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# Characteristic Properties of Centrally Symmetric Convex Sets

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**Abstract.** This is a survey on characteristic properties of centrally symmetric convex sets in the $n$-dimensional Euclidean space. These properties are formulated in terms of orthogonal projections, plane sections, shadow-boundaries, affine diameters, and tiling polytopes.

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## 1 Introduction

Among various geometric concepts of symmetry, *central symmetry* of sets is one of the most simple to deal with. Convex geometry provides an extensive study of centrally symmetric sets, including their characteristic properties. This fact became visible on the turn of 20th century (see, e. g., Bonnesen and Fenchel [13], § 14). Since then the study of such sets became an established topic of proper interest.

Besides the intuitive appeal and simplicity of description, centrally symmetric sets mark their existence in various brunches of mathematics due to the fact that many analytical or geometric problems on arbitrary sets can be essentially simplified for the case of central symmetry.

For instance, an essential part of geometric number theory deals with lattice packing of centrally symmetric convex bodies. The classical theorem of Minkowski [56, Chapter 3] asserts that if a convex body $K$ in the $n$-dimensional Euclidean space is symmetric about the origin $o$ and has volume $V(K) \geq 2^n$, then it contains at least one nonzero lattice point.

Another example give the results on measures of symmetry of convex bodies, where centrally symmetric convex bodies form the core of this discipline (see, e. g., Grünbaum [32] and Toth [81]).

Centrally symmetric polytopes naturally appear in the study of parallelohedra, that is, convex polytopes which admit tiling of the space by translates (see, e. g., Fedorov [28] and Gruber [31]), and also in the theory of zonotopes and zonoids (see, e. g., McMullen [53], Schneider and Weil [70]).

Geometric theory of vector normed spaces gives one more instance, where centrally symmetric convex bodies are often interpreted as unit balls. Consequently, properties of these spaces are closely related to the properties of the bodies.


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There are two survey articles, due to Grünbaum [32] and Heil and Martini [36], which contain a large selection of results on characteristic properties of centrally symmetric convex sets, published prior to 1993. The present paper aims to complement these surveys with additional and more recent references. While our text is divided into well-contoured groups of results, additional sporadic characteristic properties of centrally symmetry convex sets can be found in the above surveys. We also omit the topic on measures of symmetry of convex sets, which, in our opinion, deserves a separate survey.

For the uniformity of presentation, we assume throughout the text that all convex sets in question are closed and $n$-dimensional. Such a restriction does not affect the generality of the argument, since we always can consider the sets in their affine spans, where they become full-dimensional.

We conclude this section with necessary definitions and terminology (see, e. g., [78] for a detailed account).

In what follows, $\mathbb{R}^n$ stands for the $n$-dimensional Euclidean space. A *plane* $L \subset \mathbb{R}^n$ of dimension $m$, $0 \le m \le n$, is a translate of an $m$-dimensional subspace $S \subset \mathbb{R}^n$: $L = c + S$ for a suitable point $c \in \mathbb{R}^n$. A *hyperplane* in $\mathbb{R}^n$ is a plane of dimension $n-1$. Planes $L_1$ and $L_2$ are called *parallel* provided they are translates of each other. A mapping $f : \mathbb{R}^n \to \mathbb{R}^m$ is called an *affine transformation* provided $f(x) = u + g(x)$, where $u \in \mathbb{R}^m$ and $g : \mathbb{R}^n \to \mathbb{R}^m$ is a linear transformation. The origin (zero vector) of $\mathbb{R}^n$ is denoted $o$.

A set $X \subset \mathbb{R}^n$ is called *centrally symmetric* provided there is a point $v \in \mathbb{R}^n$ such that $X = 2v - X$ (in this case $v$ is the *center* of $X$). Sets $X_1$ and $X_2$ in $\mathbb{R}^n$ are called *homothetic* if $X_1 = z + \lambda X_2$ for a suitable point $z \in \mathbb{R}^n$ and a nonzero scalar $\lambda$. Furthermore, $X_1$ and $X_2$ are *directly* (*inversely*) homothetic if $\lambda > 0$ (respectively, $\lambda < 0$). In particular, $X_1$ and $X_2$ are *translates of each other* if $\lambda = 1$, and are *symmetric to each other* if $\lambda = -1$. A line $l \subset \mathbb{R}^n$ is called an *axis of symmetry* of a nonempty set $X \subset \mathbb{R}^n$ provided for every hyperplane $H$ orthogonal to $l$ the set $X \cap H$ is symmetric about the point $l \cap H$.

For a nonempty set $X \subset \mathbb{R}^n$ and a line $l \subset \mathbb{R}^n$, the (both-way infinite) *generated cylinder* $Z_l(X)$ is the union of all lines which are parallel to $l$ and meet $X$. Given a point $p \in \mathbb{R}^n$, the *generated cone* $C_p(X)$ is defined as the union of all closed halflines with endpoint $p$ which meet $X$, and the *generated double-cone* $D_p(X)$ is defined as the union of all lines through $p$ which meet $X$

Finally, $bd\,X$, $cl\,X$, and $int\,X$, denote, respectively, the boundary, closure, and interior of a set $X \subset \mathbb{R}^n$. By a *convex solid* in $\mathbb{R}^n$ we will mean an $n$-dimensional closed convex set in $\mathbb{R}^n$, possibly unbounded. A *convex body* is a bounded convex solid. An intersection of finitely many closed halfspaces of $\mathbb{R}^n$ is called a *polyhedron*, and a *polytope* is a bounded polyhedron.

## 2 Algebra of centrally symmetric sets

This section describes some basic facts on algebraic and set-theoretic operations that involve centrally symmetric sets.

**2.1.** *For a set $X \subset \mathbb{R}^n$ and a point $v \in \mathbb{R}^n$, the set $v + X - X$ is symmetric about $v$.*

**2.2.** *For a set $X \subset \mathbb{R}^n$ and a point $v \in \mathbb{R}^n$, the intersection $X \cap (2v - X)$ is symmetric about $v$.*

**2.3.** *If a set $X \subset \mathbb{R}^n$ is symmetric about a point $v \in \mathbb{R}^n$ and $f : \mathbb{R}^n \to \mathbb{R}^m$ is an affine transformation, then the image $f(X)$ is symmetric about $f(v)$.*

**2.4.** *If a set $X \subset \mathbb{R}^n$ is symmetric about a point $v \in \mathbb{R}^n$, and $l$ is a 1-dimensional subspace of $\mathbb{R}^n$, then the generated cylinder $Z_l(X)$ is symmetric about $v$ and the line $v + l$ is its axis of symmetry.*

**2.5.** *If a set $X \subset \mathbb{R}^n$ is symmetric about a point $v \in \mathbb{R}^n$, then both sets $cl X$ and $int X$ are symmetric about $v$.*

**2.6.** *If $\{X_\lambda\}$ is a family of sets in $\mathbb{R}^n$, all symmetric about a point $v \in \mathbb{R}^n$, then their intersection $\cap X_\lambda$ and their union $\cup X_\lambda$ are symmetric about $v$.*

**2.7.** *If sets $X_1$ and $X_2$ in $\mathbb{R}^n$ are symmetric about points $v_1$ and $v_2$, respectively, then their vector sum $X_1 + X_2$ is symmetric about $v_1 + v_2$.*

**2.8.** *Let $K \subset \mathbb{R}^n$ be a convex set and $X \subset \mathbb{R}^n$ be a nonempty bounded set symmetric about a point $v \in \mathbb{R}^n$. If the vector sum $K + X$ is symmetric about a point $w \in \mathbb{R}^n$, then $cl K$ is symmetric about $w - v$.*

The assertion 2.8 follows from the equality $K + X = (2(w - v) - K) + X$, and the result of Rådström [65]: If $K_1$ and $K_2$ are convex sets and $X$ is a bounded set in $\mathbb{R}^n$ such that $K_1 + X = K_2 + X$, then $cl K_1 = cl K_2$.

## 3 Orthogonal Projections

The assertion 2.3 implies that parallel projections of a centrally symmetric convex body in $\mathbb{R}^n$ on all proper planes of $\mathbb{R}^n$ again are centrally symmetric. A natural question here is whether a convex body $K \subset \mathbb{R}^n$ itself is centrally symmetric provided its parallel (or even only orthogonal) projections on all proper planes of $\mathbb{R}^n$ are centrally symmetric. Trivially, this question has a negative answer for $n = 2$, since the orthogonal projection of any convex body $K \subset \mathbb{R}^2$ on a line $l \subset \mathbb{R}^2$ is a closed segment. Nevertheless, the above question has an affirmative answer for the case $n \geq 3$, as described below.

In 1918, Blaschke and Hessenberg [10] showed that a convex body $K \subset \mathbb{R}^3$ is centrally symmetric if and only if the orthogonal projections of $K$ on all planes of $\mathbb{R}^3$ are centrally symmetric (see also Kubota [43] in terms of sections of generated cylinders). This assertion was generalized by Bonnesen and Fenchel [13, p. 132] to the case of $n \geq 3$.

**3.1** ([13])**.** *A convex body $K \subset \mathbb{R}^n$, $n \geq 3$, is centrally symmetric if and only if there is an integer $m$, $2 \leq m \leq n - 1$, such that the orthogonal projections of $K$ on all $m$-dimensional planes of $\mathbb{R}^n$ are centrally symmetric.*

The original assertion of Blaschke and Hessenberg [10] was complemented by Ôishi [63], who proved the following result: If $K \subset \mathbb{R}^3$ is a strictly convex regular body and $L$ is a plane in $\mathbb{R}^3$, then $K$ is centrally symmetric if and only if, for any line $l \subset L$, the orthogonal projection of $K$ on any plane perpendicular to $l$ is centrally symmetric.

The next refinements of 3.1 are due to Montejano [58] and Boltyanski and Jerónimo Castro [12].

**3.2** ([58]). *Let $K \subset \mathbb{R}^n$, $n \geq 3$, be a strictly convex body and $m$ be an integer, $1 \leq m \leq n-2$, and $S$ be an $m$-dimensional subspace of $\mathbb{R}^n$. Then $K$ is centrally symmetric if and only if the orthogonal projections of $K$ on all $(m+1)$-dimensional subspaces of $\mathbb{R}^n$ that contain $S$ are centrally symmetric.*

**3.3** ([58]). *Let $K \subset \mathbb{R}^n$, $n \geq 3$, be a strictly convex body and $m$ be an integer, $2n/3 \leq m \leq n-1$. Denote by $\mathcal{H}$ a family of $m$-dimensional subspaces in $\mathbb{R}^n$ such that every 1-dimensional subspace of $\mathbb{R}^n$ is contained in a suitable subspace $S \in \mathcal{H}$. Then $K$ is centrally symmetric if and only if the orthogonal projections of $K$ on all subspaces $S \in \mathcal{H}$ is centrally symmetric.*

**3.4** ([12]). *Let $K \subset \mathbb{R}^n$, $n \geq 3$, be a convex body, $m$ be an integer, $1 \leq m \leq n-2$, and $L$ be a fixed $m$-dimensional subspace of $\mathbb{R}^n$. Assume that there exists a hyperplane $H \subset \mathbb{R}^n$ such that $H$ is orthogonal to a line contained in $L$ and each support hyperplane of $K$ parallel to $H$ has only one common point with $K$. Then $K$ is centrally symmetric if and only if for every $(m+1)$-dimensional subspace $S$ containing $L$, the orthogonal projection of $K$ on $S$ is centrally symmetric.*

We observe that the assertions 3.2–3.4 do not hold if the body $K$ is not strictly convex. Indeed, if $K$ is the triangular prism in $\mathbb{R}^3$ given by

$$K = \{(x, y, z) : x \geq 0, y \geq 0, x + y \leq 1, 0 \leq z \leq 1\},$$

and if $l$ is the coordinate $z$-axis of $\mathbb{R}^3$, then orthogonal projections of $K$ on all planes through $l$ are rectangles, while $K$ has no center of symmetry.

Another approach to the proof of 3.1 is due to Rogers [66]. We will say that a collection $\mathcal{F}$ of planes in $\mathbb{R}^n$ is an $H$-family provided any pair of convex bodies $K_1$ and $K_2$ in $\mathbb{R}^n$ are directly homothetic if and only if orthogonal projections of these bodies on all planes from $\mathcal{F}$ are directly homothetic. Similarly, a collection $\mathcal{F}$ of planes in $\mathbb{R}^n$ is called an $S$-family provided any convex body $K$ in $\mathbb{R}^n$ is centrally symmetric if and only if the orthogonal projections of $K$ on all planes from $\mathcal{F}$ are centrally symmetric.

It is easy to see that any $H$-family of planes in $\mathbb{R}^n$ also is an $S$-family. Indeed, if $\mathcal{F}$ is an $H$-family and all orthogonal projections of a convex body $K \subset \mathbb{R}^n$ on the planes from $\mathcal{F}$ are centrally symmetric, then the orthogonal projections of $K$ and $-K$ are translates of each other, and thus $K$ and $-K$ are translates of each other, implying that $K$ is centrally symmetric.

Rogers [66] proved that the collection of all 2-dimensional planes in $\mathbb{R}^n$ is an $H$-family and deduced from here that this collection also is an $S$-family (thus deducing 3.1). The latter approach allows us to describe two more $S$-families in $\mathbb{R}^n$, previously known as $H$-families, due to Soltan [74] and Székely [80] (and later to Golubyatnikov [30] for the case of translates).

**3.5** ([74]). *Given a convex body $K \subset \mathbb{R}^n$, $n \geq 3$, there exists a line $l \subset \mathbb{R}^n$ (whose position depends on $K$) with the following property: $K$ is centrally symmetric if and only if the orthogonal projections of $K$ on all 2-dimensional planes of $\mathbb{R}^n$ containing $l$ are centrally symmetric.*

**3.6** ([30, 80]). *Let $\mathcal{F}$ be a collection of $(n-1)$-dimensional subspaces in $\mathbb{R}^n$, $n \geq 3$, and let $N(\mathcal{F})$ be the family of unit normals to the subspaces from $\mathcal{F}$. Then $\mathcal{F}$ is an $H$-family (and thus is an $S$-family) provided the following conditions are satisfied:*

- ($a$) *$N(\mathcal{F})$ contains three non-collinear vectors,*
- ($b$) *the closure of $N(\mathcal{F})$ meets every big $(n-2)$-dimensional sphere of the unit sphere in $\mathbb{R}^n$.*

## 4 Plane sections and ellipsoids

In 1889, Brunn [14, Chapter IV] showed that a regular convex body $K \subset \mathbb{R}^3$ is a solid ellipsoid provided all plane sections of $K$ are centrally symmetric. This assertion was generalized and sharpened in various ways, as shown below.

Using arguments of Blaschke [9, § 44], Olovjanishnikov [64] established a local characterization of convex quadrics: Let $S \subset \mathbb{R}^3$ be a three times differentiable bounded convex surface of positive curvature, and let $H$ be a tangent plane of $S$. If all plane sections of $S$, which are sufficiently close to $H$, are centrally symmetric, then a set of all points from $S$ sufficiently close to $H$ is contained in a convex quadric.

Aitchison [2] considered the following condition ($A$) on a convex body $K \subset \mathbb{R}^n$:

- ($A$) Given a unit vector $u \in \mathbb{R}^n$ and the hyperplane $H(u) \subset \mathbb{R}^n$ which supports $K$ and has outer normal $u$, there is a scalar $\varepsilon(u) > 0$ such that every nonempty section of $K$ by a hyperplane $H \subset \mathbb{R}^n$ which is parallel to $H(u)$ and lies within a distance $\varepsilon(u)$ from $H(u)$ is centrally symmetric.

Aitchison [2] showed that the boundary of a strictly convex body $K \subset \mathbb{R}^3$ satisfying the condition ($A$) consists of finitely many pieces of ellipsoids. Furthermore, if the scalar $\varepsilon(u)$ in condition ($A$) is constant, then $K$ is a solid ellipsoid (see [1]).

Based on Aitchison's argument, Burton [17, 18] proved the following deep result. We recall that a convex polytope $Q \subset \mathbb{R}^n$ is a *zonotope* if all it faces are centrally symmetric; equivalently, $Q$ is a vector sum of segments. Also, we observe that the zonotope in 4.1 can be of any dimension $m$, where $0 \leq m \leq n$.

**4.1** ([17, 18]). *A convex body $K \subset \mathbb{R}^n$, $n \geq 3$, satisfies the condition ($A$) if and only if $K$ is the vector sum of a zonotope and a solid ellipsoid. In particular, if $K$ is strictly convex, then $K$ is a solid ellipsoid.*

The following proposition gives an analog of 4.1 to the case of line-free convex solids. We recall that a convex hypersurface $S \subset \mathbb{R}^n$ is called a *convex quadric* provided there is a quadric hypersurface $T \subset \mathbb{R}^n$ and a component $U$ of $\mathbb{R}^n \setminus T$ such that $U$ is a convex set and $S = bd U$ (see [76]).

**4.2** ([77]). *Let $K \subset \mathbb{R}^n$, $n \geq 3$, be a line-free convex solid and $\delta > 0$ be a scalar such that all proper 2-dimensional sections of $K$ of diameter less than $\delta$ are centrally symmetric. Then $bd K$ is a convex quadric.*

Another characteristic property of solid ellipsoids is given by Montejano and Morales-Amaya [60].

**4.3** ([60]). *A centrally symmetric convex body $K \subset \mathbb{R}^n$, $n \geq 3$, is a solid ellipsoid if all 2-dimensional planes sufficiently close to a given diametral chord of $K$ meet $K$ along centrally symmetric sets.*

Montejano and Morales-Amaya [60] also conjectured that a convex body $K \subset \mathbb{R}^n$, $n \geq 3$, symmetric about a point $p$, is a solid ellipsoid provided there is a point $q \in \mathbb{R}^n \setminus \{p\}$ such that all 2-dimensional planes through $q$ sufficiently close to $p$ meet $K$ along centrally symmetric sets.

Olovjanishnikov [64] proved one more extension of Brunn's result from [14].

**4.4** ([64]). *A convex body $K \subset \mathbb{R}^n$, $n \geq 3$, is a solid ellipsoid provided all hyperplane sections of $K$ that divide the volume of $K$ in a given ratio $\lambda \neq 1$ are centrally symmetric.*

Another development of Brunn's ideas was initiated by Rogers [66]. Following [66], we say that a point $p \in \mathbb{R}^n$ is a *pseudocenter* of a convex body $K \subset \mathbb{R}^n$ provided all sections of $K$ by 2-dimensional planes through $p$ are centrally symmetric, not necessarily about $p$. Furthermore, a pseudocenter $p$ of $K$ is called a *false center* if $K$ is not symmetric about $p$. Rogers [66] (see also Burton [16]) proved that a convex body $K \subset \mathbb{R}^n$ with a pseudocenter $p \in int K$ has a true center of symmetry and conjectured that $K$ must be a solid ellipsoid if $p$ is a false center of $K$. This conjecture was confirmed by Aitchison, Petty, and Rogers [3].

Höbinger [38] showed that Roger's conjecture remains true if the body $K$ is regular and a false center $p$ lies anywhere in $\mathbb{R}^n$; he also asked whether the assumption on the regularity of $K$ can be removed. This question was affirmatively answered by Larman [44] (see also Burton and Mani [19] for another proof). Summing up, one can formulate the following assertion.

**4.5** ([3, 44]). *Let $K \subset \mathbb{R}^n$ be a convex body and $p$ be a point in $\mathbb{R}^n$, $n \geq 3$. If all sections of $K$ by 2-dimensional planes through $p$ are centrally symmetric, then $K$ is centrally symmetric. Furthermore, if $p$ is not a center of $K$, then $K$ is a solid ellipsoid.*

The following assertions of Montejano [59] and Jerónomi-Castro, Montejano, and Morales-Amaya [39] characterize centrally symmetric strictly convex bodies in $\mathbb{R}^3$ in terms of plane sections:

($a$) a strictly convex body $K \subset \mathbb{R}^3$ is centrally symmetric provided for every 2-dimensional subspace $S \subset \mathbb{R}^3$ one can choose continuously a centrally symmetric plane section of $K$ ([59]),

($b$) a strictly convex body $K \subset \mathbb{R}^3$ is centrally symmetric provided for every 2-dimensional subspace $S \subset \mathbb{R}^3$ one can choose continuously a pair of plane sections of $K$ which are parallel to $S$ and inversely homothetic ([39]).

## 5 Generated cylinders and shadow-boundaries

We recall that the cylinder $Z_l(K)$ generated by a convex body $K \subset \mathbb{R}^n$ in the direction of a line $l \subset \mathbb{R}^n$ is the union of all lines which are parallel to $l$ and meet $K$. The set $S_l(K) = bd\,K \cap bd\,Z_l(K)$ is called the *shadow-boundary* of $K$ in the direction $l$. Based on results from Section 2, we easily obtain the following result.

**5.1.** *For convex body $K \subset \mathbb{R}^n$, the assertions below are equivalent:*

($a$) *$K$ is centrally symmetric,*

($b$) *all generated cylinders $Z_l(K)$ are centrally symmetric,*

($c$) *all shadow-boundaries $S_l(K)$ are centrally symmetric.*

Every shadow-boundary $S_l(K)$ of a convex body $K \subset \mathbb{R}^n$ divides the boundary of $K$ into two parts, namely, the components of $bd\,K \setminus S_l(K)$. The following assertion is due to Schneider [69]; afterwards it was generalized by Averkov, Makai, and Martini [7] for the case of the $n$-dimensional Minkowski space.

**5.2** ([69])**.** *A convex body $K \subset \mathbb{R}^n$, $n \geq 2$, is centrally symmetric if and only if every its shadow-boundary $S_l(K)$ divides $bd\,K$ into two parts of equal surface area.*

An analog of 5.2 was obtained by Morales-Amaya, Jerónimo-Castro, and Verdusco Hernández [61].

**5.3** ([61])**.** *Let $K \subset \mathbb{R}^n$, $n \geq 3$, be a strictly convex body, and let $L \subset \mathbb{R}^n$ be a hypersurface, which is the image of the unit sphere such that $K$ is contained in the interior of $L$. Suppose that for every point $x \in L$ there exists another point $y \in L$ such that the generated double-cones of $K$ with apices at $x$ and $y$, differ by a translation. Then $bd\,K$ and $L$ are centrally symmetric and concentric.*

## 6 Affine diameters and central symmetry

We recall that a chord $[a, b]$ of a convex body $K \subset \mathbb{R}^n$ is an *affine diameter* of $K$ provided there are two parallel, distinct hyperplanes $H_a$ and $H_b$ both supporting $K$ such that $a \in H_a$ and $b \in H_b$. It is easy to see that any point $x \in K$ belongs to an affine diameter of $K$, and for any direction $l$ in $\mathbb{R}^n$ there is an affine diameter of $K$ parallel to $l$ (see, e. g., [73] for these and other properties of affine diameters).

In 1954, Hammer [34] proved the following assertion (see also Busemann [20, pp. 89–90] and Dharmadhikari and Jogdeo [24] for $n = 2$).

**6.1** ([34])**.** *A convex body $K \subset \mathbb{R}^n$, $n \geq 2$, is symmetric about a given point $v \in int K$ if and only if every chord of $K$ through $v$ is an affine diameter of $K$.*

The next result is due to Falkoner [27] for $n = 2$ and to Khassa [42] for all $n \geq 3$.

**6.2** ([27, 42])**.** *A convex body $K \subset \mathbb{R}^n$, $n \geq 2$, is centrally symmetric provided for every line $l$ and for every pair of parallel hyperplanes $H_1$ and $H_2$ in $\mathbb{R}^n$ both supporting $K$ and parallel to $l$, there is an affine diameter of $K$ in the direction $l$ lying mid-way between $H_1$ and $H_2$.*

The following assertion collects some known criteria of central symmetry of convex sets in the plane.

**6.3.** *A convex body $K \subset \mathbb{R}^2$ is centrally symmetric if and only if any of the following conditions holds:*

($a$) *Every chord of $K$ dividing the boundary of $K$ into two arcs of equal length is an affine diameter $K$* (Zindler [84] for the case when $K$ is strictly convex and has continuous curvature, Hammer and Smith [35] for any plane convex body).

($b$) *Every chord of $K$ that divides the area of $K$ into two parts of equal areas is an affine diameter of $K$* (Hammer and Smith [35] and Kharazishvili [41]).

($c$) *Every midcurve of $K$ bisects the area of the body* (Falkoner [27]).

($d$) *There is a unique point $v \in int K$ which belongs to three chords each halving the area of $K$* (Menon [55] and Zarankiewicz [83]).

($e$) *If, additionally, $K$ is strictly convex, then $K$ is centrally symmetric if and only if every affine diameter divides the area of $K$ in a constant ratio* (Nakajima [62]).

Hammer and Smith [35] announced that part ($a$) in 6.3 holds for the case of Minkowski norm in the plane, and this fact was proved by Averkov [6].

## 7 Central sections and symmetry

Following Grünbaum [32], a vast amount of literature is devoted to measures of symmetry on convex bodies, where centrally symmetric bodies appear in natural way. We provide below some known geometric characteristics for a given point $p \in int K$ to be the center of symmetry of the convex body $K \subset \mathbb{R}^n$.

**7.1.** *A convex body $K \subset \mathbb{R}^n$, $n \geq 3$, is symmetric about $p$ if and only if every hyperplane through $p$ divides the boundary of $K$ into two parts of equal surface area* (Funk [29] and Blaschke [8] for $n = 3$, Schneider [69] for $n \geq 3$).

**7.2.** *A convex body $K \subset \mathbb{R}^3$ is symmetric about $p$ if and only if every plane through $p$ halves the volume of $K$* (Blaschke [9, p. 250]).

**7.3.** *A convex body $K \subset \mathbb{R}^3$ is symmetric about $p$ if and only if $p$ is the gravity center of every plane section of $K$ through $p$* (Blaschke [8]).

**7.4.** *Let $K \subset \mathbb{R}^n$, $n \geq 2$, be a convex body and $p_1$ and $p_2$ be distinct points in $\mathbb{R}^n$. Then $K$ is symmetric about the midpoint $p$ of the segment $[p_1, p_2]$ if and only if every pair of parallel hyperplanes through $p_1$ and $p_2$, respectively, meet $K$ in sections of equal $(n-1)$-dimensional volume* (Rogers [67] for $n = 2$ and Larman and Tamvakis [45] for all $n \geq 2$).

**7.5.** *Let $K \subset \mathbb{R}^n$, $n \geq 2$, be a convex body, with $p \in int K$, and let $m$, $1 \leq m \leq n-1$, be an integer. Then $K$ is centered about $p$ if and only if the $m$-dimensional volume function $x \to V_m(K \cap (x + L_m))$, $x \in \mathbb{R}^n$, attains its maximum at $x = p$ for every choice of the $m$-dimensional subspace $L_m \subset \mathbb{R}^n$* (Makai, Martini, and Ödor [49]).

See also Makai and Martini [48] for an assertion of a similar to 7.5 spirit.

Given a convex body $K \subset \mathbb{R}^n$, with $o \in int K$, the *intersection body IK* of $K$ is defined by the radial function $\rho_{IK}(e) = \mathrm{Vol}_{n-1}(K \cap e^{\perp})$, $e \in \mathbb{S}^{n-1}$:

$$IK = \{te : 0 \leq t \leq \rho_{IK}(e),\ e \in \mathbb{S}^{n-1}\},$$

where $\mathrm{Vol}_{n-1}$ stands for the $(n-1)$-dimensional volume and $e^{\perp}$ denotes the $(n-1)$-dimensional subspace of $\mathbb{R}^n$ orthogonal to $e$ (see Lutwak [46]).

Similarly (see Martini [50]), the *cross-section body CK* of $K$ is defined by the radial function:

$$\rho_{CK}(e) = \max\{\mathrm{Vol}_{n-1}(K \cap (e^{\perp} + \lambda e)) : \lambda \in \mathbb{R}\},\ e \in \mathbb{S}^{n-1}.$$

**7.6.** *Let $K \subset \mathbb{R}^n$, $n \geq 2$, be a convex body, with $o \in int K$. If there is a constant $\delta > 0$ such that $IK = \delta\, CK$, then $K$ is centered about $o$* (Makai and Martini [47]).

We recall that the *Steiner symmetrization* $S_u(K)$ of a convex body $K \subset \mathbb{R}^n$ in the direction of a nonzero vector $u \in \mathbb{R}^n$ is defined to be the convex body symmetric with respect to the $(n-1)$-dimensional subspace $u^{\perp}$ and whose intersection with any line $l$ parallel to $u$ is a segment of the same length as the segment $K \cap l$. It is well-known that all Steiner symmetrizations of a convex body $K \subset \mathbb{R}^n$ also are convex bodies (see, e. g., Bonnesen and Fenchel [13], § 9).

**7.7.** *Let $K \subset \mathbb{R}^n$ be a convex body and $U \subset \mathbb{R}^n$ be an open non-empty set such that for each $u \in U$, the Steiner symmetrization $S_u(K)$ of $K$ is centrally symmetric. Then $K$ itself is centrally symmetric* (Saroglou [68]).

Following a problem of Makai, Martini, and Ödor [49] (formulated by them for the case of convex bodies), Stephen [79] proved the following assertion, where *rbd* stands for the relative boundary of a low-dimensional convex set.

**7.8.** *Let $Q \subset \mathbb{R}^n$ be a polytope, with $o \in int Q$, such that*

$$\mathrm{Vol}_{n-2}(\mathrm{rbd}\,(Q \cap e^{\perp})) = \max \mathrm{Vol}_{n-2}(\mathrm{rbd}\,(Q \cap (\lambda e + e^{\perp})))$$

*for all $e \in \mathbb{S}^{n-1}$ and $\lambda \in \mathbb{R}$. Then $Q$ is centered about $o$.*

## 8 Antipodality and symmetric intersections of translates

Kharazishvili [40] proved that a convex body $K \subset \mathbb{R}^n$ is a parallelotope if and only if there is a real number $\lambda \in (0, 1)$ such that all intersections $K \cap (x + \lambda K)$, $x \in \mathbb{R}^n$, are centrally symmetric. This assertion was generalized in [72,75], as follows.

In a standard way, a subset $F$ of a convex body $K \subset \mathbb{R}^n$ is called an *exposed face* of $K$ provided there is a hyperplane $H$ supporting $K$ such that $F = K \cap H$. If an exposed face $F$ of $K$ consists of a single point (respectively, of a line segment), then it is called an *exposed point* (respectively, an *exposed line segment*).

For any hyperplane $H$ supporting $K$, denote by $H'$ the hyperplane parallel to $H$ and supporting $K$ from the opposite side. In this case, the exposed face $F' = K \cap H'$ of $K$ is called *associate* to the exposed face $F = M \cap H$. He next assertion immediately follows from arguments of the papers [72, 75].

**8.1.** *A convex body $K \subset \mathbb{R}^n$ is centrally symmetric if and only if the following two conditions are satisfied:*

($a$) *every exposed point $a$ of $K$ has an associate exposed point $a'$ such that $(-K) \cap (a' + W)$ is a translate of $K \cap (a + W)$ for a suitable neighborhood $W$ of $o$,*

($b$) *every exposed line segment $[a, b]$ of $K$ has an associate exposed line segment $[a', b']$ which is a translate of $[a, b]$ ($a - b$ and $a' - b'$ have the same direction) and such that, for a suitable neighborhood $W$ of $o$, the sets $K \cap (a' + W)$ and $K \cap (b' + W)$ are symmetric to $K \cap (b + W)$ and $K \cap (a + W)$, respectively.*

The assertion 8.1 provides the main tool for the proof of the following result.

**8.2** ([72, 75])**.** *For a pair of line-free convex solids $K$ and $K'$ in $\mathbb{R}^n$, the following three conditions are equivalent.*

($a$) *All intersections $K \cap (x + K')$, $x \in \mathbb{R}^n$, are centrally symmetric.*

($b$) *All $n$-dimensional intersections $K \cap (x + K')$, $x \in \mathbb{R}^n$, are centrally symmetric.*

($c$) *$K$ and $K'$ are represented as direct vector sums $K = P \oplus Q$ and $K' = P' \oplus Q'$ such that conditions ($i$) and ($ii$) below are satisfied:*

  ($i$) *$P$ is a line-free closed convex set of some dimension $m$, $0 \le m \le n$, and $P' = z - P$ for a suitable point $z \in \mathbb{R}^n$,*

  ($ii$) *$Q$ and $Q'$ are compatible generalized parallelotopes, both of dimension $n - m$.*

Here a *generalized parallelotope* is a direct vector sum of finitely many line segments or closed halflines. Two generalized parallelotopes $Q$ and $Q'$ of the same dimension $m$ are called *isothetic* provided they can be represented as direct vector sums

$$Q = Q_1 + \cdots + Q_m, \quad Q' = Q'_1 + \cdots + Q'_m,$$

where $Q_i$ and $Q'_i$ are parallel one-dimensional convex sets, i.e., each of $Q_i, Q'_i$ is either a line segment or a closed halfline, $i = 1, \ldots, m$. Finally, isothetic generalized parallelotopes $Q$ and $Q'$ are called *compatible*, if, for each $i = 1, \ldots, m$, either at least one of $Q_i, Q'_i$ is a segment, or both $Q_i, Q'_i$ are closed halflines with opposite directions.

The next assertion is proved by Minkowski [57, p. 118] for $n = 3$ and generalized by Bonnesen and Fenchel [13, p. 133] for all $n \geq 3$.

**8.3** ([13, 57])**.** *If the boundary of an $n$-polytope $Q \subset \mathbb{R}^n$, $n \geq 3$, consists of pairwise parallel $(n-1)$-dimensional faces of equal areas, then $Q$ is centrally symmetric.*

Given an $n$-polytope $Q \subset \mathbb{R}^n$ and a nonzero vector $e \in \mathbb{R}^n$, we denote by $H_e$ and $H'_e$ the parallel hyperplanes both supporting $Q$ and having outward normals $e$ and $-e$, respectively. Furthermore, let $F_e = H_e \cap Q$ and $F'_e = H'_e \cap Q$. The following result is proved by Bolker [11], who also posed a list of related open problems.

**8.4** ([11])**.** *A polytope $Q \subset \mathbb{R}^n$, $n \geq 3$, is centrally symmetric if and only if any of the following conditions $(a)$–$(c)$ holds.*

$(a)$ *For all $e \in \mathbb{R}^n$, the vector sum $F_e + F'_e$ is centrally symmetric.*

$(b)$ *If $\dim F'_e \leq \dim F_e = 2$, then $F_e + F'_e$ is centrally symmetric.*

$(c)$ *If $\dim F'_e \leq \dim F_e = 1$, then $F_e$ and $F'_e$ are line segments of the same length.*

## 9 Tiling and symmetric polytopes

We recall that an $n$-polytope $Q \subset \mathbb{R}^n$ is called a *parallelohedron* provided the whole space $\mathbb{R}^n$ can be tiled by translates of $Q$. Their study was originated by Fedorov [28, Section IV], who described parallelohedra in dimensions two and three.

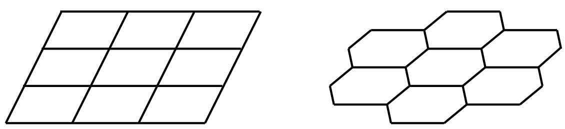

Figure 1. Parallelogons in $\mathbb{R}^2$.

In the plane, there are two types of such parallelogons: parallelograms and centrally symmetric hexagons (see Figure 1). In the 3-space, there are five types of such parallelohedra: parallelepipeds, hexagonal prisms, rhombic dodecahedrons, elongated dodecahedrons, and truncated octahedrons (see Figure 2).

Fedorov assumed without proof that every parallelohedron in $\mathbb{R}^3$ is centrally symmetric. This fact was established by Minkowski [57]. Delaunay [23] and Alexandrov [5] observed that Minkowski's argument can be routinely extended to the case of higher dimensions and formulated the following necessary conditions for an $n$-polytope $Q \subset \mathbb{R}^n$ to be a parallelohedron:

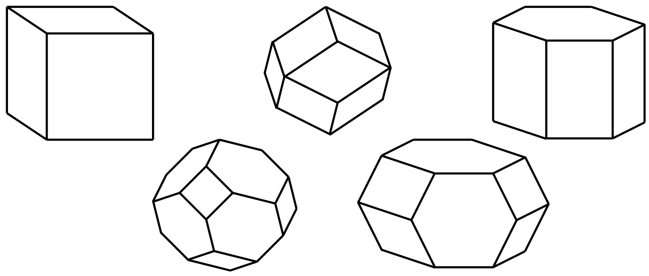

Figure 2. Parallelohedra in $\mathbb{R}^3$.

(P1) $Q$ is centrally symmetric,

(P2) every $(n-1)$-dimensional face of $Q$ is centrally symmetric,

(P3) every parallel projection of $Q$ on a 2-dimensional plane along an $(n-2)$-dimensional face of $Q$ is either a parallelogram or a centrally symmetric hexagon.

Based on these observations, Venkov [82, Theorem 5] (later also McMullen [54] and Dolbilin [25]) proved the following assertion.

**9.1** ([82])**.** *An $n$-polytope $Q \subset \mathbb{R}^n$, $n \geq 3$, is a parallelohedron if and only if it satisfies the above conditions* (P1)–(P3).

In fact, condition (P1) can be omitted in 9.1, because (P2) implies (P1) for the case of any $n$-polytope in $\mathbb{R}^n$. This result was proved by Alexandrov [4] for the case $n = 3$ (see Burckhardt [15] for another proof, also [22]). A more general statement was obtained by Shephard [71] (see also McMullen [52] for another proof).

**9.2** ([71])**.** *If every $j$-face of an $n$-polytope $Q \subset \mathbb{R}^n$, $n \geq 3$, is centrally symmetric, where $j$ is a given integer satisfying $2 \leq j \leq n-2$, then the $k$-faces of $Q$ are also centrally symmetric for all $k$ such that $j \leq k \leq n$; in particular, $Q$ is itself centrally symmetric.*

McMullen [51] further refined 9.2 (see also Dolbilin and Kozachok [26]).

**9.3** ([51])**.** *Let $Q \subset \mathbb{R}^n$ be an $n$-polytope, $n \geq 4$, such that for some integer $j$ satisfying $2 \leq j \leq n-2$, all $j$-faces of $Q$ are centrally symmetric. Then all faces of $Q$ of each dimension are centrally symmetric.*

The inequality $j \leq n-2$ in 9.2 and 9.3 cannot be replaced the weaker condition $j \leq n-1$ (see [51]). Indeed, a routine verification shows that all $(n-1)$-dimensional faces of the $n$-polytope

$$Q = \{(x_1, \dots, x_n) \in \mathbb{R}^n : |x_i| \leq 2,\ i = 1, \dots, n,\ \sum_{i=1}^{n} |x_i| \leq n\}$$

are centrally symmetric. On the other hand, the $(n-2)$-face $F_1 \cap F_2$ of $Q$ is not centrally symmetric, where

$$F_1 = \{(x_1, \ldots, x_n) \in Q : x_n = 2\}, \quad F_2 = \{(x_1, \ldots, x_n) \in Q : \sum_{i=1}^{n} x_i = n\}.$$

Another example of a similar spirit is given in [26]: if $Q_4$ is the regular 4-polytope with 24 three-dimensional faces, then all these faces are octahedra, while its 2-dimensional faces are triangles.

## 10 Various results

Chakerian and Klamkin [21] established a necessary and sufficient condition under which a compact set (not necessarily convex) is centrally symmetric.

**10.1** ([21])**.** *A compact set $X \subset \mathbb{R}^n$ is centrally symmetric if and only if for every three-point set $F = \{a, b, c\}$ contained in $X$ there exists a point $v \in \mathbb{R}^n$, depending on $F$, such that the set $F' = 2v - F$ also is contained in $X$.*

Boltyanski and Jerónimo Castro [12] proved a dual assertion of similar spirit.

**10.2** ([12])**.** *A compact convex set $K \subset \mathbb{R}^n$ is centrally symmetric if and only if for every $n$-dimensional simplex $T \subset \mathbb{R}^n$ that contains $K$ there exists a point $v$, depending on $T$, such that the simplex $T' = 2v - T$ also contains $K$.*

Two more results of Minkowski [57, p. 119] and Hadwiger [33] (for $n = 3$) and Hertel [37] (for $n = 4$) are related to partitions of symmetric polytopes.

**10.3** ([57])**.** *If a convex body $K \subset \mathbb{R}^n$, $n \geq 2$, can be expressed as the union of finitely many pairwise non-overlapping centrally symmetric polytopes, then $K$ is a centrally symmetric polytope itself.*

**10.4** ([33, 37])**.** *An $n$-polytope in $\mathbb{R}^n$, $n = 3, 4$, is equivalent to a cube, by dissection and translation, if and only if it is centrally symmetric and has centrally symmetric $(n-1)$-dimensional faces.*

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