(t, x_1(t - \sigma(t,\mu)))d\mu \bigg) + c(t) \int_a^b x_2(t - \sigma(t,\mu))d\mu \\ & \leqslant \frac{M}{m} \left\| T_1 \left(a(t)c(t) \int_a^b x_1(t - \sigma(t,\mu))d\mu - \int_a^b f(t, x_1(t - \sigma(t,\mu)))d\mu \right) \right\| \\ & = \frac{M}{m} \sup_{t \in [0,T]} \bigg| \int_t^{t+T} G_1(t,s) \int_a^b \Big[f(s, x_1(s - \sigma(s,\mu))) \\ & - a(s)q(s)x_1(s - \sigma(s,\mu)) \Big] d\mu ds \bigg| \\ & \leqslant \frac{M}{m} \int_t^{t+T} G_1(t,s) m N_2 ds = N_2 \end{aligned}$$

and

$$(Qx_1)(t) + (Sx_2)(t) = P_1 \bigg(a(t)c(t) \int_a^b x_1(t - \sigma(t,\mu))d\mu$$

$$
\begin{aligned}
&- \int_a^b f(t, x_1(t-\sigma(t,\mu)))d\mu\Big) + c(t)\int_a^b x_2(t-\sigma(t,\mu))d\mu \\
&\geqslant T_1\left(a(t)c(t)\int_a^b x_1(t-\sigma(t,\mu))d\mu - \int_a^b f(t, x_1(t-\sigma(t,\mu)))d\mu\right) \\
&+ c(t)(b-a)N_2 \\
&\geqslant \int_t^{t+T} G_1(t,s)\int_a^b \Big[f(s, x_1(s-\sigma(s,\mu))) - a(s)q(s)x_1(s-\sigma(s,\mu))\Big]d\mu ds \\
&+ c_0N_2 \geqslant \int_t^{t+T} G_1(t,s)(N_1 - c_0N_2)Mds + c_0N_2 = N_1.
\end{aligned}
$$

Thus, we have $Qx_1 + Sx_2 \in \Omega$ for all $x_1, x_2 \in \Omega$. Now we show S is a contraction operator on Ω. In fact, for $x_1, x_2 \in \Omega$, we have

$$
\begin{aligned}
|(Sx_1)(t) - (Sx_2)(t))| &= \left|c(t)\int_a^b x_1(t-\sigma(t,\mu))d\mu - c(t)\int_a^b x_2(t-\sigma(t,\mu))d\mu\right| \\
&\leqslant -c(t)\int_a^b |x_1(t-\sigma(t,\mu)) - x_2(t-\sigma(t,\mu))|d\mu.
\end{aligned}
$$

By taking the supremum norm, we conclude that

$$\|Sx_1 - Sx_2\| \leqslant -c_0\|x_1 - x_2\|,$$

which implies that S is a contraction mapping on Ω. It is known that P_1 is completely continuous by Lemma 3, so is Q. Then, from Lemma 7, (1) has at least one T-periodic solution $x(t)$. □

Theorem 3. *Let $M < (\frac{\pi}{T})^2$. Suppose that $0 \leqslant c(t)(b-a) \leqslant c_1 < 1$ and there exist constants N_1 and N_2 with $0 < N_1 < N_2$ such that*

$$\frac{N_1M}{(b-a)} \leqslant f(t,x) - c(t)a(t)x \leqslant \frac{mN_2(1-c_1)}{(b-a)}, \quad \forall (t,x) \in [0,T] \times [N_1, N_2].$$

Then (2) has at least one positive T-periodic solution $x(t)$ such that $N_1 \leqslant x(t) \leqslant N_2$.

Theorem 4. *Let $M < (\frac{\pi}{T})^2$. Suppose that $-1 < c_0 \leqslant c(t)(b-a) \leqslant 0$, $-c_0M < m$ and there exist constants N_1 and N_2 with $0 < N_1 < N_2$ such that*

$$\frac{(N_1 - c_0N_2)M}{(b-a)} \leqslant f(t,x) - c(t)a(t)x \leqslant \frac{mN_2}{(b-a)}, \quad \forall (t,x) \in [0,T] \times [N_1, N_2].$$

Then (2) has at least one positive T-periodic solution $x(t)$ such that $N_1 \leqslant x(t) \leqslant N_2$.

The proofs of Theorem 3 and Theorem 4 are omitted, as they are very similar to those of Theorem 1 and Theorem 2, respectively.

Example 1. Consider the following equation

$$\left[x(t) - \frac{\exp(\cos t)}{100}\int_{\pi/4}^{\pi/2} x(t-10\mu-\sin t)d\mu\right]'' = \left(1+\frac{\cos t}{10}\right)x(t)$$
$$-\int_{\pi/4}^{\pi/2}(17+\exp(\cos t)+\cos(x^5(t-10\mu-\sin t)))d\mu. \qquad (11)$$

Comparing (1) to (11), we see $T = 2\pi$, $c(t) = \frac{\exp(\cos t)}{100}$, $a(t) = \left(1+\frac{\cos t}{10}\right)$, $f(t,x) = 17+\exp(\cos t)+\cos(x^5)$, $\sigma(t,\mu) = 10\mu+\sin t$, $a = \pi/4$ and $b = \pi/2$. It can be verified that the conditions of Theorem 1 are satisfied with $N_1 = 10$ and $N_2 = 20$.Therefore, (11) has at least one positive T-periodic solution.

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