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The application of $14D$ Ricci-flat metrics to the equations of rotation of a top

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Abstract. With a help of Ricci-flat metric of 14-dim Riemann space the solutions of Navier-Stokes system of equations to the theory of rotation of a top are applied.

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1 Introduction

The Navier-Stokes system of equations

$$\frac{\partial}{\partial t}\vec{U} + \left(\vec{U}\cdot\vec{\nabla}\right)\vec{U} - \mu\Delta\vec{U} + \vec{\nabla}P = 0, \qquad \vec{\nabla}\cdot\vec{U} = 0, \tag{1}$$

where $\vec{U} = [U(\vec{x},t), V(\vec{x},t), W(\vec{x},t)]$ is the fluid velocity, $P = P(\vec{x},t)$ is the pressure, μ is the viscosity and $\vec{x} = (x,y,z)$ has numerous applications.

The problem of their integration and classification of solutions is important to the modern mathematical and theoretical physics. We will use methods of the Riemannian geometry to studies properties of the system (1).

2 Ricci-flat metric associated with the NS-system

Let us introduce into consideration 14-dim space in local coordinates $(x,y,z,t,\eta,\rho,m,u,v,w,p,\xi,\chi,n)$.

Theorem 1. *The metric of the form*

$$\begin{aligned} ds^2 = {} & 2\,dx^2 + 2\,dxdy + 2\,dxdu + 2\,dy^2 + 2\,dydz + 2\,dydv + 2\,dz^2 + 2\,dzdw + \\ & +2\,dtdp + 2\,d\eta d\xi + 2\,d\rho d\chi + 2\,dmdn + A dt^2 + B d\eta^2 + C d\rho^2 + E dm^2, \end{aligned} \tag{2}$$

where

$$A = 2 - U\left(x,y,z,t\right)u - V\left(x,y,z,t\right)v - W\left(x,y,z,t\right)w,$$

$$B = \left(-UW + \mu\frac{\partial}{\partial z}U\right)w + \left(-UV + \mu\frac{\partial}{\partial y}U\right)v + \left(\mu\frac{\partial}{\partial x}U - (U)^2 - P\right)u - Up,$$

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$$C = \left(-VW + \mu\,\frac{\partial}{\partial z}V\right)w + \left(\mu\,\frac{\partial}{\partial y}V - (V)^2 - P\right)v + \left(-UV + \mu\,\frac{\partial}{\partial x}V\right)u - Vp,$$

$$E = \left(-\mu\,\frac{\partial}{\partial x}U - \mu\,\frac{\partial}{\partial y}V - (W)^2 - P\right)w + \left(-VW + \mu\,\frac{\partial}{\partial y}W\right)v +$$

$$\left(-UW + \mu\,\frac{\partial}{\partial x}W\right)u - Wp$$

is the Ricci-flat on solutions of Navier-Stokes system of equations (1).

The introduced 14-dimensional space consists of $6D$-space in plane coordinates $(\eta, \rho, m, \xi, \chi, n)$ and of dual 4-dimensional subspaces in Euler coordinates (x, y, z, t) and of Lagrangian coordinates- (u, v, w, p).

This communication presents the results on a study of the properties of incompressible fluid flows based on multidimensional Riemannian metrics of type (1) proposed by the author [1], [2]. As a basic example of nontrivial flow described by a first-order ODE system is considered

$$\begin{aligned}
\frac{d}{dt}x\,(t) &= U(x, y, z, t) = A(t)yz + L(t),\\
\frac{d}{dt}y\,(t) &= V(x, y, z, t) = A(t)xz + M(t),\\
\frac{d}{dt}z\,(t) &= W(x, y, z, t) = A(t)xy + N(t)
\end{aligned} \tag{3}$$

with the new functions $(L(t), M(t), N(t))$.

In this case the Ricci-flat metric looks as

$$\begin{gathered}
ds^2 = 2\,dx^2 + 2\,dxdy + 2\,dxdu + 2\,dy^2 + 2\,dydz + 2\,dydv + 2\,dz^2 + 2\,dzdw +\\
+2\,dp\,dt + 2\,d\eta\,d\xi + 2\,d\rho\,d\chi + 2\,dm\,dn +\\
((-A(t)xy - N(t))\,w + (-A(t)xz - Mt))\,v + (-A(t)yz - L(t))\,u + 2)\,dt^2 +\\
(K_1)\,d\eta^2 + (K_2)\,d\rho^2 + (K_3)\,dm^2
\end{gathered} \tag{4}$$

where the coefficients K_i, depends on coordinates and the pressure $P = P(x, y, z, t)$

$$P\,(x, y, z, t) = -1/2\,\left(A(t)\right)^2 z^2\left(x^2 + y^2\right) +$$

$$+\left(-L(t)A(t)y - M(t)A(t)x - \left(\frac{d}{dt}A(t)\right)xy - \frac{d}{dt}N(t)\right)z -$$

$$-1/2\,\left(A(t)\right)^2 x^2y^2 + \left(-A(t)yN(t) - \frac{d}{dt}L(t)\right)x - \left(\frac{d}{dt}M(t)\right)y + F_3(t).$$

Next, we consider a special case of solving system of equations (3) of the form

$$\frac{d}{dt}x\,(t) = y(t)z\,(t) + L, \quad \frac{d}{dt}y\,(t) = x(t)z\,(t)\,, \quad \frac{d}{dt}z\,(t) = x(t)y\,(t)\,. \tag{5}$$

The partial differential equation corresponding to the system (5) has a form

$$\left(\frac{\partial}{\partial x}z\left(x,y\right)\right)\left(yz\left(x,y\right)+L\right)+\left(\frac{\partial}{\partial y}z\left(x,y\right)\right)xz\left(x,y\right)-xy=0 \tag{6}$$

and its solutions are expressed through solutions of the second-order ODE

$$\begin{gathered}\frac{d^2}{dx^2}y\left(x\right)+\frac{\left(\left(y\left(x\right)\right)^4+L^2\right)\left(\frac{d}{dx}y\left(x\right)\right)^3}{L^2x}-3\,\frac{\left(y\left(x\right)\right)^3\left(\frac{d}{dx}y\left(x\right)\right)^2}{L^2}+\\ \frac{\left(3\,\left(y\left(x\right)\right)^2x^2-L^2\right)\frac{d}{dx}y\left(x\right)}{L^2x}-\frac{y\left(x\right)x^2}{L^2}=0.\end{gathered} \tag{7}$$

On the base of the Liouville theory of Invariants the equation (7) are investigated and the following algebraic ODE is derived

$$\begin{gathered}\left(\frac{d}{dx}y\left(x\right)\right)^2+\frac{x^2\left(\left(y\left(x\right)\right)^2+C_1\right)\left(L\left(y\left(x\right)\right)^3-\left(y\left(x\right)\right)^2-C_1\right)}{y\left(x\right)\Delta}-\\ 2\,\frac{x\left(\left(y\left(x\right)\right)^2+C_1\right)\left(L\left(y\left(x\right)\right)^3+L^2y\left(x\right)-\left(y\left(x\right)\right)^2-L-C_1\right)\frac{d}{dx}y\left(x\right)}{\Delta}=0,\end{gathered} \tag{8}$$

where

$$\Delta = Ly(x)^6+2L^2y(x)^4+Ly(x)^4C_1+L^3y(x)^2+2L^2y(x)^2C_1-y(x)^5+$$

$$+L^3C_1-2Ly(x)^3-2C_1y(x)^3-L^2y(x)-2Ly(x)C_1-y(x){C_1}^2.$$

With equation (8) written in the new variables $\frac{d}{dx}y\left(x\right)=S,\ y\left(x\right)=T$

$$\begin{gathered}LS^2T^7+\left(-2\,LSx-S^2\right)T^6+\left(2\,L^2S^2+LS^2C_1+Lx^2+2\,Sx\right)T^5+\\ \left(-2\,L^2Sx-2\,LSC_1\,x-2\,LS^2-2\,S^2C_1-x^2\right)T^4+\\ \left(L^3S^2+2\,L^2S^2C_1+LC_1\,x^2+2\,LSx+4\,SC_1\,x\right)T^3+\\ \left(-2\,L^2SC_1\,x-L^2S^2-2\,LS^2C_1-S^2{C_1}^2-2\,C_1\,x^2\right)T^2+\\ \left(L^3S^2C_1+2\,LSC_1\,x+2\,S{C_1}^2x\right)T-{C_1}^2x^2=0\end{gathered} \tag{9}$$

an algebraic curve is associated, whose properties depend on the values of parameter C_1 and L.

The genus of the algebraic curve (9) in variables (S) and (T) $g = 1$, the J-invariant=1728, and so, it is an elliptic curve, which in the general case is parametrized by the doubly periodic Weierstrasse function. The values of parameters $L = (9/100) \times 5^{1/3}$ and $C_1 = -0.262983858339810$, $C_1 = -0.214099234924727$, $C_1 = -0.635931322599648$ correspond to degeneracies of the curve, which can be associated with a special regime of liquid- flow of rotation of the top.

In the case of the Kovalevsky top $A = 2C, B = 2C, y_0 = 0, z_0 = 0$ the Euler-Poisson system of equations

$$\frac{d}{dt}x(t) = -\frac{(C-B)\,yz}{A} + \frac{Mg\,(y_0\,\omega 2(t) - z_0\,\omega 1(t))}{A},$$

$$\frac{d}{dt}y(t) = -\frac{(A-C)\,x(t)\,z(t)}{B} + \frac{z_0\,\omega(t) - x_0\,\omega 2(t)}{B},$$

$$\frac{d}{dt}z(t) = -\frac{(B-A)\,x(t)\,y(t)}{C} + \frac{x_0\,\omega 1(t) - y_0\,\omega(t)}{C},$$

$$\frac{d}{dt}\omega(t) = z(t)\,\omega 1(t) - y(t)\,\omega 2(t), \quad \frac{d}{dt}\omega 1(t) = x(t)\,\omega 2(t) - z(t)\,\omega(t),$$

$$\frac{d}{dt}\omega 2(t) = y(t)\,\omega(t) - x(t)\,\omega_1(t), \tag{10}$$

leads to the relations

$$x(t) = \frac{y(t)\,\omega(t)\,\omega_2(t) + \omega(t)\,\frac{d}{dt}\omega(t) + \omega_1(t)\,\frac{d}{dt}\omega_1(t)}{\omega_2(t)\,\omega_1(t)}, \tag{11}$$

$$z(t) = \frac{y(t)\,\omega_2(t) + \frac{d}{dt}\omega(t)}{\omega 1(t)}, \tag{12}$$

$$y(t) = -2\,\frac{C\left(\frac{d}{dt}\omega_1(t)\right)\omega(t)\,\omega_1(t)}{\omega 2(t)\left(2\,C\left(-(\omega_1(t))^2 - (\omega_2(t))^2 - 1\right) + 2\,C\,(\omega_1(t))^2 + C\,(\omega_2(t))^2\right)} -$$
$$\frac{\left(2\,C\left(-(\omega_1(t))^2 - (\omega_2(t))^2 - 1\right) + C\,(\omega_2(t))^2\right)\frac{d}{dt}\omega(t)}{\omega_2(t)\left(2\,C\left(-(\omega_1(t))^2 - (\omega_2(t))^2 - 1\right) + 2\,C\,(\omega_1(t))^2 + C\,(\omega_2(t))^2\right)} +$$
$$+\frac{C_2\,\omega 1(t)}{2\,C\left(-(\omega_1(t))^2 - (\omega_2(t))^2 - 1\right) + 2\,C\,(\omega_1(t))^2 + C\,(\omega_2(t))^2}. \tag{13}$$

Together with the known first integrals of system (10)

$$(\omega(t))^2 + (\omega_1(t))^2 + (\omega_2(t))^2 - 1,\ Ax(t)\,\omega(t) + By(t)\,\omega_1(t) + Cz(t)\,\omega_2(t) - C_2,$$

$$A\,(x(t))^2 + B\,(y(t))^2 + C\,(z(t))^2 - 2\,Mg\,(x_0\,\omega(t) + y_0\,\omega 1(t) + z_0\,\omega 2(t)) + C_1,$$

$$\left((x(t))^2 + (y(t))^2 + c\omega(t)\right)^2 + (2\,x(t)\,y(t) + c\omega_1(t))^2 = K^2 \tag{14}$$

the components of fluid-flow velocity and pressure $P(x, y, z, t)$ are determined using the Ricci-flat metric on solutions of the system (1).

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