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The codimension of the phase portraits for degenerate quadratic differential systems

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Abstract. In this paper we present a complete study of degenerate quadratic differential systems, i.e. the polynomials from right-hand sides of these systems are not co-prime. We give the complete set of their phase portraits together with the necessary and sufficient conditions for the realization of each one of them. These conditions are given in using invariant polynomials and we present here the "bifurcation" diagram directly in the space $\mathbb{R}^{12}$ of the whole set of the parameters of the quadratic systems.

This paper is part of a project whose ultimate goal is the complete classification of all topologically distinct phase portraits of quadratic systems modulo limit cycles. We also provide a label for each phase portrait inside the global codification related to the global configurations of singularities and their topological codimensions.

Mathematics subject classification: Primary 58K45, 34C05, 34A34.
Keywords and phrases: quadratic vector field, infinite and finite singularities, codimension, affine invariant polynomial, Poincaré compactification, configuration of singularities, topological equivalence relation.

1 Introduction and the statement of the main result

We consider here differential systems of the form

$$\frac{dx}{dt} = p(x,y), \qquad \frac{dy}{dt} = q(x,y), \tag{1}$$

where p, $q \in \mathbf{R}[x,y]$, i.e. p, q are polynomials in x, y with real coefficients. We call *degree* of a system (1) the integer $n = \max(\deg p, \deg q)$. In particular we call *quadratic* a differential system (1) with $n = 2$. We denote here by **QS** the whole class of real quadratic differential systems.

The motivation of the authors for writing this paper is to advance towards the complete classification of topologically distinct phase portraits of quadratic differential systems modulo limit cycles. During the last decades, more than one thousand of papers have been dedicated to quadratic systems producing many hundreds of phase portraits.

The approaches used by these papers have been of different kinds:

- Some papers considered phase portraits of quadratic systems under very restricted conditions like: existence of centers (see [26,30,32]), chordal quadratic systems (see [19,24]), systems with just one finite singularity (see [17,25]) and many others.

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- Some papers considered phase portraits of a large family of quadratic systems with few restrictions obtaining the full corresponding bifurcations diagrams like: systems with a finite nilpotent singularity [22], systems with a weak focus of order two [4], systems with two saddle-nodes [11, 12, 15] and many others.

- Some papers considered phase portraits of the most generic classes like: structurally stable phase portraits of quadratic systems (i.e. of codimension 0) [2], structurally unstable phase portraits of quadratic systems of codimension 1 [3], and some papers dealing with phase portraits of codimension 2 [1, 13, 14].

All these approaches for the study of quadratic systems are important and useful. But none of them alone can completely determine all the possible phase portraits. The interaction among all three approaches is needed for obtaining the full classification.

The class of degenerate quadratic systems (denoted by **QSD**) is among the most restrictive systems and has not yet been completely classified.

The local behavior of the trajectories in the vicinity of the line at infinity in the generic case of quadratic systems with finite number of singularities (finite and infinite) has been studied in [27] where 40 classes were detected.

In [5] we added the systems with infinite number of infinite singularities (6 classes) as well as the systems with infinite number of finite singularities (30 classes denoted by QD_1^∞–QD_{30}^∞). However in the second case one class was omitted and in fact we have 31 topologically distinct phase portraits in the vicinity at infinity for the class **QSD** which we present here in Figure 1 using, for the new class, the notation QD_{31}^∞.

Remark 1. *We note that the existence of the class QD_{31}^∞ was already detected in [7] and in paper [6] where one configuration of singularities for systems in* **QSD** *(where the common factor is formed by two complex parallel lines) did not correspond to any of the 30 initial classes. However QD_{31}^∞ was not formally presented until now. We point out that the existence of this class allows us to detect that the topological configuration (192) from [6] is topologically equivalent to topological configuration (137). This happens because the infinite intersection of two complex parallel lines produces a real singularity topologically equivalent to an intricate singularity with two hyperbolic sectors. More detailed explanation could be found in [8, Remark 2] where we already decided not to shift the code of configurations from (193) up to (208) and left the gap related to (192) considering this class empty.*

In [7] the authors gave the set of all global geometric configurations of singularities for the whole family of quadratic systems including systems in **QSD** (in total 1764 cases).

In [6] the authors grouped the set of 1764 geometric configurations into 208 global topologically distinct configurations of singularities. During this process some

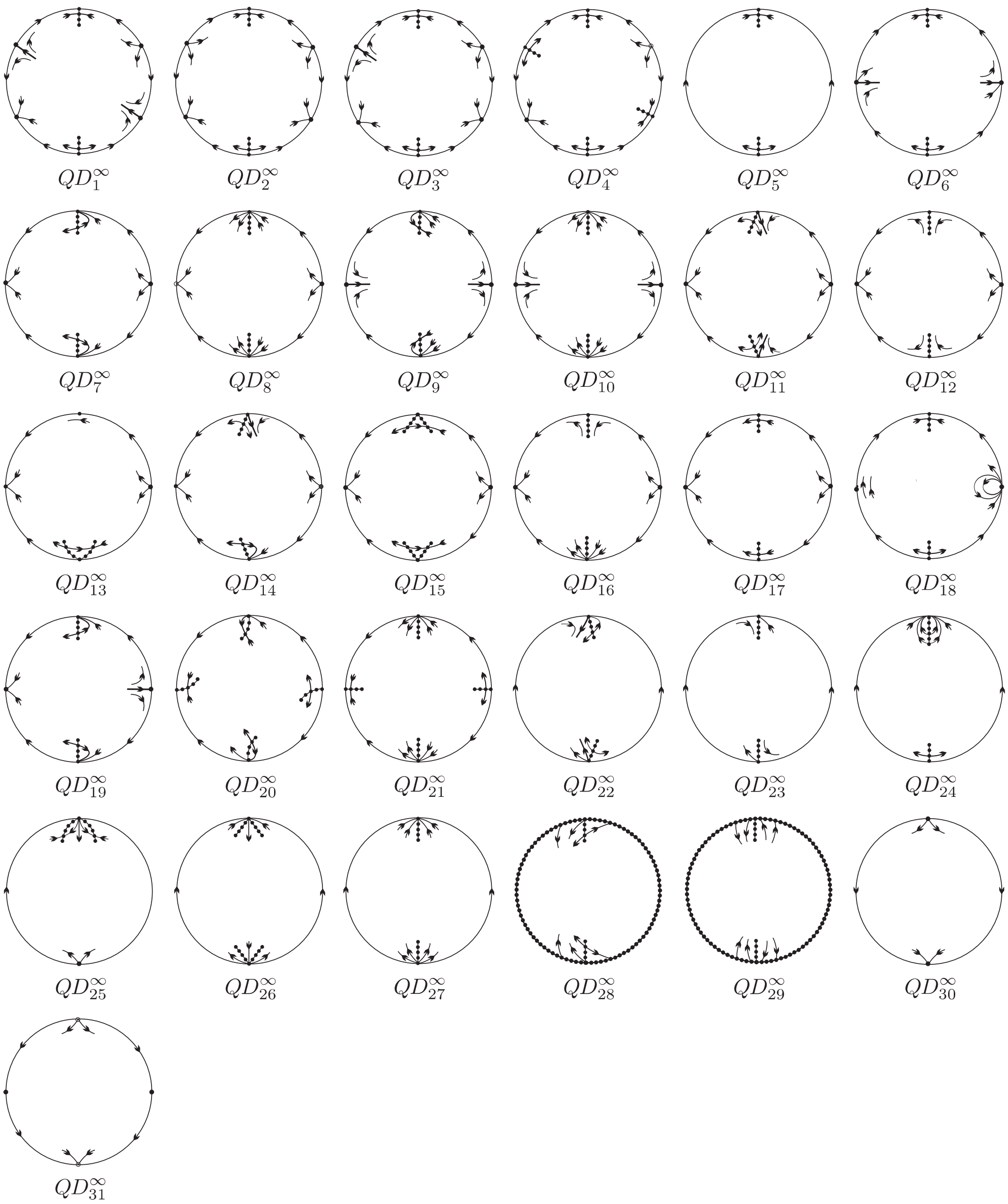

Figure 1. Topologically distinct local configurations of infinite singular points of systems in **QSD**.

geometric configurations of singularities in **QSD** became identified with other configurations having a finite number of singularities. These cases are the ones that produce the phase portraits in Figure 2.

The reader must notice that the phase portrait $QS47_1^{(2)}$ is drawn here as coming from a degenerate system and the finite singularity looks like an intricate singularity

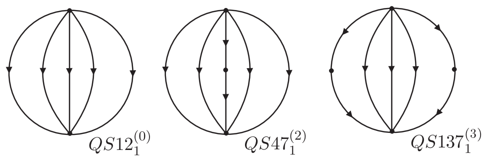

Figure 2. Phase portraits of systems in **QSD** topologically equivalent to phase portraits of non-degenerate systems.

with two hyperbolic sectors. But the most generic topological representative of $QS47_1^{(2)}$ is with a nilpotent cusp which is a singularity of codimension two.

Our main result is:

Main Theorem. *The following statements hold:*

(i) The family **QSD** *possesses a total of 41 topologically distinct phase portraits given in Figures 2 and 3. Moreover for each one of these phase portraits its codimension is determined.*

(ii) The topological classification is done using algebraic invariant polynomials and hence it is independent of the normal forms in which the systems may be presented.

(iii) The bifurcation diagram of the phase portraits of systems in the family **QSD** *is done in the twelve-dimensional parameter space $\mathbb{R}^{12}$ and it is presented in Figure 4. This diagram gives us an algorithm to determine for any given system its corresponding phase portrait.*

Remark 2. *The phase portraits from Figure 2 are those which are topologically equivalent to non-degenerate phase portraits. So they have a lower codimension than the one they would have if they where realizable only as degenerate phase portraits. The phase portraits presented in Figure 3 are those which imply the existence of an infinite number of real finite singularities.*

In order to construct the set of phase portraits of the family of systems in **QSD** there are at least three ways to do it, which we describe below.

1. One can start from the ten topologically different phase portraits of linear systems in the Poincaré disc adding one straight line filled up with singularities (from now on a *singular line*). One must also consider the constant systems adding a conic filled up with singularities (from now on a *singular conic*).
2. One can start from the 31 topologically distinct local configurations of infinite singularities given in Figure 1 and complete them with the possible singular curves compatible with each QD_1^{∞}–QD_{31}^{∞} and add a compatible finite singularity if necessary.
3. One may consider each one of the geometrical configurations of singularities from [7] of systems in **QSD** (or topological configurations from [6]) and find the possible phase portrait generated by each one of them.

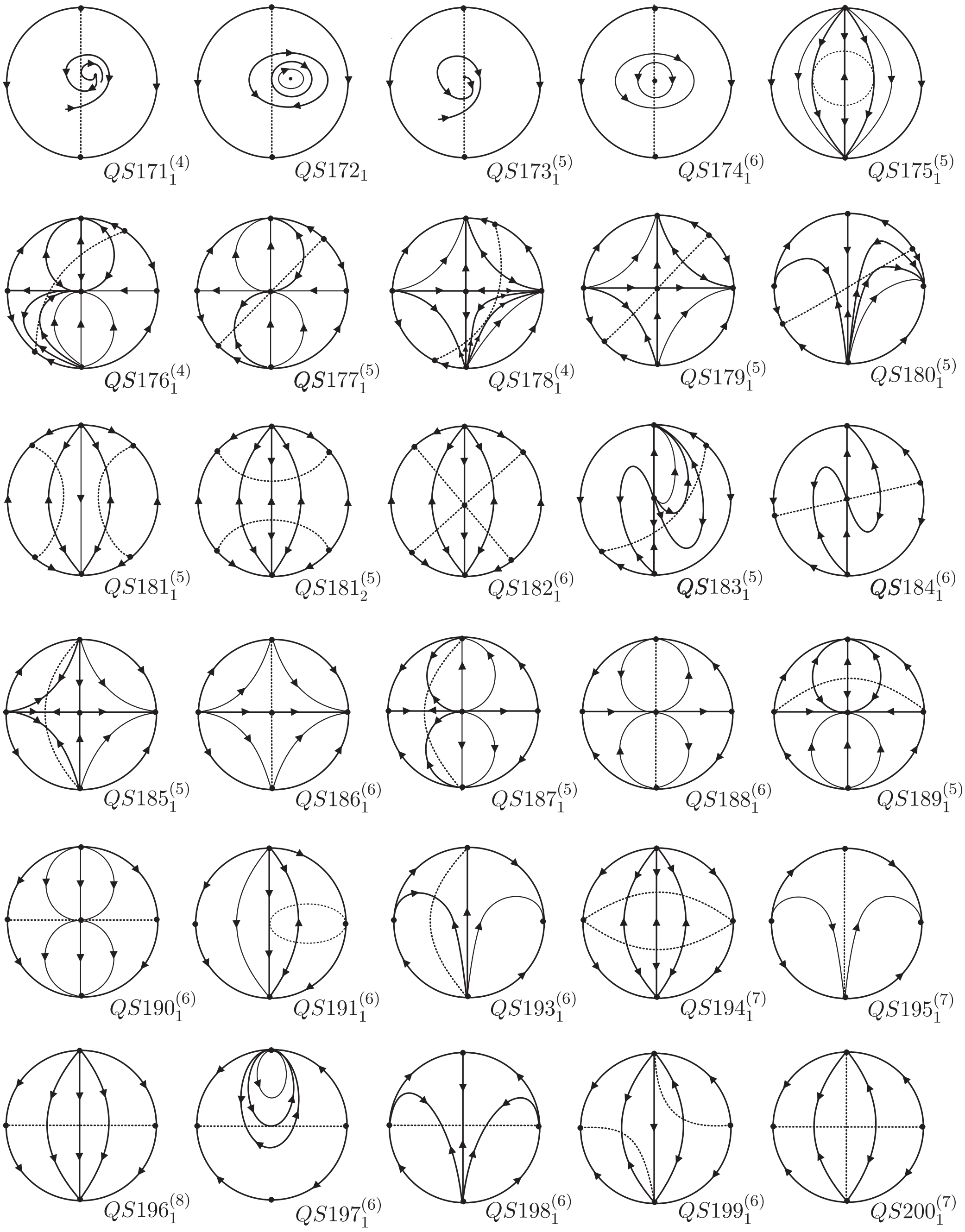

Figure 3. Phase portraits of systems in **QSD** with an infinite number of real singularities.

We have followed all three ways and obtained the same results concerning the phase portraits. However if one wants to obtain the "bifurcation" diagram, i.e. to determine the necessary and sufficient conditions for the realization of each one of the obtained phase portraits, the third way is compulsory.

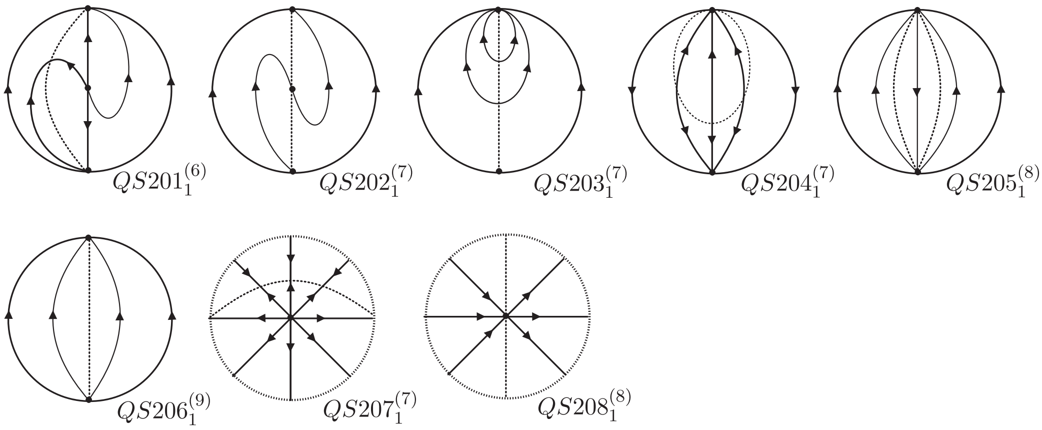

FIGURE 3 (*cont.*) Phase portraits of systems in **QSD** with an infinite number of real singularities.

This article is organized as follows. In Section 2 we give notations for singularities and for the phase portraits, we describe the source from which we take the concept of codimension and we construct affine invariant polynomials which completely classify the systems in the family **QSD**.

Section 3 is dedicated to the proof of Main Theorem.

2 Preliminaries

In this section we bring some concepts and notations from other papers which will be used here.

2.1 Notations for singularities

In the book [7] we defined some new concepts for singularities closely related to the Jacobian matrix because they are more convenient for the geometrical classification rather than the classical concepts. The new definitions are:

We call *elemental* a singular point with its two eigenvalues not zero.

We call *semi-elemental* a singular point with exactly one of its eigenvalues equal to zero.

We maintain the name of *nilpotent* for a singular point with its two eigenvalues zero but with its Jacobian matrix at this point not identically zero.

We call *intricate* a singular point with its Jacobian matrix identically zero.

Since in this paper we deal with the class **QSD**, we may shorten the description and simplify it for this specific class. Some concepts such as for example order of weak singularities are not needed so we remove them. However, since the notation of the most degenerate configuration includes many other simpler notations, we must incorporate most part of what is exposed in [7].

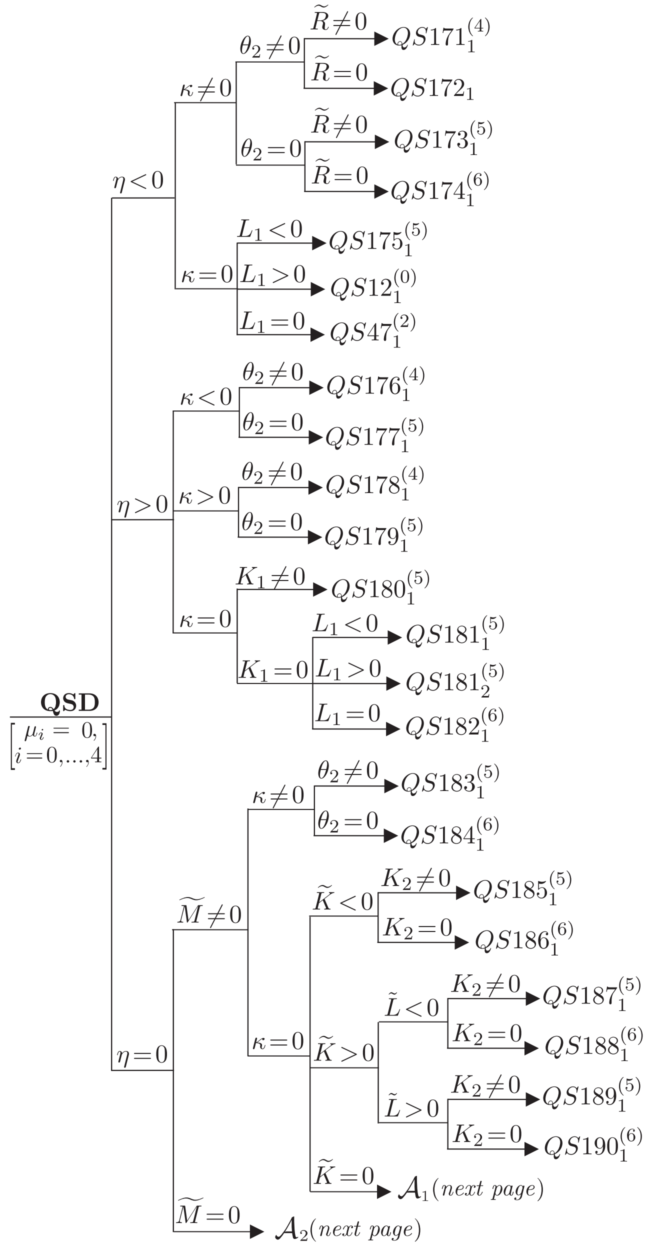

Figure 4. Diagram for the phase portraits of systems in **QSD**.

First we start describing the finite and infinite singularities, denoting the first ones with lower case letters and the second with capital letters. When describing in a sequence both finite and infinite singular points, we will always place first the finite ones and only later the infinite ones, separating them by a semicolon ';'. Even though finite and infinite singular points may either be real or complex, from the topological viewpoint, only the real ones are interesting and only these will be listed.

Elemental singularities: We use the letters 's','S' for "saddles"; 'n', 'N' for

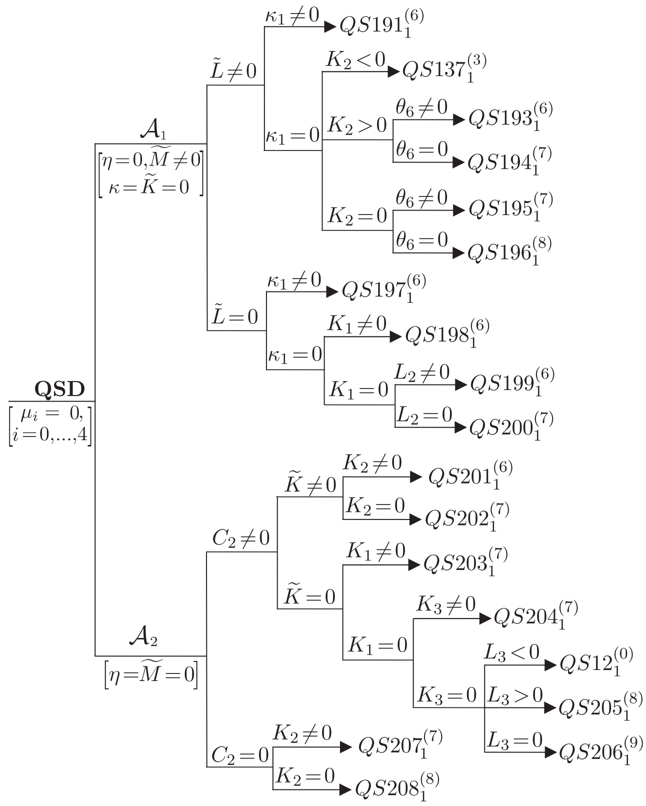

FIGURE 4 (*cont.*) Diagram for the phase portraits of systems in **QSD**.

"nodes"; 'f' for "foci" and 'c' for "centers". We will also denote by 'a' (anti-saddle) for either a focus or any type of node when the local phase portraits are topologically equivalent. Since the number of characteristic directions of nodes is critical in the class **QSD**, we need to keep the following notations:

- 'n' for a node with two distinct eigenvalues;
- 'n^d' (a one-direction node) for a node with two identical eigenvalues whose Jacobian matrix cannot be diagonal;
- 'n^*' (a star node) for a node with two identical eigenvalues whose Jacobian matrix is diagonal.

Moreover, in the case of an elemental infinite node, there is a geometrical feature that distinguishes whether all orbits except one arrive tangent to infinity or to an affine direction, and this concept is also important in the class **QSD**.

So we must use the notations 'N^∞', 'N^f', 'N^d' and 'N^∞' for infinite nodes as they were defined in [7].

All non-elemental singular points are multiple points. Even though multiplicity is in most cases irrelevant for the local topological phase portrait, for some infinite singularities the type of multiplicity could be relevant and we must point out the way we denote these cases. We denote by '$\binom{a}{b}$...' the maximum number a (respectively b) of finite (respectively infinite) singularities which can be obtained by perturbation of the multiple point. For example, '$\binom{1}{1}SN$' and '$\binom{0}{2}SN$' correspond to two saddle-nodes at infinity which are locally topologically distinct.

Semi-elemental singularities: They can either be nodes, saddles or saddle-nodes, finite or infinite. However semi-elemental nodes and saddles are respectively topologically equivalent with elemental nodes and saddles. So we will use the same notation as if they were elemental ones. The only new semi-elemental singularity is the saddle-node which we denote by 'sn'. As indicated above for infinite saddle-nodes SN we will also keep the multiplicity. Moreover, as in [7] we also need the notation '$\binom{1}{1}NS$' for some infinite saddle-nodes.

Nilpotent singularities: They can either be saddles, nodes, saddle-nodes, elliptic-saddles, cusps, foci or centers. The first four of these could be at infinity. The only finite nilpotent points for which we need to introduce notation are the elliptic-saddles and cusps which we denote respectively by es and cp.

In the case of nilpotent infinite points, the relative positions of the sectors with respect to the line at infinity can produce topologically different phase portraits. This forces us to use a notation for these points similar to the notation which we will use for the intricate points.

Intricate singularities: It is known that the neighborhood of any singular point of a polynomial vector field (except for foci and centers) is formed by a finite number of sectors which could only be of three types: parabolic, hyperbolic and elliptic (see [18]). Then, a reasonable way to describe intricate and nilpotent points at infinity is to use a sequence formed by the types of their sectors. In the book [7] we use a geometrical notion of sector which is more subtle but which looses part of its meaning in the topological setting. More precisely any two adjacent parabolic geometrical sectors merge into one, and parabolic sectors adjacent to elliptic ones can be omitted. To lighten the notation, we make the convention to eliminate the parabolic sectors adjacent to the elliptic sectors.

Thus in quadratic systems, we have just four topological possibilities for finite intricate singular points of multiplicity four (see [10]) which are the following ones:

a) $phph$; b) hh; c) $hhhhhh$; d) ee.

It is worth noting that the singularity hh is topologically equivalent with cp.

For intricate and nilpotent singular points at infinity, we insert a dash (hyphen) between the sectors to split those which appear on one side or the other of the equator of the sphere. In this way we will distinguish between $\binom{2}{2}P-HHP$ and $\binom{2}{2}PH-PH$. When describing a single finite nilpotent or intricate singular point, one can always apply an affine change of coordinates to the system, so it does not really matter which sector starts the sequence, or the direction (clockwise or counter-clockwise) we choose. If it is an infinite nilpotent or intricate singular point, then we will always start with a sector bordering the infinity (to avoid using two dashes).

The lack of finite singular points after the removal of degeneracies will be encapsulated in the notation $\scriptstyle\emptyset$ (i.e. small size $\emptyset$). In similar cases when we need to point out the lack of an infinite singular point, we will use the symbol $\emptyset$.

Finally there is also the possibility that we have an infinite number of finite or of infinite singular points. In the first case, this means that the quadratic polynomials defining the differential system are not coprime. Their common factor may produce a line or conic with real coefficients filled up with singular points. This is mainly the class under study in this paper.

Line at infinity filled up with singularities: It is known that any such system has in a sufficiently small neighborhood of infinity one of 6 topological distinct phase portraits (see [27]). The way to determine these portraits is by studying the reduced systems on the infinite local charts after removing the degeneracy of the systems within these charts. Following [7] we use the notation $[\infty;\, \emptyset]$, $[\infty;\, N]$, $[\infty;\, N^d]$, $[\infty;\, S]$, $[\infty;\, C]$, $[\infty;\, \binom{2}{0}SN]$ or $[\infty;\, \binom{3}{0}ES]$ indicating the singularities obtained after removing the line filled with singularities.

Degenerate systems: We will denote with the symbol $\ominus$ the case when the polynomials defining the system have a common factor. The degeneracy can be produced by a common factor of degree one which defines a straight line or a common quadratic factor which defines a conic. Following [7] we will indicate each case by the following symbols:

- $\ominus[|]$ for a real straight line;
- $\ominus[\circ]$ for a real ellipse;
- $\ominus[©]$ for a complex ellipse (i.e. an irreducible conic over $\mathbb{R}$ which has only complex points);
- $\ominus[)(]$ for a hyperbola;
- $\ominus[\cup]$ for a parabola;
- $\ominus[\times]$ for two real straight lines intersecting at a finite point;
- $\ominus[\cdot]$ for two complex straight lines which intersect at a real finite point;
- $\ominus[||]$ for two real parallel lines;
- $\ominus[||^c]$ for two complex parallel lines;
- $\ominus[|_2]$ for a double real straight line.

It is worth noticing that the degeneracy $\ominus[©]$ implies the non-existence of real singularities, so we have a chordal system whose phase portrait is equivalent to the case when there are four complex finite singularities and one real infinite singularity. Moreover, the degeneracy $\ominus[\cdot]$ implies the existence of one finite real singularity and its local phase portrait is the same as that of a singularity hh which turns out to be topologically equivalent to a cusp. And finally, we point out that degeneracy $\ominus[||^c]$ produces one real singularity with configuration $H - H$ at infinity.

Moreover, we also want to determine whether after removing the common factor of the polynomials, singular points remain on the curve defined by this common factor. If the reduced system has no finite singularity on this curve, we will use the symbol $\emptyset$ to describe this situation. If some singular points remain we will use the corresponding notation of their various kinds. In this situation, the geometrical properties of the singularity that remains after the removal of the degeneracy may produce topologically different phenomena, even if they are topologically equivalent singularities. So, we will need to keep the geometrical information associated to that singularity. Some examples of the way we denote the complete notation are:

- $\big(\ominus\,[|];\emptyset\big)$ denotes the presence of a real straight line filled up with singular points such that the reduced system has no singularity on this line;
- $\big(\ominus\,[|];f\big)$ denotes the presence of the same straight line such that the reduced system has a strong focus on this line;
- $\big(\ominus\,[|];n^d\big)$ denotes the presence of the same straight line such that the reduced system has a node n^d on this line;
- $\big(\ominus\,[\cup];\emptyset\big)$ denotes the presence of a parabola filled up with singularities such that no singular point of the reduced system is situated on this parabola.

Degenerate systems with a non-isolated infinite singular point, which however is isolated on the line at infinity: The existence of a common factor of the polynomials defining the differential system also affects the infinite singular points. We point out that the projective completion of a real affine line filled up with singular points has a point on the line at infinity which will then be also a non-isolated singularity.

There is a detailed description of this notation in [7]. In case that after the removal of the finite degeneracy, a singular point at infinity remains on the same place, we must denote it with all its geometrical properties since they may influence the local topological phase portrait. We give below some examples:

- $N, S, \big(\ominus\,[|];\emptyset\big)$ means that the system has at infinity a node, a saddle, and one pair of non-isolated singular point which is part of a real straight line filled up with singularities (other that the line at infinity), and that the reduced linear system has no infinite singular point in that position;
- $S, \big(\ominus\,[|];N^*\big)$ means that the system has a saddle at infinity, and one pair of non-isolated singular point which is part of a real straight line filled up with singularities (other that the line at infinity), and that the reduced linear system has a star node in that position;
- $S, \big(\ominus\,[\,)(\,];\emptyset,\emptyset\big)$ means that the system has a saddle at infinity, and two pairs of non-isolated singular points which are part of a hyperbola filled up with singularities, and that the reduced constant system has no singularities in those positions;
- $\big(\ominus\,[\times];N^*,\emptyset\big)$ means that the system has two pairs of non-isolated singular points at infinity which are part of two real intersecting straight lines filled up with singularities, and that the reduced constant system has a star node in one of those positions and no singularity in the other;
- $S, \big(\ominus\,[\circ];\emptyset,\emptyset\big)$ means that the system has a saddle at infinity, and two pairs of non-isolated (complex) singular points which are the two points at infinity on the complexification of a (real) ellipse, and the reduced constant system has no singularities in those positions.

- S, $\left(\ominus\,[|];N_3^{\infty}\right)$ means that the system has a saddle at infinity, and one pair of non-isolated singular point which is part of a real straight line filled up with singularities (other that the line at infinity), and that the reduced linear system has in that position a node such that none of the eigenvectors of the node coincides with the line of singularities and all the orbits (except one) arriving at the node are tangent to the line at infinity.

Degenerate systems with the line at infinity filled up with singularities: According to [7] there are only two geometrical configurations of this class which are also topologically distinct, and which produce just the two phase portraits $QS207_1^{(7)}$ and $QS208_1^{(8)}$ given in Figure 3. The notations of configurations of infinite singularities are $[\infty;\ \left(\ominus\,[|];\ \emptyset_3\right)]$ for picture $QS207_1^{(7)}$ and $[\infty;\ \left(\ominus\,[|];\ \emptyset_2\right)]$, for picture $QS208_1^{(8)}$as explained in Figure 3.21 of [7].

On the link http://mat.uab.cat/∼artes/articles/notation.pdf we offer a table with the geometrical notations of singularities (from which the topological one can be easily extracted) for an easy access during unlimited time in principle.

2.2 Codimension

In paper [8] the concept of codimension applied to polynomial differential systems was developed covering different equivalence classes (topological or geometrical). Using the new definition of codimension given in [8] one can assign a codimension to singularities, to global configurations of singularities or to phase portraits in the Poincaré disc.

In the paper [8] a topological codimension was given to each one of the 207 global topologically distinct configurations of singularities, except those with the centers for the reasons explained there. The topological codimension of a phase portrait is greater than or equal to the topological codimension of its configuration of singularities. More precisely it is greater if the phase portrait has one (or more) non-forced separatrix connection. In the case of systems in **QSD** there does not exist any non-forced separatrix connection and therefore the topological codimension of a phase portrait coincides always with the topological codimension of its configuration of singularities.

In this current paper we indicate the topological codimension (modulo limit cycles) of each phase portrait of a system in **QSD**.

2.3 Notations for phase portraits

In the paper [8] a new notation to identify every phase portrait of a quadratic system was proposed.

Notation 1. *We denote each phase portrait as $QSr_a^{(b)}$ where QS stands for "quadratic differential system", r is the number of the configuration of singularities from [6], 'b' is the topological codimension of the phase portrait and 'a' is simply a* *cardinal to enumerate the different phase portraits which have the same configuration and codimension.*

This notation has already been widely used in paper [9] where every phase portrait having a nilpotent or intricate infinite singularity has already received its definitive name. In the current paper we continue using the style described in Notation 1.

The final goal of this general project is to obtain all the phase portraits of quadratic systems modulo limit cycles and recognize each one of them by an individual name.

2.4 Invariant polynomials associated to the systems in the class QSD

Consider real quadratic systems of the form

$$\begin{aligned} \frac{dx}{dt} &= p_0 + p_1(x,y) + p_2(x,y) \equiv P(x,y), \\ \frac{dy}{dt} &= q_0 + q_1(x,y) + q_2(x,y) \equiv Q(x,y), \end{aligned} \tag{2}$$

with homogeneous polynomials p_i and q_i $(i = 0,1,2)$ of degree i in x, y:

$$\begin{aligned} p_0 &= a_{00}, \quad p_1(x,y) = a_{10}x + a_{01}y, \quad p_2(x,y) = a_{20}x^2 + 2a_{11}xy + a_{02}y^2, \\ q_0 &= b_{00}, \quad q_1(x,y) = b_{10}x + b_{01}y, \quad q_2(x,y) = b_{20}x^2 + 2b_{11}xy + b_{02}y^2. \end{aligned}$$

It is known that on the set of quadratic systems the group $Aff(2,\mathbb{R})$ of affine transformations of the plane acts (cf. [27]). For every subgroup $G \subseteq Aff(2,\mathbb{R})$ we have an induced action of G on **QS**. We can identify the set **QS** of systems (2) with a subset of $\mathbb{R}^{12}$ via the map $\mathbf{QS} \longrightarrow \mathbb{R}^{12}$ which associates to each system (2) the 12–tuple $\tilde{a} = (a_{00}, a_{10}, a_{01}, a_{20}, a_{11}, a_{02}, b_{00}, b_{10}, b_{01}, b_{20}, b_{11}, b_{02})$ of its coefficients. We associate to this group action polynomials in x, y and parameters which behave well with respect to this action, the GL–comitants (GL–invariants), the T–comitants (affine invariants) and the CT–comitants. For their definitions as well as their detailed constructions we refer the reader to the paper [27] (see also [7]).

Following [7] (see also [16]) we apply the differential operator $\mathcal{L} = x \cdot \mathbf{L}_2 - y \cdot \mathbf{L}_1$ acting on $\mathbb{R}[\tilde{a}, x, y]$ with

$$\begin{aligned} \mathbf{L}_1 &= 2a_{00}\frac{\partial}{\partial a_{10}} + a_{10}\frac{\partial}{\partial a_{20}} + \frac{1}{2}a_{01}\frac{\partial}{\partial a_{11}} + 2b_{00}\frac{\partial}{\partial b_{10}} + b_{10}\frac{\partial}{\partial b_{20}} + \frac{1}{2}b_{01}\frac{\partial}{\partial b_{11}}, \\ \mathbf{L}_2 &= 2a_{00}\frac{\partial}{\partial a_{01}} + a_{01}\frac{\partial}{\partial a_{02}} + \frac{1}{2}a_{10}\frac{\partial}{\partial a_{11}} + 2b_{00}\frac{\partial}{\partial b_{01}} + b_{01}\frac{\partial}{\partial b_{02}} + \frac{1}{2}b_{10}\frac{\partial}{\partial b_{11}}, \end{aligned}$$

to construct several invariant polynomials which are needed here. More precisely using the operator $\mathcal{L}$ and the affine invariant $\mu_0 = \mathrm{Res}_x\big(p_2(\tilde{a},x,y), q_2(\tilde{a},x,y)\big)/y^4$ we construct the following polynomials

$$\mu_i(\tilde{a},x,y) = \frac{1}{i!}\mathcal{L}^{(i)}(\mu_0), \; i = 1,..,4, \quad \text{where} \;\; \mathcal{L}^{(i)}(\mu_0) = \mathcal{L}(\mathcal{L}^{(i-1)}(\mu_0)).$$

According to [7, Lemma 5.2] these invariant polynomials are responsible for the total multiplicity of the finite singular points of an arbitrary quadratic system. In particular by [7, Lemma 5.2, statement (iii)] we have the following

Lemma 1. *An arbitrary quadratic system belongs to the family* **QSD** *(i.e. it is degenerate) if and only if* $\mu_i = 0$ *for every* $i = 0, 1, 2, 3, 4$.

We single out the following five polynomials, basic ingredients in constructing invariant polynomials for systems (2):

$$\begin{aligned} &C_i(\tilde{a}, x, y) = yp_i(x, y) - xq_i(x, y), \ (i = 0, 1, 2)\\ &D_i(\tilde{a}, x, y) = \frac{\partial p_i}{\partial x} + \frac{\partial q_i}{\partial y}, \ (i = 1, 2). \end{aligned} \tag{3}$$

As it was shown in [29] these polynomials of degree one in the coefficients of systems (2) are GL–comitants of these systems. Let $f,\ g \in \mathbb{R}[\tilde{a}, x, y]$ and

$$(f, g)^{(k)} = \sum_{h=0}^{k} (-1)^h \binom{k}{h} \frac{\partial^k f}{\partial x^{k-h} \partial y^h} \ \frac{\partial^k g}{\partial x^h \partial y^{k-h}}.$$

The polynomial $(f, g)^{(k)} \in \mathbb{R}[\tilde{a}, x, y]$ is called *the transvectant of index* k *of* (f, g) (cf.[21],[23])).

Proposition 1 (see [31])**.** *Any* GL*–comitant of systems* (2) *can be constructed from the elements* (3) *by using the operations:* $+$, $-$, $\times$, *and by applying the differential operation* $(*, *)^{(k)}$.

Remark 3. *We point out that the elements* (3) *generate the whole set of* GL*–comitants and hence also the set of affine comitants as well as the set of* T*-comitants.*

We construct the following GL–comitants of the second degree with respect to the coefficients of the initial systems

$$\begin{aligned} &T_1 = (C_0, C_1)^{(1)}, \quad T_2 = (C_0, C_2)^{(1)}, \quad T_3 = (C_0, D_2)^{(1)},\\ &T_4 = (C_1, C_1)^{(2)}, \quad T_5 = (C_1, C_2)^{(1)}, \quad T_6 = (C_1, C_2)^{(2)},\\ &T_7 = (C_1, D_2)^{(1)}, \quad T_8 = (C_2, C_2)^{(2)}, \quad T_9 = (C_2, D_2)^{(1)}. \end{aligned} \tag{4}$$

Using these GL–comitants as well as the polynomials (3) we construct the additional invariant polynomials (see also [27])

$$
\begin{aligned}
\widetilde{M}(\tilde{a},x,y) =&(C_2,C_2)^{(2)} \equiv 2\mathrm{Hess}\big(C_2(\tilde{a},x,y)\big);\\
\eta(\tilde{a}) =&(\widetilde{M},\widetilde{M})^{(2)}/384 \equiv \mathrm{Discrim}\big(C_2(\tilde{a},x,y)\big);\\
\widetilde{K}(\tilde{a},x,y) =&\big[T_8+4T_9+4D_2^2\big]/18;\\
\widetilde{H}(\tilde{a},x,y) =&\big[T_8-8T_9-2D_2^2\big]/18;\\
\widetilde{D}(\tilde{a},x,y) =&\big[2C_0T_8-16C_0T_9-4C_0D_2^2-C_1T_6+6C_1T_7-6D_1T_5+6C_1D_1D_2\\
&-9D_1^2C_2-(C_1,T_5)^{(1)}\big]/36;\\
K_1(\tilde{a},x,y) =&\big[T_5+2C_1D_2-3C_2D_1\big]/6;\\
K_2(\tilde{a},x,y) =&4(T_2,\widetilde{M}-2\widetilde{K})^{(1)}+3D_1(C_1,\widetilde{M}-2\widetilde{K})^{(1)}-(\widetilde{M}-2\widetilde{K})\big(16T_3-3T_4/2+3D_1^2\big);\\
K_3(\tilde{a},x,y) =&C_2^2(4T_3+3T_4)+C_2(3C_0\widetilde{K}-2C_1T_7)+2K_1(3K_1-C_1D_2);\\
\tilde{L}(\tilde{a},x,y) =&4\widetilde{K}+8\widetilde{H}-\widetilde{M};\\
L_1(\tilde{a},x,y) =&(C_2,\widetilde{D})^{(2)};\\
L_2(\tilde{a},x,y) =&(C_2,\widetilde{D})^{(1)};\\
L_3(\tilde{a},x,y) =&C_1^2-4C_0C_2;\\
\widetilde{R}(\tilde{a},x,y) =&\tilde{L}+8\widetilde{K};\\
\kappa(\tilde{a}) =&(\widetilde{M},\widetilde{K})^{(2)}/4;\\
\kappa_1(\tilde{a}) =&(\widetilde{M},C_1)^{(2)};\\
\theta_2(\tilde{a}) =&\big(C_1,\widetilde{N}\big)^{(2)}/16;\\
\theta_6(\tilde{a},x,y) =&C_1T_8-2C_2T_6.
\end{aligned}
$$

3 Proof of the Main Theorem

We start by reproducing here Diagram 12.2 from [7] as diagram in Figure 5. We apply to this diagram some few modifications:

- Since for the systems in **QSD** with an infinite number of real finite singularities there is a bijective map between the set of geometric configurations of singularities and the set of topological configurations of singularities, we have added the code of the topological configuration given in [6] to the corresponding geometric configuration.
- In the case of configurations of singularities of systems in **QSD** with a complex singular conic, we have added the code and configuration of the non-degenerate topologically equivalent configurations.
- We have removed from the configurations of singularities the complex infinite singularities which are irrelevant for systems in **QSD**.
- We have corrected an error in [6] which was already pointed out in [8]; more precisely the geometric configuration which received the code (192) in [6] is

topologically equivalent to configuration (137) as we indicate in diagram from Figure 5.

Remark 4. *Some of the configurations of singularities in Figure 5 correspond to systems with a singular line and no finite singular point. However removing this singular line, the resulting linear system has a finite singularity. For example this occurs in (173), (174) and others. For shortness we will say that the finite singularity is located "under" the singular line.*

Remark 5. *Some of the configurations of singularities in Figure 5 correspond to systems with a singular curve (line or conic) which intersects the line at infinity. Removing this singular curve the resulting linear or constant system has an infinite singularity at one of the intersection points of the singular curve with the infinite line. For example this occurs in (185), (193) and others. For shortness we will say that the infinite singularity is located "under" the singular curve.*

In what follows we examine case by case the configurations provided by the diagram of Figure 5 and present the corresponding phase portrait (given in Figures 2 and 3) with one exception when we have two phase portraits generated by the same configuration of singularities.

(171) This configuration has a finite focus and a singular line which does not pass through it. The only possible phase portrait is $QS171_1^{(4)}$. The codimension value 4 assigned to this phase portrait comes directly from the topological codimension of the configuration of singularities (171).

(172) We have a center and a singular line which does not pass through it. As a result we get a single phase portrait $QS172_1$ also known as Vul_{29} (see [30]). We do not assign a codimension to this phase portrait because we do not assign codimensions to configurations with centers.

(173) We have a singular line and a focus "under" it. The unique phase portrait is $QS173_1^{(5)}$.

(174) We have a singular line and a center "under" it. The unique phase portrait is $QS174_1^{(6)}$.

(175) This configuration has a real ellipse as a singular curve. Therefore the system obtained after removing this singular curve is constant. The unique phase portrait is $QS175_1^{(5)}$.

(12) This configuration has a complex ellipse as a singular curve. Therefore there is no finite real singular points and this configuration is topologically equivalent to (12). The unique phase portrait is $QS12_1^{(0)}$. The topological codimension of this portrait is 0 because it is structurally stable even though the geometrical codimension of the original configuration of singularities is much higher. This same configuration (12) will appear again but for a different geometric configuration.

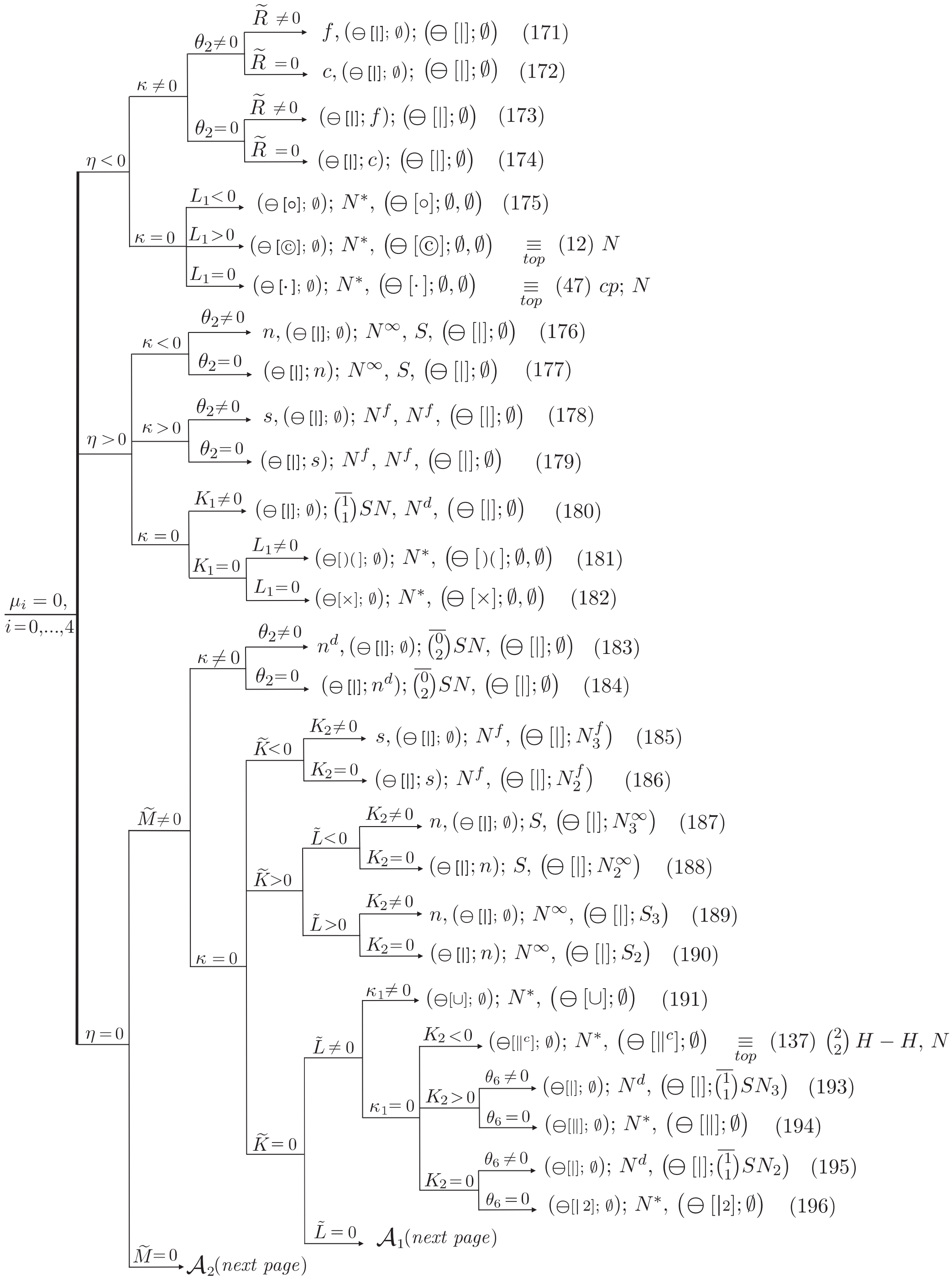

Figure 5. Diagram for the geometric configurations of singularities of systems in **QSD**.

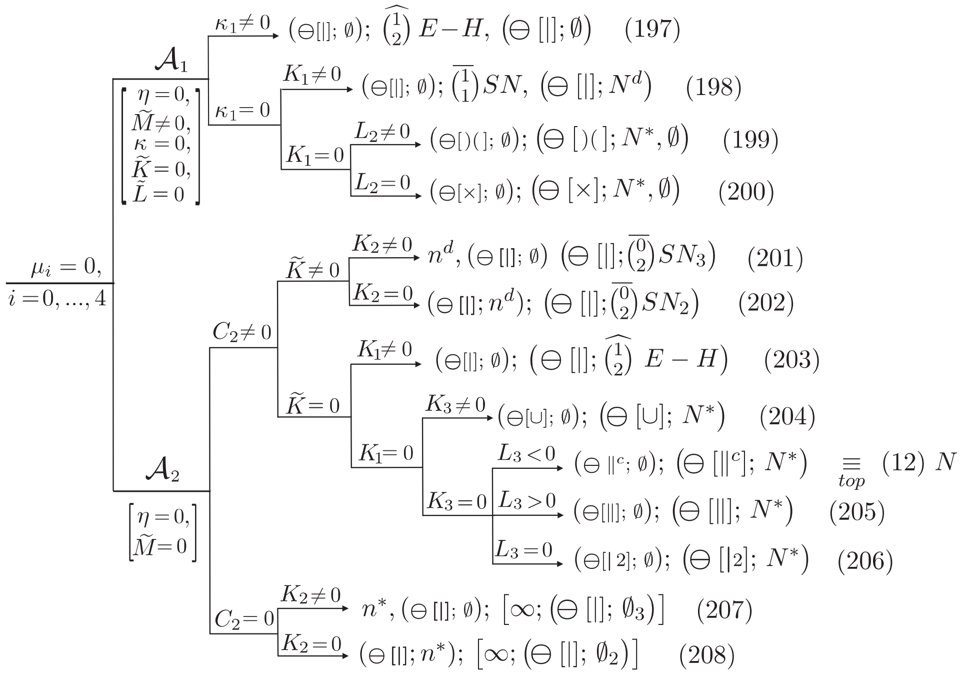

FIGURE 5 (*cont.*) Diagram for the geometric configurations of singularities of systems in **QSD**.

(47) This configuration has two complex lines (intersecting at a finite point) as a singular curve. The neighborhood of this singular point is formed by two hyperbolic sectors and looks like an intricate singular point $hh_{(4)}$. Moreover such a singular point is topologically equivalent to a nilpotent cusp and the global configuration is topologically equivalent to (47). The unique phase portrait is $QS47_1^{(2)}$. The topological codimension of this portrait is 2 because this is the codimension of the cusp.

(176) This configuration has a finite generic node (with two different eigenvalues) and a singular line which does not pass through it. Moreover the singular line is not parallel to the invariant lines generated by the eigenvectors of the finite node. The only possible phase portrait is $QS176_1^{(4)}$.

(177) We have a singular line and a generic node "under" it. Moreover the singular line is not parallel to the invariant lines generated by the eigenvectors of this node. The unique phase portrait is $QS177_1^{(5)}$.

(178) This configuration has a finite saddle and a singular line which does not pass through it and it is not parallel to the invariant lines generated by the eigenvectors of the saddle. The only possible phase portrait is $QS178_1^{(4)}$.

(179) We have a singular line and a generic saddle "under" it. Moreover the singular line is not parallel to the invariant lines generated by the eigenvectors of this saddle. The unique phase portrait is $QS179_1^{(5)}$.

(180) This configuration has no finite isolated singularities and a singular line. The linear system after the removal of this line has an infinite semi-elemental $\overline{\binom{1}{1}}\, SN$ and a node N^d, i.e. a node with coinciding eigenvalues and non-diagonalizable Jacobian matrix. The infinite singular points are not "under" the singular line. This leads to the unique phase portrait $QS180_1^{(5)}$.

(181) This configuration has a real hyperbola as a singular curve and an isolated infinite singularity which is a star node. This is the unique configuration for a system in **QSD** which produces two different phase portraits. It depends on whether the infinite star node is located between the two infinite singular points produced by the same component of the hyperbola or not. In the first case we get the phase portrait $QS181_2^{(5)}$, whereas in the second case we obtain $QS181_1^{(5)}$.

(182) This configuration has two complex singular straight lines intersecting at a real finite singularity. The constant flow that remains after removing the singular curve has an infinite singularity which does not coincide with any of the infinite singularities produced by the singular curve. The only possible phase portrait is $QS182_1^{(6)}$.

(183) This configuration has a finite one-direction node (n^d) and a singular line which does not pass through it. Moreover the singular line is not parallel to the invariant line generated by the eigenvector of the finite node. The only possible phase portrait is $QS183_1^{(5)}$.

(184) We have a singular line and a finite one-direction node "under" it. Moreover the singular line is not parallel to the invariant line generated by the eigenvector of the finite node. This leads to the unique phase portrait $QS184_1^{(6)}$.

(185) This configuration has a finite saddle and a singular line which does not pass through it. But the singular line is parallel to one of the invariant lines generated by the eigenvectors of the saddle. The only possible phase portrait is $QS185_1^{(5)}$.

(186) We have a singular line and a generic saddle "under" it. Moreover the singular line coincides with one of the invariant lines generated by the eigenvectors of this saddle. The unique phase portrait is $QS186_1^{(6)}$.

(187) This configuration has a finite generic node (with two different eigenvalues) and a singular line which does not pass through it. Moreover the singular line is parallel to the invariant line generated by the eigenvector corresponding to eigenvalue of higher absolute value of the finite node. The only possible phase portrait is $QS187_1^{(5)}$.

(188) We have a singular line and a generic node "under" it. Moreover the singular line coincides with the invariant line generated by the eigenvector corresponding to the eigenvalue of higher absolute value of the finite node. This leads to the unique phase portrait $QS188_1^{(6)}$.

(189) This configuration has a finite generic node (with two different eigenvalues) and a singular line which does not pass through it. Moreover the singular line is parallel to the invariant line generated by the eigenvector corresponding to the eigenvalue of lower absolute value of the finite node. The only possible phase portrait is $QS189_1^{(5)}$.

(190) We have a singular line and a generic node "under" it. Moreover the singular line coincides with the invariant line generated by the eigenvector corresponding to the eigenvalue of lower absolute value of the finite node. This leads to the unique phase portrait $QS190_1^{(6)}$.

(191) This configuration has a real parabola as a singular curve. Moreover the system obtained after removing this singular curve is constant and it has an infinite singularity not contained in the singular parabola. The unique phase portrait is $QS191_1^{(6)}$.

(192)→(137) This configuration has two parallel complex singular straight lines. Moreover the system obtained after removing this singular curve is constant and it has an infinite singularity not contained in the singular curve. The intersection between two parallel complex singular straight lines is a real singular point at infinity whose neighborhood behaves like an intricate singularity with two hyperbolic sectors. Then the geometrical configuration of singularities is topologically equivalent to (137) and we get the unique phase portrait $QS137_1^{(3)}$. We point out that this configuration was not detected being topologically equivalent to (137) in [6]. The error was detected in [8] where we decided to leave the code (192) unassigned.

(193) This configuration has no finite isolated singularities and a singular line. The linear system after the removal of this line has an infinite semi-elemental singularity $\overline{\binom{1}{1}}\, SN$ and a node N^d. However the infinite saddle-node is "under" the singular line which does not coincide with the invariant line of the linear system. This leads to the unique phase portrait $QS193_1^{(6)}$.

(194) This configuration has two real parallel singular straight lines. Moreover the system obtained after removing this singular curve is constant and it has an infinite singularity not contained in the singular curve. The unique phase portrait is $QS194_1^{(7)}$.

(195) This configuration has no finite isolated singularities and a singular line. The linear system after the removal of this line has an infinite semi-elemental singularity $\overline{\binom{1}{1}}\, SN$ and a node N^d. However the infinite saddle-node is "under" the singular line which coincides with the invariant line of the linear system. This leads to the unique phase portrait $QS195_1^{(7)}$.

(196) This configuration has a double real singular straight line. Moreover the system obtained after removing this singular line is constant and it has an infinite singularity not contained in the singular line. The unique phase portrait is $QS196_1^{(8)}$.

(197) This configuration has no finite isolated singularities and a singular line. The linear system after the removal of this line has an infinite nilpotent elliptic saddle $\widehat{\binom{1}{2}}\, E-H$. However the infinite nilpotent saddle is not "under" the singular line. This leads to the unique phase portrait $QS197_1^{(6)}$.

(198) This configuration has no finite isolated singularities and a singular line. The linear system after the removal of this line has an infinite semi-elemental singularity $\overline{\binom{1}{1}}\, SN$ and a node N^d. However the infinite node is "under" the singular line. It is important to notice that all orbits that arrive at this infinite node are tangent to the infinite line[1]. So every orbit going out from the infinite node must cross exactly once the singular line.

This leads to the unique phase portrait $QS198_1^{(6)}$.

(199) This configuration has a real hyperbola as a singular curve and an infinite star node "under" it. Then the only phase portrait is $QS199_1^{(6)}$.

(200) This configuration has real non-parallel straight lines as a singular curve, and an infinite star node "under" it. This leads to the unique phase portrait $QS200_1^{(7)}$.

(201) This configuration has a finite one-direction node (n^d) and a singular line which does not pass through it. However the singular line is parallel to the invariant line generated by the eigenvector of the finite node. The only possible phase portrait is $QS201_1^{(6)}$.

(202) We have a singular line and a finite one-direction node "under" it. Moreover the singular line coincides with the invariant line generated by the eigenvector of the finite node. This leads to the unique phase portrait $QS202_1^{(7)}$.

(203) This configuration has no finite isolated singularities and a singular line. The linear system after the removal of this line has an infinite nilpotent elliptic saddle $\widehat{\binom{1}{2}}\, E-H$. Moreover the infinite nilpotent elliptic saddle is "under" the singular line. Then we get to the unique phase portrait $QS203_1^{(7)}$.

(204) This configuration has a real parabola as a singular curve. Moreover the system obtained after removing this singular curve is constant and it has an infinite singularity "under" the singular parabola. The unique phase portrait is $QS204_1^{(7)}$.

(12) This configuration has two parallel complex singular straight lines. Moreover the system obtained after removing this singular curve is constant and it has an infinite singularity which is a star node "under" the singular curve. In this case the coincidence of the real infinite star node of the constant system with the real infinite singularity produced by the intersection of the complex parallel lines, produces a point whose neighborhood behaves like a normal star node. Thus this configuration is topologically equivalent to (12) and the unique phase portrait is $QS12_1^{(0)}$.

[1]The infinite node in configurations (180), (193) and (195) is also an N^d. But since it is not under the singular line, this is not relevant.

(205) This configuration has two real parallel singular straight lines. Moreover the system obtained after removing this singular curve is constant and it has an infinite singularity "under" the singular curve. The unique phase portrait is $QS205_1^{(8)}$.

(206) This configuration has a double real singular straight line. Moreover the system obtained after removing this singular line is constant and it has an infinite singularity "under" the singular line. The only possible phase portrait is $QS206_1^{(9)}$.

(207) This configuration has the line at infinity filled up with singularities, a real singular straight line and a finite star node. The unique phase portrait is $QS207_1^{(7)}$.

(208) This configuration has the line at infinity filled up with singularities and a real singular straight line with a finite star node "under" it. This leads to the unique phase portrait $QS208_1^{(8)}$.

Since we examined all the branches of the diagram given in Figure 5 we conclude that the Main Theorem is proved.

4 Conclusions

In this paper we have already classified all the phase portraits of quadratic systems belonging to the family **QSD**, i.e. degenerate systems. Among them we have found two systems with the infinite line filled up with singularities. We have also "baptized" them with a definitive name according to our Notation 1. We think that it is an opportunity to include here the rest of phase portraits of quadratic systems that have an infinite number of singularity at infinity. This family of systems was already investigated in [20, 28], where 11 phase portraits for non-degenerate systems in this class were detected. Here in Figure 6 these phase portraits are presented with their definitive names.

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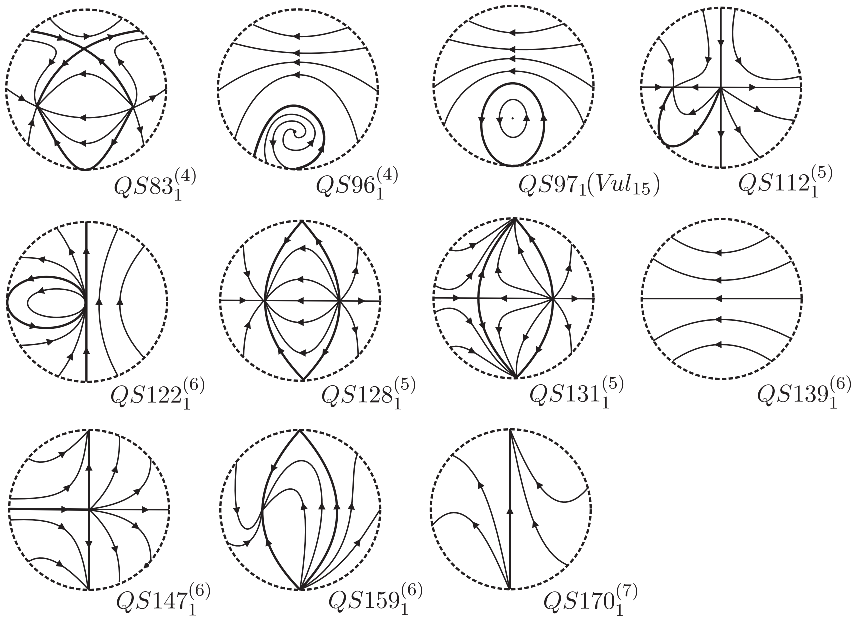

Figure 6. Phase portraits of non-degenerate quadratic systems with infinite singular line.

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