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Some Characterizations of Generalized Middle Bol Loops

Benard Osoba, Tèmítọ́pẹ́ Gbọ́láhàn Jaiyéọlá, Abednigo Orobosa Isere and Yakub Tunde Oyebo

Abstract. In this work, the two identities that characterize middle Bol loop (MBL) were separately generalized to define generalized middle Bol loops (GMBL1 and GMBL2) and their relationship was unraveled for possible equivalence. Thereafter, autotopic characterizations of identities GMBL1 and GMBL2 in a loop were carried out. Banking on these, the Bryant-Schneider group characterization of generalized middle Bol loop was carried out. Furthermore, an autotopic-split and right (left) pseudo-automorphic characterization of generalized middle Bol loop were carried out. Furthermore, a right (left) pseudo-automorphic characterization of the isotopy-isomorphy of generalized middle Bol loop was also carried out. This is a sort of 'G-generalized middle Bol loop' characterization based on pseudo-automorphic companionship of elements of GMBL as already established for 'G-loops'.

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1 Introduction

Let G be a set that is non-empty. Define the binary operation " $\odot$ " on G. The pair $(G, \odot)$ is known as a groupoid or magma if $x \odot y \in G$ for every $x, y \in G$. A quasigroup $(G, \odot)$ is defined, if the equations $v \odot x = u$ and $y \odot v = u$ have unique solutions $x, y \in G$ for all $v, u \in G$.

Let $(G, \odot)$ be a quasigroup and let there exist a unique element $i \in G$ called the identity element such that for all $x \in G, x \odot i = i \odot x = x$, then $(G, \odot)$ is called a loop. At times, we shall write xy instead of $x \odot y$ and stipulate that " $\odot$ " has lower priority than juxtaposition among factors to be multiplied.

Let $(G, \odot)$ be a groupoid and v be a fixed element in G, then the left and right translations L_v and R_v of v are respectively defined by $xL_v = v \odot x$ and $xR_v = x \odot v$ for all $x \in G$.

Sometimes we shall use vu instead of $v \odot u$ and say that juxtaposition of the factors to be multiplied has greater significance than " $\odot$ ".

Now, it is clear that a groupoid is a quasigroup if its left and right translations are permutations. A quasigroup has inverse mappings L_v^{-1} and R_v^{-1} because its left and right translations are bijective.

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Let

$$v\backslash u = uL_v^{-1} = vM_u \qquad \text{and} \qquad v/u = vR_u^{-1} = uM_v^{-1}$$

and note that

$$v\backslash u = w \Longleftrightarrow a \odot w = u \qquad \text{and} \qquad v/u = w \Longleftrightarrow w \odot u = v.$$

Thus, for any quasigroup $(G, \odot)$, we have two new binary operations: right division $(/)$ and left division $(\backslash)$, where M_v is the middle translation for any fixed $v \in G$. Consequently, $(G, \backslash)$ and $(G, /)$ are also quasigroups. Using the operations $(\backslash)$ and $(/)$, the definition of a loop can be restated as follows.

Definition 1. A *loop* $(G, \odot, /, \backslash, i)$ is a set G together with three binary operations $(\odot)$, $(/)$, $(\backslash)$ and one nullary operation i such that

(i) $a \odot (a\backslash b) = b$, $(b/a) \odot a = b$ for all $a, b \in G$,

(ii) $a\backslash(a \odot b) = b$, $(b \odot a)/a = b$ for all $a, b \in G$ and

(iii) $a\backslash a = i = b/b$ or $i \odot a = a = a \odot i$ for all $a, b \in G$.

For further results on some varieties of loops and their characterizations, see [9–15,31]. In a loop $(G, \odot)$ with identity element i, the *left inverse element* of $x \in G$ is the element $xJ_\lambda = x^\lambda \in G$ such that

$$x^\lambda \odot x = i$$

while the *right inverse element* of $x \in G$ is the element $xJ_\rho = x^\rho \in G$ such that

$$x \odot x^\rho = i.$$

If $J_\rho = J_\lambda$, then we shall simply write J and call it the inverse mapping. Thus, $x^\lambda = x^\rho = x^{-1}$ will then be called the inverse element of $x \in G$.

Definition 2. A loop $(G, \odot)$ is called a middle Bol loop (MBL) if

$$(a/b)(c\backslash a) = (a/(cb))a \text{ or } (a/b)(c\backslash a) = a((cb)\backslash a) \tag{1}$$

for all $a, b, c \in G$.

For more on quasigroups and loops, check [17,38,40].

Middle Bol loops (MBL) were investigated at first by V. D. Belousov [4], who established identities (1) characterising loops which satisfy the anti-automorphic inverse property. After Belousov's excellent characterisation and the establishment of the groundwork of the structure of middle Bol loop, Gwaramija provided an isostrophic connection between the right (left) and middle Bol loops [8].

Grecu [7] showed that right multiplication group of a middle Bol loop coincides with the left multiplication group of the corresponding right Bol loop. Thereafter, Syrbu [41,42] examined MBL in respect to the universality of the flexible law, which led to the reappearance of an MBL in the literature in 1994 and 1996.

Kuznetsov [26], while investigating gyrogroups in 2003, established several algebraic properties of the MBL and developed a method for creating an MBL from a gyrogroup, while the connections of right Bol loops with its MBL were established in 2010 by Syrbu [43]. It was revealed that two MBLs are isomorphic if and only if the corresponding left (right) Bol loops are isomorphic. Drapal and Shcherbacov [5] in 2010, found a new identity of MBL.

Jaiyéọlá et al. [20] introduced the holomorphic structure of an MBL in 2017 and established that the holomorph of a commutative loop is a commutative MBL if and only if the loop is an MBL and its automorphism group is abelian. Generalized Bol loops were examined by Adeniran et al. [1], Jaiyéọlá and Popoola [24].

Osoba et al. [28, 29] investigated further the multiplication group of middle Bol loop in relation to left Bol loop and the relationship of multiplication groups and isostrophic quasigroups respectively while Jaiyéọlá [22, 23] studied second Smarandache Bol loops. The Smarandache nuclei of second Smarandache Bol loops were further studied by Osoba [30] while core of second Smarandache Bol loops was investigated in [33].

In 2015, Jaiyéọlá et al. [19] presented new algebraic properties of MBL and further showed the necessary and sufficient conditions for the existence of inverse property, alternative property and flexible property through a bi-variate mapping of an MBL. The authors [21] in 2021 announced additional properties of MBL, while the new algebraic connections between right and middle Bol loops and their cores were unveiled by Osoba and Jaiyéọlá [32] in 2022. More results on the algebraic properties of middle Bol loops using its parastrophes were presented by Oyebo and Osoba in [35]. The paper revealed some of the algebraic properties the parastrophic structures of MBL share with its underlying structure. The connections between middle Bol loop and right Bol loop with their crypto-automorphism features were unveiled in [34]. In [16,18,27], the Bryant-Schneider groups of some varieties of loops were studied, while in [25], the Bryant-Schneider group of middle Bol loop with some of the isostrophy-group invariance results were studied. It was further shown that the subgroups of the Bryant-Schneider group of an MBL were isomorphic to the automorphism and pseudoaumorphism groups of its corresponding right (left) Bol loop.

A generalized middle Bol loop GMBL characterized by

$$\text{(GMBL1)} \qquad (x/y) \odot (z^\alpha \backslash x^\alpha) = x \odot ((z^\alpha \odot y)\backslash x^\alpha) \qquad (2)$$

was first introduced in [2], as a consequence of a generalized Moufang loop with universal $\alpha-$elastic property where the map $\alpha : G \to G$ is a homomorphism. Thus, if $\alpha : x \mapsto x$, then the identity of generalized middle Bol loop reduces to the identity of middle Bol loop. The authors in [3] presented the basic algebraic properties of generalized middle Bol loop. Work on generalized middle Bol loop was further revealed by Osoba et al. [36].

In this current study, we shall generalize the equivalent identities of middle Bol loops in Definition 2 to identities (2) and (3) where α is a self-map on the loop.

$$\text{(GMBL2)} \qquad (x/y) \odot (z^{\alpha} \backslash x^{\alpha}) = (x/(z^{\alpha} \odot y)) \odot x^{\alpha} \tag{3}$$

They shall be called generalized middle Bol loop identity 1 (GMBL1) and generalized middle Bol loop identity 2 (GMBL2) respectively.

In this paper, the Bryant-Schneider group and pseudo-automorphism of a generalized middle Bol loop (GMBL) are studied. A subgroup of a Bryant-Schneider group of a GMBL is found. Some elements of a Bryant-Schneider group of a GMBL are shown to be crypto-automorphisms. The necessary and sufficient condition(s) for a LP-isotope of a GMBL $(G, \odot)$ to be isomorphic to $(G, \circ)$ is shown. Some further characterizations of right (left) pseudo-automorphism of a GMBL are also presented.

2 Preliminaries

We now introduce some basic notions and concepts in quasigroup and loop theory which are of importance to this work.

Definition 3. Let G be a non-empty set, the set of all permutations on G forms a group $SYM(G)$ called the symmetric group of G. Let $(G, \odot)$ be a loop and let $A, B, C \in SYM(G)$. If

$$xA \odot yB = (x \odot y)C \ \forall \ x, y \in G,$$

then the triple (A, B, C) is called an autotopism and such triples form a group $AUT(G, \odot)$ called the autotopism group of $(G, \odot)$. If $A = B = C$, then A is called an automorphism of $(G, \odot)$ whose set forms a group $AUM(G, \odot)$ called the automorphism group of $(G, \odot)$.

Definition 4. Let $(G, \odot)$ be a loop.

1. $\phi \in SYM(G)$ is called a left pseudo-automorphism with companion $a \in G$ if $(\phi L_a, \phi, \phi L_a) \in AUT(G, \odot)$. The set of such maps forms a group.

2. $\phi \in SYM(G)$ is called a right pseudo-automorphism with companion $a \in G$ if $(\phi, \phi R_a, \phi R_a) \in AUT(G, \odot)$. The set of such maps forms a group.

3. $\phi \in SYM(G)$ is called a crypto-automorphism with companions $a, b \in G$ if $(R_a\phi, L_b\phi, \phi) \in AUT(G, \odot)$. The set of such maps forms a group. See [34].

4. A mapping $\phi \in SYM(G)$ such that $(\phi R_g^{-1}, \phi L_f^{-1}, \phi) \in AUT(G, \odot)$ for some $f, g \in G$ is called a Bryant-Schneider map of $(G, \odot)$. The set of such maps forms a group called the Bryant-Schneider group $BS(G, \odot)$ of $(G, \odot)$. See [37].

From Definition 4, it is clearly seen that

$$(\phi R_g^{-1}, \phi L_f^{-1}, \phi) = (\phi, \phi, \phi)(R_g^{-1}, L_f^{-1}, I),$$

which implies that ϕ is an isomorphism of $(G, \odot)$ onto some f, g-isotope of it.

Theorem 1. *[37] Let the set* $BS(G, \odot) = \{\phi \in SYM(G) : \exists f, g \in G \ni (\phi R_g^{-1}, \phi L_f^{-1}, \phi) \in AUT(G, \odot)\}$*, then* $BS(G, \odot) \leq SYM(G)$.

Theorem 1 is associated with Theorem 2.

Theorem 2. *(Pflugfelder [38]) Let* $(G, \odot)$ *and* $(H, \circ)$ *be two isotopic loops. For some* $f, g \in G$*, there exists an* f, g*-principal isotope* $(G, *)$ *of* $(G, \odot)$ *such that* $(H, \circ) \cong (G, *)$.

Definition 5. Let $(G, \odot)$ be a quasigroup with fixed elements $a, b \in G$. The isotope of the form (R_a^{-1}, L_b^{-1}, I) is called LP-isotope.

Definition 6. A quasigroup $(G, \odot)$ is said to have the

1. left inverse property (LIP) if there exists a mapping $J_\lambda : x \mapsto x^\lambda$ such that $x^\lambda \odot (x \odot y) = y$ for all $x, y \in G$,
2. right inverse property (RIP) if there exists a mapping $J_\rho : x \mapsto x^\rho$ such that $(y \odot x) \odot x^\rho = y$ for all $x, y \in G$,
3. inverse property (IP) if it has both the LIP and RIP,
4. flexible or elastic property if $(x \odot y) \odot x = x \odot (y \odot x)$ holds for all $x, y \in G$,
5. $\alpha-$elastic property if $(x \odot y) \odot x^\alpha = x \odot (y \odot x^\alpha)$ holds for all $x, y \in G$.

Definition 7. Let $(G, \odot)$ be a loop.

1. $\mathbf{N}_l = \{v \in G : (v \odot x) \odot y = v \odot (x \odot y) \ \forall \ x, y \in G\}$ is called the left nucleus of G .
2. $\mathbf{N}_r = \{v \in G : y \odot (x \odot v) = (y \odot x) \odot v \ \forall \ x, y \in G\}$ is called the right nucleus of G.
3. $\mathbf{N}_m = \{v \in G : (y \odot v) \odot x = y \odot (v \odot x) \ \forall \ x, y \in G\}$ is called the middle nucleus of G .

Definition 8. A loop $(G, \odot)$ is said to be

1. commutative loop if $R_x = L_x$ for all $x \in G$,
2. an anti-automorphic inverse property loop (AAIPL) if $(xy)^\rho = y^\rho x^\rho$ for all $x, y \in G$.

3 Main Results

3.1 Autotopic characterization of generalized middle Bol loop

First, we need to unravel the relationship between identities (2) and (3) and their possible equivalence.

Lemma 1. *Let $(G, \odot)$ be a loop. Let x, y, z be arbitrary elements in G.*

1. *If $(G, \odot)$ obeys identity GMBL1 such that $\alpha : i \mapsto i$, then*
   (a) $(x/y) \odot x^\alpha = x \odot (y \backslash x^\alpha)$,
   (b) $y^\lambda \odot (z^\alpha)^\rho = (z^\alpha \odot y)^\rho$,
   (c) $y^\lambda = y^\rho$.
2. *If $(G, \odot)$ obeys identity GMBL2 such that $\alpha : i \mapsto i$, then*
   (a) $x \odot (z^\alpha \backslash x^\alpha) = (x/z^\alpha) \odot x^\alpha$,
   (b) $y^\lambda \odot (z^\alpha)^\rho = (z^\alpha \odot y)^\lambda$,
   (c) $(z^\alpha)^\lambda = (z^\alpha)^\rho$.
3. *If $(G, \odot)$ obeys identity GMBL1 such that α is bijective and $\alpha : i \mapsto i$, then*
   (a) $(x/y) \odot x^\alpha = x \odot (y \backslash x^\alpha)$,
   (b) $y^\lambda \odot z^\rho = (z \odot y)^\rho$,
   (c) $y^\lambda = y^\rho$.
4. *If $(G, \odot)$ obeys identity GMBL2 such that α is bijective and $\alpha : i \mapsto i$, then*
   (a) $x \odot (z \backslash x^\alpha) = (x/z) \odot x^\alpha$,
   (b) $y^\lambda \odot z^\rho = (z \odot y)^\lambda$,
   (c) $z^\lambda = z^\rho$.
5. *Let $\alpha : i \mapsto i$. Then $(G, \odot)$ obeys identity GMBL1 if and only if $(G, \odot)$ obeys identity GMBL2 and $(x/y) \odot x^\alpha = x \odot (y \backslash x^\alpha)$.*
6. *Let α be bijective such that $\alpha : i \mapsto i$. Then $(G, \odot)$ obeys identity GMBL1 if and only if $(G, \odot)$ obeys identity GMBL2.*

Proof. 1. Assume that $(G, \odot)$ obeys the identity GMBL1 such that $\alpha : i \mapsto i$.

(a) Put $z = i$ in (2) to get $(x/y) \odot (i^\alpha \backslash x^\alpha) = x \odot ((i^\alpha \odot y) \backslash x^\alpha)$ which gives $(x/y) \odot x^\alpha = x \odot (y \backslash x^\alpha)$.
(b) In (2), put $x = i$ to get $(i/y) \odot (z^\alpha \backslash i^\alpha) = i \odot ((z^\alpha \odot y) \backslash i^\alpha)$ to get $y^\lambda \odot (z^\alpha)^\rho = (z^\alpha \odot y)^\rho$.
(c) In (b), put $z = i$ to get $y^\lambda = y^\rho$.

2. Assume that $(G,\odot)$ obeys the identity GMBL2 such that $\alpha : i \mapsto i$. Do similarly as done in item 1 to prove (a), (b) and (c).

3. Assume that $(G,\odot)$ obeys identity GMBL1 such that α is bijective and $\alpha : i \mapsto i$. Then the proofs of (a), (b) and (c) follow up from 1.

4. Assume that $(G,\odot)$ obeys identity GMBL2 such that α is bijective and $\alpha : i \mapsto i$. Then the proofs of (a), (b) and (c) follow up from 2.

5. Let $\alpha : i \mapsto i$. If $(G,\odot)$ obeys identity GMBL1, then it obeys identity GMBL2 because it satisfies $(x/y) \odot x^\alpha = x \odot (y\backslash x^\alpha)$ by 1. The converse follows by reversing the process.

6. This follows from item 5.

□

We now look at the autotopic characterization of identities GMBL1 and GMBL2 in a loop.

Lemma 2. *Let $(G,\odot)$ be a loop. Let x, y, z be arbitrary elements in G.*

1. *Let α be an onto itself map on G such that $\alpha : i \mapsto i$.*
   (a) *$(G,\odot)$ obeys identity GMBL1 if and only if $(J_\rho M_x^{-1}, J_\lambda M_{x^\alpha}, J_\lambda M_{x^\alpha} L_x) \in AUT(G,\odot)$ and $y^\lambda \odot (z^\alpha)^\rho = (z^\alpha \odot y)^\rho$.*
   (b) *$(G,\odot)$ obeys identity GMBL2 if and only if $(J_\rho M_x^{-1}, J_\lambda M_{x^\alpha}, J_\rho M_{x^\alpha}^{-1} R_x) \in AUT(G,\odot)$ and $y^\lambda \odot (z^\alpha)^\rho = (z^\alpha \odot y)^\lambda$.*

2. *Let α be bijective such that $\alpha : i \mapsto i$. The following are equivalent.*
   (a) *$(G,\odot)$ obeys identity GMBL1.*
   (b) *$(G,\odot)$ obeys identity GMBL2.*
   (c) *$(J_\rho M_x^{-1}, J_\lambda M_{x^\alpha}, J_\lambda M_{x^\alpha} L_x) \in AUT(G,\odot)$ and $y^\lambda \odot z^\rho = (z \odot y)^\rho$.*
   (d) *$(J_\rho M_x^{-1}, J_\lambda M_{x^\alpha}, J_\rho M_{x^\alpha}^{-1} R_x) \in AUT(G,\odot)$ and $y^\lambda \odot z^\rho = (z \odot y)^\lambda$.*

Proof. We shall be employing Lemma 1.

1. (a) $(G,\odot)$ obeys identity GMBL1 if and only if

$$\begin{aligned} x(y^\alpha z\backslash x^\alpha) = (x/z)(y^\alpha\backslash x^\alpha) &\Leftrightarrow zM_x^{-1} \odot y^\alpha M_{x^\alpha} = (y^\alpha \odot z)J_\rho J_\lambda M_{x^\alpha} L_x \\ &\Leftrightarrow zM_x^{-1} \odot y^\alpha M_{x^\alpha} = (zJ_\lambda \odot y^\alpha J_\rho)J_\lambda M_{x^\alpha} L_x \\ &\Leftrightarrow zJ_\rho M_x^{-1} \odot y' J_\lambda M_{x^\alpha} = (z \odot y')J_\lambda M_{x^\alpha} L_x \end{aligned}$$

   with $z \mapsto zJ_\rho$ and $y^\alpha J_\rho \mapsto y'$. Thus, $(J_\rho M_x^{-1}, J_\lambda M_{x^\alpha}, J_\lambda M_{x^\alpha} L_x) \in AUT(G,\odot)$.

   (b) Use similar steps of the proof of (a) for identity GMBL2.

2. This is a follow-up to 1.

□

Henceforth, we shall assume that in a loop that obeys identities GMBL1 or GMBL2, α is bijective such that $\alpha : i \mapsto i$. Thus, we shall simply refer to such loop as a generalized middle Bol loop (GMBL) based on Lemma 2(2).

Lemma 3. *Let (ρ, ψ, ϕ) be an autotopism of a GMBL $(G, \odot)$. Then $(J\psi J, J\rho J, J\phi J)$ is also an autotopism of $(G, \odot)$.*

Proof. Let $(G, \odot)$ be a GMBL and $(\rho, \psi, \phi) \in AUT(G, \odot)$, then for all $x, y \in G$, we have

$$x\rho \odot y\psi = (x \odot y)\phi \Longrightarrow [x\rho \odot y\psi]J = (x \odot y)\phi J \Longrightarrow [(y\psi)J \odot (x\rho)J] = (x \odot y)\phi J.$$

Set $y \mapsto yJ$ and $x \mapsto xJ$, to get

$$yJ\psi J \odot xJ\rho J = [(xJ \odot yJ)\phi]J \implies yJ\psi J \odot xJ\rho J = [(y \odot x)J\phi]J.$$

Thus, $(J\psi J, J\rho J, J\phi J) \in AUT(G, \odot)$. □

3.2 Bryant-Schneider group characterization of generalized middle Bol loop

Using the results in Section 3.1, we shall carry out the Bryant-Schneider group characterization of generalized middle Bol loop.

Theorem 3. *Let $(G, \odot)$ be a generalized middle Bol loop and let $\phi \in BS(G, \odot)$ be such that $\phi : i \mapsto i$. For some $f, g \in G$:*

1. $R_g^{-1} = M_{f^\alpha}^{-1} R_f L_{f/f^\rho}^{-1} M_{f^\alpha}^{-1}$ *and* $L_f^{-1} = M_{g^\alpha}^{-1} R_g R_{g^\lambda \backslash g^\alpha}^{-1} M_g$.
2. $\phi = \phi(f, g) \equiv \phi(f, f^\rho)$ *and* $\phi \equiv \phi(f, g) = \phi(g^\lambda, g)$.

Proof. Suppose that $(G, \odot)$ is a GMBL, then $\beta = (JM_x^{-1}, JM_{x^\alpha}, JM_{x^\alpha}^{-1} R_x)$ is an autotopism of $(G, \odot)$ for all $x \in G$. Since $\gamma = (\phi R_g^{-1}, \phi L_f^{-1}, \phi) \in AUT(G, \odot)$ for some $f, g \in G$, then by Lemma 3,

$$\gamma = (J\phi L_f^{-1} J, J\phi R_g^{-1} J, J\phi J) \in AUT(G, \odot) \text{ for some } f, g \in G. \tag{4}$$

The product

$$\gamma\beta = (J\phi L_f^{-1} JJM_x^{-1}, J\phi R_g^{-1} JJM_{x^\alpha}, J\phi JJM_{x^\alpha}^{-1} R_x)$$

$$= (J\phi L_f^{-1} M_x^{-1}, J\phi R_g^{-1} M_{x^\alpha}, J\phi M_{x^\alpha}^{-1} R_x) \tag{5}$$

is also autotopism of $(G, \odot)$. Writing this in equivalent form, for all $a, b \in G$, we get

$$aJ\phi L_f^{-1} M_x^{-1} \odot bJ\phi R_g^{-1} M_{x^\alpha} = (a \odot b)J\phi M_{x^\alpha}^{-1} R_x \Longrightarrow$$

$$a^{-1}\phi L_f^{-1}M_x^{-1}\odot b^{-1}\phi R_g^{-1}M_{x^\alpha}=(a\odot b)^{-1}\phi M_{x^\alpha}^{-1}R_x\Longrightarrow$$

$$x/(f\backslash(a^{-1})\phi)\odot((b^{-1}\phi)/g)\backslash x^\alpha=(x^\alpha/((a\odot b)^{-1})\phi)\odot x. \tag{6}$$

Set $a=i$, the identity element in G and $x=f$ in (6), we have

$$\begin{aligned}[f/(f\backslash i)]\odot(b^{-1}\phi)/g)\backslash f^\alpha &= (f^\alpha/(b^{-1}\phi)f\\ \Longrightarrow f/f^\rho\odot b^{-1}\phi R_g^{-1}M_{f^\alpha} &= b^{-1}\phi M_{f^\alpha}^{-1}R_f\\ \Longrightarrow R_g^{-1}M_{f^\alpha}L_{f/f^\rho} &= M_{f^\alpha}^{-1}R_f\\ \Longrightarrow R_g^{-1} &= M_{f^\alpha}^{-1}R_fL_{f/f^\rho}^{-1}M_{f^\alpha}^{-1}\end{aligned}$$

for some $f\in G$.

For any $t\in G$, $tR_g^{-1}=tM_{f^\alpha}^{-1}R_fL_{f/f^\rho}^{-1}M_{f^\alpha}^{-1}\Longrightarrow t/g=f^\alpha/\Big[(f/f^\rho)\backslash\big((f^\alpha/t)f\big)\Big]$. Put $t=f^\alpha$ in the last equality to get $f^\alpha/g=f^\alpha/\big[(f/f^\rho)\backslash f\big]=f^\alpha/f^\rho\Longrightarrow g=f^\rho$. Thus, $\phi=\phi(f,g)\equiv\phi(f,f^\rho)$. Taking the same steps by setting $b=i$ and $x=g$ in (6), we have

$$\begin{aligned}g/(f\backslash(a^{-1})\phi)\odot(i/g)\backslash g^\alpha &= (g^\alpha/a^{-1}\phi)\odot g\\ a^{-1}\phi L_f^{-1}M_g^{-1}R_{g^\lambda\backslash g^\alpha} &= a^{-1}\phi M_{g^\alpha}^{-1}R_g\\ L_f^{-1}M_g^{-1}R_{g^\lambda\backslash g^\alpha} &= M_{g^\alpha}^{-1}R_g\\ L_f^{-1} &= M_{g^\alpha}^{-1}R_gR_{g^\lambda\backslash g^\alpha}^{-1}M_g.\end{aligned}$$

So, for any $t\in G$, we have
$tL_f^{-1}=tM_{g^\alpha}^{-1}R_gR_{g^\lambda\backslash g^\alpha}^{-1}M_g\Longrightarrow f\backslash t=\Big[\big((g^\alpha\backslash t)g\big)/(g^\lambda\backslash g^\alpha)\Big]\backslash g$. Put $t=g^\alpha$, we have $f\backslash g^\alpha=\big[g/(g^\lambda\backslash g^\alpha)\big]\backslash g=g^\lambda\backslash g^\alpha$. That is, $f\backslash g^\alpha=g^\lambda\backslash g^\alpha\Longrightarrow f=g^\lambda$. Thus, $\phi\equiv\phi(f,g)=\phi(g^\lambda,g)$. □

Corollary 1. *Let $(G,\odot)$ be a GMBL. Any $\phi\in BS(G,\odot)$ such that $\phi:i\mapsto i$ induces $\Psi=J\phi M_g^{-1}R_g\in SYM(G)$ for some $g\in G$ and the following hold:*

1. *$\Psi\in BS(G,\odot)$.*
2. *Ψ^{-1} is a crypto-automorphism with companions $g^\lambda\backslash g^\alpha$ and g/g^ρ.*

Proof. Substituting $R_g^{-1}=M_{f^\alpha}^{-1}R_fL_{f/f^\rho}^{-1}M_{f^\alpha}^{-1}$ and $L_f^{-1}=M_{g^\alpha}^{-1}R_gR_{g^\lambda\backslash g^\alpha}^{-1}M_g$ in (5), we get $(J\phi M_{g^\alpha}^{-1}R_gR_{g^\lambda\backslash g^\alpha}^{-1}M_gM_x^{-1},J\phi M_{f^\alpha}^{-1}R_fL_{f/f^\rho}^{-1}M_{f^\alpha}^{-1}M_{x^\alpha},J\phi M_{x^\alpha}^{-1}R_x)$ which is an autotopism of $(G,\odot)$.

Put $x=g$ to get

$$(J\phi M_{g^\alpha}^{-1}R_gR_{g^\lambda\backslash g^\alpha}^{-1},J\phi M_{f^\alpha}^{-1}R_fL_{f/f^\rho}^{-1}M_{f^\alpha}^{-1}M_{g^\alpha},J\phi M_{g^\alpha}^{-1}R_g)$$

which is an autotopism of $(G,\odot)$, for some $g\in G$. Putting $f=g$ gives

$$(J\phi M_{g^\alpha}^{-1}R_gR_{g^\lambda\backslash g^\alpha}^{-1},J\phi M_{g^\alpha}^{-1}R_gL_{g/g^\rho}^{-1},J\phi M_{g^\alpha}^{-1}R_g)\in AUT(G,\odot).$$

Letting $\Psi = J\phi M_g^{-1}R_g$, gives $(\Psi R^{-1}_{g^\lambda\backslash g^\alpha}, \Psi L^{-1}_{g/g^\rho}, \Psi)$ is also autotopism of $(G,\odot)$. Thus, $(\Psi R^{-1}_{g^\lambda\backslash g^\alpha}, \Psi L^{-1}_{g/g^\rho}, \Psi)^{-1} = (R_{g^\lambda\backslash g^\alpha}\Psi^{-1}, L_{g/g^\rho}\Psi^{-1}, \Psi^{-1}) \in AUT(G,\odot)$. □

Theorem 4. *Let $(G,\odot)$ be a GMBL. Then*

$$BS'(G,\odot) = \Big\{\phi \in BS(G,\odot) \mid \phi : i \mapsto i \text{ and } (a\phi)^{-1} = \big(a^{-1}\big)\phi\Big\}$$
$$= \Big\{\phi \in SYM(G)|\exists\ f \in G\ \ni\ \big(\phi R^{-1}_{f^\rho}, \phi L^{-1}_f, \phi\big) \in AUT(G),\ i\phi = i \text{ and}$$
$$(a\phi)^{-1} = (a^{-1})\phi\ \forall\ a \in G\Big\} = \Big\{\phi \in SYM(G)|\exists\ g \in G\ \ni\ \big(\phi R^{-1}_g, \phi L^{-1}_{g^\lambda}, \phi\big)$$
$$\in AUT(G),\ i\phi = i \text{ and } (a\phi)^{-1} = (a^{-1})\phi\ \forall\ a \in G\Big\} \le BS(G,\odot).$$

Proof. Let $BS'(G,\odot) = \Big\{\phi \in BS(G,\odot) \mid \phi : i \mapsto i \text{ and } (a\phi)^{-1} = \big(a^{-1}\big)\phi\Big\} \subseteq BS(G,\odot)$. Following from Theorem 3,

$$BS'(G,\odot) = \Big\{\phi \in BS(G,\odot) \mid \phi : i \mapsto i \text{ and } (a\phi)^{-1} = \big(a^{-1}\big)\phi\Big\}$$
$$= \Big\{\phi \in SYM(G)|\exists\ f \in G\ \ni\ \big(\phi R^{-1}_{f^\rho}, \phi L^{-1}_f, \phi\big) \in AUT(G),\ i\phi = i \text{ and}$$
$$(a\phi)^{-1} = (a^{-1})\phi\ \forall\ a \in G\Big\} = \Big\{\phi \in SYM(G)|\exists\ g \in G\ \ni\ \big(\phi R^{-1}_g, \phi L^{-1}_{g^\lambda}, \phi\big)$$
$$\in AUT(G),\ i\phi = i \text{ and } (a\phi)^{-1} = (a^{-1})\phi\ \forall\ a \in G\Big\}.$$

Let $\mathcal{L}$ denote the identity map on G such that $i\mathcal{L} = i$ and $(f\mathcal{L})^{-1} = (f^{-1})\mathcal{L}\ \forall\ f \in G$ and $(\mathcal{L}R_i^{-1}, \mathcal{L}L_i^{-1}, \mathcal{L}) = (\mathcal{L},\mathcal{L},\mathcal{L}) \in AUT(G,\odot)$. So, $\mathcal{L} \in BS'(G,\odot)$. Hence, $BS'(G,\odot)$ is not empty.

Let $\psi, \pi \in BS'(G,\odot)$. Then, $\psi,\pi \in BS(G,\odot)$ and $i\psi = i$ and $(a\psi)^{-1} = (a^{-1})\psi,\ i\pi = i$ and $(a\pi)^{-1} = (a^{-1})\pi\ \forall\ a \in G$.

Furthermore, there exist $f_1, g_1, f_{11}, g_{11} \in G$ with $g_1 = f_1^\rho, f_{11} = g_{11}^\lambda$ such that

$$A = (\psi R^{-1}_{g_1}, \psi L^{-1}_{f_1}, \psi), B = (\pi R^{-1}_{g_{11}}, \pi L^{-1}_{f_{11}}, \pi), B^{-1} =$$
$$(R_{g_{11}}\pi^{-1}, L_{f_{11}}\pi^{-1}, \pi^{-1}) \in AUT(G,\odot).$$
$$AB^{-1} = (\psi R^{-1}_{g_1}, \psi L^{-1}_{f_1}, \psi)(R_{g_{11}}\pi^{-1}, L_{f_{11}}\pi^{-1}, \pi^{-1}) =$$
$$(\psi R^{-1}_{g_1}R_{g_{11}}\pi^{-1}, \psi L^{-1}_{f_1}L_{f_{11}}\pi^{-1}, \psi\pi^{-1}) \in AUT(G,\odot).$$

Let $\mu = \pi R^{-1}_{g_1}R_{g_{11}}\pi^{-1}$ and $\nu = \pi L^{-1}_{f_1}L_{f_{11}}\pi^{-1}$ so that $(\psi\pi^{-1}\mu, \psi\pi^{-1}\nu, \psi\pi^{-1}) \in AUT(G,\odot)$ if and only if for all $a, b \in G$

$$a\psi\pi^{-1}\mu \odot b\psi\pi^{-1}\nu = (a \odot b)\psi\pi^{-1}. \tag{7}$$

Setting $a = i$ in G and replacing b by $b\pi\psi^{-1}$ in (7), we have

$$(i\psi\pi^{-1}\mu) \odot (b\nu) = b \Longrightarrow b\nu L_{(i\psi\pi^{-1}\mu)} = b \Longrightarrow \nu = L^{-1}_{(i\psi\pi^{-1}\mu)}.$$

Similarly, setting $b = i$ in G and replacing a by $a\pi\psi^{-1}$ in (7), we have

$$(a\mu) \odot (i\psi\pi^{-1}\nu) = a \Longrightarrow a\mu R_{(i\psi\pi^{-1}\nu)} = a \Longrightarrow \mu = R^{-1}_{(i\psi\pi^{-1}\nu)}.$$

Thus, $g = i\psi\pi^{-1}\nu = i\nu = i\pi L^{-1}_{f_1} L_{f_{11}}\pi^{-1} = [f_{11} \odot (f_1\backslash i)]\pi^{-1} = [f_{11} \odot f_1^\rho]\pi^{-1}$ and $f = i\psi\pi^{-1}\mu = i\mu = i\pi R^{-1}_{f_1^\rho} R_{f_{11}^\rho}\pi^{-1} = iR^{-1}_{f_1^\rho} R_{f_{11}^\rho}\pi^{-1} = [(i/f_1^\rho) \odot f_{11}^\rho]\pi^{-1} = (f_1 \odot f_{11}^\rho)\pi^{-1}$. Then, $f^\rho = \underbrace{[(f_1 \odot f_{11}^\rho)\pi^{-1}]^{-1}}_{\text{since } x^\rho = x^\lambda} = (f_1 \odot f_{11}^\rho)^{-1}\pi^{-1} = (f_{11} \odot f_1^\rho)\pi^{-1} = g$. Hence,

$AB^{-1} = (\psi\pi^{-1}\mu, \psi\pi^{-1}\nu, \psi\pi^{-1}) = (\psi\pi^{-1}R^{-1}_{f^\rho}, \psi\pi^{-1}L^{-1}_f, \psi\pi^{-1})$ is an autotopism of $(G, \odot)$, $i\psi\pi^{-1} = i$ and $(a^{-1})\psi\pi^{-1} = (a\psi\pi^{-1})^{-1}$ $\forall$ $a \in G$. So, $\psi\pi^{-1} \in BS'(G, \odot)$.

Also, $AB^{-1} = (\psi\pi^{-1}R^{-1}_g, \psi\pi^{-1}L^{-1}_{g^\lambda}, \psi\pi^{-1}) \in AUT(G, \odot)$. Therefore, $BS'(G, \odot) \leq BS(G, \odot)$. □

Corollary 2. *Let $(G, \odot)$ be a GMBL. Then, $AUM(G, \odot) \leq BS'(G, \odot) \leq BS(G, \odot)$.*

Proof. This follows from Theorem 4. □

Theorem 5. *Let $(G, \odot)$ be a GMBL, then $JM_{x^\alpha}L_x \in BS(G, \odot)$ for every $x \in G$.*

Proof. Let $(G, \odot)$ be a generalized middle Bol loop, then $(JM_x^{-1}, JM_{x^\alpha}, JM_{x^\alpha}L_x) \in AUT(G, \odot)$ for all $x \in G$. Choose $JM_{x^\alpha}L_x = \phi$ to get the triple $(JM_x^{-1}, JM_{x^\alpha}, \phi)$ which is an autotopism of $(G, \odot)$. Using the identical relation, we have

$$aJM_x^{-1} \odot bJM_{x^\alpha} = (a \odot b)\phi \ \forall\ a, b \in G. \tag{8}$$

Put $a = i$ in (8), the identity element in G to get $b\phi = iJM_x^{-1} \odot bJM_{x^\alpha} = x/i \odot bJM_{x^\alpha} = bJM_{x^\alpha}L_x \Rightarrow JM_{x^\alpha}L_x = \phi$ for all $b \in G$. So,

$$JM_{x^\alpha} = \phi L_x^{-1} \ \forall\ x \in G. \tag{9}$$

Also, put $b = i$ in (8), to get $a\phi = aJM_x^{-1} \odot iJM_{x^\alpha} = aJM_x^{-1} \odot (i\backslash x^\alpha) = aJM_x^{-1}R_{x^\alpha}$ for all $a \in G$. So,

$$\phi R^{-1}_{x^\alpha} = JM_x^{-1} \ \forall\ x \in G. \tag{10}$$

Using (9) and (10) in (8), we get that $(\phi R^{-1}_{x^\alpha}, \phi L_x^{-1}, \phi)$ is an autotopism of $(G, \odot)$. So, $\phi \in BS(G, \odot)$. □

Corollary 3. *Let $(G, \odot)$ be an MBL. Then, $JM_xL_x \in BS(G, \odot)$ for every $x \in G$.*

Proof. Consequence of Theorem 5. □

3.3 Pseudo-automorphic characterization of generalized middle Bol loop

Using the results in Section 3.1, we shall carry out an autotopic-split and right (left) pseudo-automorphic characterization of generalized middle Bol loop. Furthermore, we shall carry out a right (left) pseudo-automorphic characterization of the isotopy-isomorphy of generalized middle Bol loop.

Theorem 6. *Let $(G,\odot)$ be a GMBL. For any $x \in \mathbf{N}_m$, $\phi = JM_x^{-1}R_x^{-1}$ is a right pseudo-automorphism of $(G,\odot)$ with companion $x \odot x^\alpha$.*

Proof. Let $(G,\odot)$ be a generalized middle Bol loop with $x \in \mathbf{N}_m$, then $(JM_x^{-1}, JM_{x^\alpha}, JM_{x^\alpha}L_x)$ and (R_x^{-1}, L_x, i) are autotopisms of $(G,\odot)$. The product

$$(JM_x^{-1}, JM_{x^\alpha}, JM_{x^\alpha}L_x)(R_x^{-1}, L_x, i) = \\ (JM_x^{-1}R_x^{-1}, JM_{x^\alpha}L_x, JM_{x^\alpha}L_x). \in AUT(G,\odot). \tag{11}$$

Replacing $JM_x^{-1}R_x^{-1}$ with ϕ in (11), we get $(\phi, JM_{x^\alpha}L_x, JM_{x^\alpha}L_x) \in AUT(G,\odot)$. Writing the identical relation, we have

$$a\phi \odot bJM_{x^\alpha}L_x = (a \odot b)JM_{x^\alpha}L_x \tag{12}$$

for all $a, b \in G$. Put $b = i$ in (12), where i is the identity element in G, to get

$$aJM_{x^\alpha}L_x = a\phi \odot iJM_{x^\alpha}L_x = a\phi \odot xx^\alpha = a\phi R_{xx^\alpha}$$

for all $a \in G$ and for all $x \in \mathbf{N}_m$. Then, $(\phi, \phi R_{xx^\alpha}, \phi R_{xx^\alpha}) \in AUT(G,\odot)$, that is ϕ is right pseudo-automorphism of $(G,\odot)$ with companion xx^α. □

Corollary 4. *Let $(G,\odot)$ be a middle Bol loop. For any $x \in \mathbf{N}_m$, $\phi = JM_x^{-1}R_x^{-1}$ is a right pseudo-automorphism of $(G,\odot)$ with companion x^2.*

Proof. Consequence of Theorem 6 by $x^\alpha = x$. □

Theorem 7. *Let $(G, i, \odot)$ be a generalized middle Bol loop and let $K = (A, B, C) \in AUT(G,\odot)$. Then, there exists a right pseudo-automorphism ϕ with companion $y^{-1}\backslash x^{(\alpha)^{-1}}$, where $iA = x$ and $iB = y$ such that*

$$K = (\phi, \phi R_{y^{-1}\backslash(x^\alpha)^{-1}}, \phi R_{y^{-1}\backslash(x^\alpha)^{-1}})(M_{x^{-1}}J, M^{-1}_{(x^\alpha)^{-1}}J, L^{-1}_{x^{-1}}M^{-1}_{(x^\alpha)^{-1}}J).$$

Proof. Let $(G,\odot)$ be a GMBL, then $(JM^{-1}_{x^{-1}}, JM_{(x^\alpha)^{-1}}, JM_{(x^\alpha)^{-1}}L_{x^{-1}}) \in AUT(G,\odot)$ $\forall\ x\ \in G$. If $K = (A, B, C) \in AUT(G,\odot)$, then the product

$$(A, B, C)(JM^{-1}_{x^{-1}}, JM_{(x^\alpha)^{-1}}, JM_{(x^\alpha)^{-1}}L_{x^{-1}}) = \\ (AJM^{-1}_{x^{-1}}, BJM_{(x^\alpha)^{-1}}, CJM_{(x^\alpha)^{-1}}L_{x^{-1}}) \in AUT(G,\odot)\ \forall\ x \in G. \tag{13}$$

Replacing $AJM^{-1}_{x^{-1}}$ with ϕ, and $i\phi = iAJM^{-1}_{x^{-1}} = x^{-1}/(iA)^{-1} = x^{-1}/x^{-1} = i$. Then $(\phi, BJM_{(x^\alpha)^{-1}}, CJM_{(x^\alpha)^{-1}}L_{x^{-1}}) \in AUT(G,\odot)\ \forall\ x \in G$. So,

$$a\phi \odot bBJM_{(x^{\alpha})^{-1}} = (a \odot b)CJM_{(x^{\alpha})^{-1}}L_{x^{-1}} \tag{14}$$

for all $a, b \in G$. Let $a = i$, where i is identity element in $(G, \odot)$ to get

$$i\phi \odot bBJM_{(x^{\alpha})^{-1}} = bCJM_{(x^{\alpha})^{-1}}L_{x^{-1}}$$

for all $b \in G$, this implies that

$$bBJM_{(x^{\alpha})^{-1}} = bCJM_{(x^{\alpha})^{-1}}L_{x^{-1}} \Rightarrow BJM_{(x^{\alpha})^{-1}} = CJM_{(x^{\alpha})^{-1}}L_{x^{-1}}$$

for all $x \in G$.

Using the last equality, we obtain $(\phi, BJM_{(x^{\alpha})^{-1}}, BJM_{(x^{\alpha})^{-1}}) \in AUT(G, \odot)\ \forall\, x \in G$. So, for all $a, b \in G$, the identical relation

$$a\phi \odot bBJM_{(x^{\alpha})^{-1}} = (a \odot b)BJM_{(x^{\alpha})^{-1}} \tag{15}$$

holds. Letting $b = i$ in (15), we have

$$\begin{aligned} a\phi \odot iBJM_{(x^{\alpha})^{-1}} &= aBJM_{(x^{\alpha})^{-1}} \\ \Rightarrow a\phi \odot y^{-1}M_{(x^{\alpha})^{-1}} &= aBJM_{(x^{\alpha})^{-1}} \\ \Rightarrow a\phi \odot y^{-1}\backslash(x^{\alpha})^{-1} &= aBJM_{(x^{\alpha})^{-1}} \\ \Rightarrow \phi R_{y^{-1}\backslash(x^{\alpha})^{-1}} &= BJM_{(x^{\alpha})^{-1}}. \end{aligned}$$

That is, $K = (\phi, \phi R_{y^{-1}\backslash(x^{\alpha})^{-1}}, \phi R_{y^{-1}\backslash(x^{\alpha})^{-1}}) \in AUT(G, \odot)$, this implies that ϕ is a right pseudo-automorphism with companion $y^{-1}\backslash x^{(\alpha)^{-1}}$. So,

$$K = (A, B, C)(JM_{x^{-1}}^{-1}, JM_{(x^{\alpha})^{-1}}, JM_{(\alpha)^{-1}}L_{x^{-1}}) = \left(\phi, \phi R_{y^{-1}\backslash(x^{\alpha})^{-1}}, \phi R_{y^{-1}\backslash(x^{\alpha})^{-1}}\right).$$

That is,

$$(A, B, C) = (\phi, \phi R_{y^{-1}\backslash(x^{\alpha})^{-1}}, \phi R_{y^{-1}\backslash(x^{\alpha})^{-1}})(M_{x^{-1}}J, M_{(x^{\alpha})^{-1}}^{-1}J, L_{x^{-1}}^{-1}M_{(x^{\alpha})^{-1}}^{-1}J).$$

□

Theorem 8. *Let $(G, i, \odot)$ be a generalized middle Bol loop and let $K = (A, B, C) \in AUT(G, \odot)$. Then, there exists a left pseudo-automorphism ϕ with companion $y^{-1}/x^{(\alpha)^{-1}}$, where $iA = x$ and $iB = y$ such that*

$$K = (\phi L_{y^{-1}/(x^{\alpha})^{-1}}, \phi, \phi L_{y^{-1}/(x^{\alpha})^{-1}})(M_{x^{-1}}J, M_{(x^{\alpha})^{-1}}^{-1}J, L_{x^{-1}}^{-1}M_{(x^{\alpha})^{-1}}^{-1}J).$$

Proof. Follow similar steps to Theorem 7. □

Theorem 9. *Let $(G, \odot)$ be a generalized middle Bol loop, $x, y \in G$, and let $a \circ b = aR_x^{-1} \odot bL_y^{-1}$, for all $a, b \in G$. Then, $(G, \circ) \cong (G, \odot)$ if and only if there exists*

1. *right pseudo-automorphism of* $(G, \odot)$ *with companion* $x^{-1}\backslash y^{(\alpha)^{-1}}$ *or*
2. *left pseudo-automorphism of* $(G, \odot)$ *with companion* $y^{-1}/x^{(\alpha)^{-1}}$.

Proof. 1. Let C be isomorphism between $(G, \circ)$ and $(G, \odot)$. Then

$$(a \odot b)C = aC \circ bC = aCR_x^{-1} \odot bCL_y^{-1} = aB \odot bA,$$

for all $a, b \in G$, where $B = CR_x^{-1}$ and $A = CL_y^{-1}$. So the triple $(B, A, C) \in AUT(G, \odot)$.
Let i and e denote the identity elements in $(G, \odot)$ and $(G, \circ)$ respectively. So, since C is an isomorphism between them, we have $iC = e$, where $e = y \odot x$, and

$$iA = iCR_x^{-1} = eR_x^{-1} = (y \odot x)R_x^{-1} = y, \tag{16}$$

$$iB = iCL_y^{-1} = eL_y^{-1} = (y \odot x)L_y^{-1} = x. \tag{17}$$

Here, from Theorem 7, (16) and (17), we have the equality

$$(B, A, C) = (\phi, \phi R_{x^{-1}\backslash(y^\alpha)^{-1}}, \phi R_{x^{-1}\backslash(y^\alpha)^{-1}})(M_{x^{-1}}J, M^{-1}_{(x^\alpha)^{-1}}J, L^{-1}_{x^{-1}}M^{-1}_{(x^\alpha)^{-1}}J)$$

where ϕ is a right pseudo-automorphism of $(G, \odot)$ with companion $x^{-1}\backslash y^{(\alpha)^{-1}}$.
Conversely, suppose that ϕ is a right pseudo-automorphism of $(G, \odot)$ with companion $x^{-1}\backslash y^{(\alpha)^{-1}}$. Then, $(\phi, \phi R_{x^{-1}\backslash(y^\alpha)^{-1}}, \phi R_{x^{-1}\backslash(y^\alpha)^{-1}}) \in AUT(G, \odot)$.
On the other hand, since $(G, \odot)$ is GMBL, we get that

$$(JM^{-1}_{x^{-1}}, JM_{(x^\alpha)^{-1}}, JM_{(\alpha)^{-1}}L_{x^{-1}})^{-1} =$$
$$(M_{y^{-1}}J, M^{-1}_{(y^\alpha)^{-1}}J, L^{-1}_{y^{-1}}M^{-1}_{(y^\alpha)^{-1}}J) \in AUT(G, \odot).$$

So,
$$(B, A, C) = (\phi, \phi R_{x^{-1}\backslash(y^\alpha)^{-1}}, \phi R_{x^{-1}\backslash(y^\alpha)^{-1}})(M_{y^{-1}}J, M^{-1}_{(y^\alpha)^{-1}}J, L^{-1}_{y^{-1}}M^{-1}_{(y^\alpha)^{-1}}J)$$
$$= (\phi M_{y^{-1}}J, \phi R_{x^{-1}\backslash(y^\alpha)^{-1}}M^{-1}_{(y^\alpha)^{-1}}J, \phi R_{x^{-1}\backslash(y^\alpha)^{-1}}L^{-1}_{y^{-1}}M^{-1}_{(y^\alpha)^{-1}}J) \in AUT(G, \odot).$$
Writing the identical relation, we have

$$a\phi M_{y^{-1}}J \odot b\phi R_{x^{-1}\backslash(y^\alpha)^{-1}}M^{-1}_{(y^\alpha)^{-1}}J =$$
$$(a \odot b)\phi R_{x^{-1}\backslash(y^\alpha)^{-1}}L^{-1}_{y^{-1}}M^{-1}_{(y^\alpha)^{-1}}J \Rightarrow (a \odot b)C = aB \odot bA \ \forall \ a, b \in G \tag{18}$$

where
$C = \phi R_{x^{-1}\backslash(y^\alpha)^{-1}}L^{-1}_{y^{-1}}M^{-1}_{(y^\alpha)^{-1}}J, B = \phi M_{y^{-1}}J$ and $A = \phi R_{x^{-1}\backslash(y^\alpha)^{-1}}M^{-1}_{(y^\alpha)^{-1}}J$.
Also,
$$iB = i\phi M_{y^{-1}}J = (y^{-1})^{-1} = y.$$

$$iA = i\phi R_{x^{-1}\backslash(y^\alpha)^{-1}}M^{-1}_{(y^\alpha)^{-1}}J = [(y^\alpha)^{-1}/(x^{-1}\backslash(y^\alpha)^{-1}]^{-1} = (x^{-1})^{-1} = x$$

Hence, let $[(y^\alpha)^{-1}/(x^{-1}\backslash(y^\alpha)^{-1})] = t \Rightarrow (y^\alpha)^{-1} = t \odot (x^{-1}\backslash(y^\alpha)^{-1}$.
Putting $s = x^{-1}\backslash(y^\alpha)^{-1} \Rightarrow x^{-1} \odot s = (y^\alpha)^{-1}$, we get that $x^{-1} \odot s = t \odot s$, so $t = x^{-1}$. Now, put $b = i$ in (18), to obtain

$$aC = aB \odot iA \Rightarrow aC = aBR_x \ \forall \ a \in G$$
$$CR_x^{-1} = B. \tag{19}$$

Put $a = i$ in (18) we get

$$bC = iB \odot bA \Rightarrow bC = bAL_y \ \forall \ b \in G$$
$$A = CL_y^{-1}. \tag{20}$$

So, from equations (18), (19) , and (20) we have

$$(a \odot b)C = aB \odot bA = aCR_x^{-1} \odot bCL_y^{-1} = aC \circ bC.$$

for all $a, b \in G$. Thus, $(G, \circ) \cong (G, \odot)$.

2. Follows similar argument with 1.

□

4 Conclusion

From this study, it can be concluded that the two identities that characterize middle Bol loop can be separately generalized to define generalized middle Bol loops (GMBL1 and GMBL2) which are distinct if the generalizing map (α) only fixes the identity element of the loop but equivalent if the generalizing map (α) fixes the identity and is bijective. Also, an autotopic-split and right (left) pseudo-automorphic characterization of generalized middle Bol loop was made feasible and a right (left) pseudo-automorphic characterization of the isotopy-isomorphy of generalized middle Bol loop was also made possible. This is a sort of ‘G-generalized middle Bol loop’ characterization based on pseudo-automorphic companionship of elements of GMBL as we have it for ‘G-loops’.

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Benard Osoba
College of Natural and Applied Sciences, Bells University of Technology, Ota, Ogun State, Nigeria. E-mail: *benardomth@gmail.com, b_osoba@bellsuniversity.edu.ng*

Tèmítọ́pẹ́ Gbọ́láhàn Jaiyéọlá
Obafemi Awolowo University, Ile-Ife 220005, Nigeria.
University of Lagos, Akoka, Nigeria. E-mail:
tjayeola@oauife.edu.ng, tgjaiyeola@unilag.edu.ng

Abednego Orobosa Isere
Ambrose Alli University, Ekpoma 310001, Nigeria.
E-mail: *abednis@yahoo.co.uk, isereao@aauekpoma.edu.ng*

Yakub Tunde Oyebo
Lagos State University, Ojo, Lagos State, Nigeria.
E-mail: *oyeboyt@yahoo.com, yakub.oyebo@lasu.edu.ng*