

On LCA groups with local ring of continuous endomorphisms

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Abstract. In this paper, we determine the locally compact abelian (LCA) groups whose ring of continuous endomorphisms is local.

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1 Introduction

A ring is local if it is unital and its non-invertible elements form an ideal. In [19], A. Orsatti tackled the problem of characterizing the abelian groups with local endomorphism ring, and obtained some partial results. Later R. Ware [21] considered a similar problem in the context of modules. Specifically, A. Orsatti has observed that an abelian group with local endomorphism ring is necessarily indecomposable, and has proved that in the class of abelian groups with non-zero torsion subgroup the only groups having local endomorphism ring are the co-cyclic groups. This reduces the considered problem to the case of indecomposable torsion-free groups. Further, A. Orsatti established that a torsion-free abelian group of rank one has a local endomorphism ring if and only if it is isomorphic to the additive group of a local subring of the field of rational numbers. Excluding the additive group of rational numbers, for any such a group X there is a prime number p with the property that the type of X is represented by the sequence $(\infty, \dots, \infty, 0, \infty, \dots)$, where 0 occupies the same position as does p in the ascending sequence $(p_n)_{n \geq 1}$ of prime numbers. Since any abelian group isomorphic with a pure subgroup of the group $\mathbb{J}_p$ of p -adic integers has the same type, A. Orsatti initiated the study of endomorphism rings of these groups. By a result of J. W. Armstrong [2], this class of groups can also be described as the class of homogeneous with the type of $\mathbb{J}_p$ purely indecomposable groups. A. Orsatti proved that the pure subgroups of finite rank of $\mathbb{J}_p$ have local endomorphism ring. He also showed that for any cardinal number $\mu \leq \aleph_1$, there exist pure subgroups of rank μ in $\mathbb{J}_p$ whose endomorphism ring is local.

The work of A. Orsatti was followed up by R.B. Warfield and P. Krylov. In [22], R.B. Warfield proved that an indecomposable abelian group has local endomorphism ring if and only if it has the finite exchange property, and thus obtained a complete

solution to A. Orsatti's problem. The full characterization of pure subgroups of $\mathbb{J}_p$ with local endomorphism ring has been obtained by P. Krylov in [12]. This result can also be found in [13, Theorem 19.10].

In the present paper, we extend these investigations by considering a similar problem in the more general context of LCA groups. Precisely speaking, we shall deal with the problem of determining the LCA groups X whose ring of continuous endomorphisms, $E(X)$, is local. Our main results, established in Theorem 8 and Theorem 10, generalize the above mentioned results of P. Krylov and R.B. Warfield to the case of LCA groups.

2 Notation

In the following, $\mathbb{P}$ stands for the set of prime numbers, $\mathbb{N}$ for the set of natural numbers (including zero), and $\mathbb{N}_0 = \mathbb{N} \setminus \{0\}$. We denote by $\mathbb{R}$ the group of reals with its usual topology and by $\mathbb{Q}$ the group of rationals, taken discrete. For $p \in \mathbb{P}$, we denote by $\mathbb{Q}_p$ the group of p -adic numbers, by $\mathbb{Z}_p$ the group of p -adic integers, both with their usual topologies, and by $\mathbb{J}_p$ the group of p -adic integers taken discrete. We also denote by $\mathbb{I}_p$ the group of rational numbers with denominators prime to p , by $\mathbb{Z}(p^\infty)$ the quasi-cyclic group corresponding to p , and by $\mathbb{Z}(p^n)$, $n \in \mathbb{N}$, the cyclic group of order p^n , all with the discrete topology.

Let $\mathcal{L}$ be the class of locally compact abelian groups. For $X \in \mathcal{L}$, we denote by $E(X)$ the ring of all continuous endomorphisms of X , by $J(E(X))$ the Jacobson radical of $E(X)$, and by $D(X)$ the minimal divisible extension of X , topologized in such a way that X becomes open in $D(X)$. Further, we let 1_X , $c(X)$, $d(X)$, $k(X)$, $m(X)$, $t(X)$, and X^* denote respectively the identity map on X , the connected component of X , the maximal divisible subgroup of X , the subgroup of compact elements of X , the smallest closed subgroup K of X such that the quotient group X/K is torsion-free, the torsion subgroup of X , and the character group of X . If S is a subset of X and C is a closed subgroup of X , we write $\bar{S}$ for the closure of S in X , $\langle S \rangle$ for the subgroup of X generated by S , $A(X^*, C)$ for the annihilator of C in X^* , and X/C for the quotient group of X modulo C taken with the quotient topology. For $p \in \mathbb{P}$, $t_p(X)$ stands for the p -primary component of $t(X)$ and X_p for the topological p -primary component of X , i.e. $X_p = \{x \in X \mid \lim_{n \rightarrow \infty} (n!)x = 0\}$. For $n \in \mathbb{N}_0$, we let $X[n] = \{x \in X \mid nx = 0\}$ and $nX = \{nx \mid x \in X\}$. We also write $\oplus$ for topological direct sum, $\dot{+}$ for algebraic direct sum, and $\cong$ for topological group isomorphism.

3 Discrete and compact abelian groups

In this section, we apply the results of A. Orsatti and P. Krylov to exhibit examples of non-discrete groups in $\mathcal{L}$ having local ring of continuous endomorphisms.

We will need the anti-isomorphism defined in the following lemma from [17].

Lemma 1. *Let $X \in \mathcal{L}$. If $E(X)$ is endowed with the compact-open topology, then $E(X)$ is a topological ring, and the mapping $u \mapsto u^*$, where $u^*(\gamma) = \gamma \circ u$ for all $u \in E(X)$ and $\gamma \in X^*$, is a topological ring anti-isomorphism from $E(X)$ onto $E(X^*)$.*

We start with the following result obtained by A. Orsatti.

Theorem 1 (A. Orsatti). *Let $X \in \mathcal{L}$ be discrete.*

- (i) *If $t(X) \neq \{0\}$, the ring $E(X)$ is local if and only if X is isomorphic with $\mathbb{Z}(p^n)$, where $p \in \mathbb{P}$ and $n \in \mathbb{N} \cup \{\infty\}$.*
- (ii) *If X is of rank one, the ring $E(X)$ is local if and only if X is isomorphic with either $\mathbb{Q}$ or $\mathbb{I}_p$ for some $p \in \mathbb{P}$.*

As a consequence of Theorem 1, we get two classes of compact abelian groups in which the groups with local ring of continuous endomorphisms can be completely described.

Corollary 1. *Let $X \in \mathcal{L}$ be compact.*

- (i) *If X has proper closed subgroups of finite index, the ring $E(X)$ is local if and only if X is topologically isomorphic with either $\mathbb{Z}_p$ or $\mathbb{Z}(p^n)$, where $p \in \mathbb{P}$ and $n \in \mathbb{N}$.*
- (ii) *If X is of dimension one, the ring $E(X)$ is local if and only if X is topologically isomorphic with either $\mathbb{Q}^*$ or $\mathbb{I}_p^*$ for some $p \in \mathbb{P}$.*

Proof. It is clear that X has proper closed subgroups of finite index if and only if X^* has non-zero elements of finite order [10, (23.25) and (24.11)]. Also, X is of dimension one if and only if X^* is of rank one [10, (24.28)]. Taking into account the fact that $E(X)$ and $E(X^*)$ are anti-isomorphic, the result follows from Theorem 1 and duality. $\square$

We continue with

Theorem 2 (A. Orsatti). *Let $p \in \mathbb{P}$. Every finite rank, pure subgroup of $\mathbb{J}_p$ has local endomorphism ring.*

Recall that a closed subgroup C of a group $X \in \mathcal{L}$ is said to be of finite codimension if X/C is of finite dimension. We have

Corollary 2. *Let $p \in \mathbb{P}$. For every closed pure subgroup of finite codimension C of $\mathbb{J}_p^*$, $E(\mathbb{J}_p^*/C)$ is local.*

As we already mentioned, P. Krylov gave in [12, Theorem 1] (see also [13, Theorem 19.10]) a complete characterization of pure subgroups of $\mathbb{J}_p$, $p \in \mathbb{P}$, which have local endomorphism ring. To state this result, we need some definitions. A unital ring E is strongly homogeneous if every element $a \in E$ can be written in the form

$a = nu$, where $n \in \mathbb{N}_0$ and u is invertible in E . A discrete valuation domain is a local, principal ideal, commutative domain. For an abelian group X , the pseudo-socle of X , denoted by $\text{soc}(X)$, is the pure subgroup of X generated by all minimal pure fully invariant subgroups of X . It is to be noted also that if X is a non-zero pure subgroup of $\mathbb{J}_p$, then every endomorphism $f \in E(X)$ extends uniquely to an endomorphism $\bar{f} \in E(\mathbb{J}_p)$. In the following theorem, $E(X)$ is identified with its image in $E(\mathbb{J}_p)$ under the ring homomorphism $f \mapsto \bar{f}$.

Theorem 3 (P. Krylov). *Let $p \in \mathbb{P}$. For a non-zero pure subgroup X of $\mathbb{J}_p$, the following conditions are equivalent:*

- (i) $X = \text{soc}(X)$.
- (ii) $\text{soc}(X) \neq \{0\}$.
- (iii) $E(X)$ is a strongly homogeneous ring.
- (iv) Every element of $E(X)$ that is invertible in $E(\mathbb{J}_p)$ is invertible in $E(X)$.
- (v) X does not have proper pure subgroups isomorphic with X .
- (vi) $E(X)$ is a discrete valuation domain.
- (vii) $E(X)$ is a local ring.
- (viii) $J(E(X)) = pE(X)$.

We conclude this section by stating the dual of Theorem 3. Before this, however, two additional definitions must be introduced. A closed subgroup C of a group $X \in \mathcal{L}$ is said to be topologically fully invariant in X if $f(C) \subseteq C$ for all $f \in E(X)$. The pseudo-radical, $\text{rad}(X)$, of a connected group $X \in \mathcal{L}$ is the connected component of the intersection of all maximal connected topologically fully invariant closed subgroups of X . If X has no maximal connected topologically fully invariant closed subgroups, then $\text{rad}(X)$ is taken to be all of X .

Corollary 3. *Let $p \in \mathbb{P}$, let C be a closed, pure, proper subgroup of $\mathbb{J}_p^*$, and let $\varphi: \mathbb{J}_p^* \rightarrow \mathbb{J}_p^*/C$ be the canonical projection. For the quotient group $X = \mathbb{J}_p^*/C$, the following conditions are equivalent:*

- (i) $\text{rad}(X) = \{0\}$.
- (ii) $\text{rad}(X) \neq X$.
- (iii) $E(X)$ is a strongly homogeneous ring.
- (iv) For every $f \in E(X)$, there exists a unique $\tilde{f} \in E(\mathbb{J}_p^*)$ such that $\varphi \circ \tilde{f} = f \circ \varphi$, and if $\tilde{f}$ is invertible in $E(\mathbb{J}_p^*)$, then f is invertible in $E(X)$.
- (v) X does not have closed, pure, proper subgroups A such that X/A is isomorphic with X .

- (vi) $E(X)$ is a discrete valuation domain.
- (vii) $E(X)$ is a local ring.
- (viii) $J(E(X)) = pE(X)$.

Proof. Let $G = A(\mathbb{J}_p, C)$. Then G is a non-zero [10, (23.24)(b)] pure [10, Corollary 7.6] subgroup of $\mathbb{J}_p$, and $G \cong (\mathbb{J}_p^*/C)^* = X^*$ [10, (23.25)]. Now, since $E(X)$ and $E(X^*)$ are anti-isomorphic, it is clear that $E(X^*)$ is local if and only if $E(X)$ is local. On the other hand, by [10, (23.17) and (24.25)], $\mathbb{J}_p^*$ is compact and connected, so $X = \mathbb{J}_p^*/C$ is compact and connected as well. It follows from [15, Corollary, p. 369] and [10, (24.25)] that a closed subgroup of X is pure in X if and only if it is connected. Let $\mathcal{Max}(X)$ be the set of all maximal, connected, topologically fully invariant, closed subgroups of X , and let $\mathcal{Min}(X^*)$ be the set of all minimal, pure, fully invariant subgroups of X^* . It is clear from the preceding that $M \in \mathcal{Max}(X)$ if and only if $A(X^*, M) \in \mathcal{Min}(X^*)$, and $L \in \mathcal{Min}(X^*)$ if and only if $A(X, L) \in \mathcal{Max}(X)$. Consequently, by [7, Proposition 3.3.3],

$$\begin{aligned} A(X^*, \text{rad}(X)) &= A(X^*, \cap_{M \in \mathcal{Max}(X)} M) \\ &= \sum_{M \in \mathcal{Max}(X)} A(X^*, M) = \sum_{L \in \mathcal{Min}(X^*)} L = \text{soc}(X^*), \end{aligned}$$

and hence $\text{rad}(X) = \{0\}$ if and only if $\text{soc}(X^*) = X^*$.

Now the result follows easily from Theorem 3 and duality. $\square$

4 Residual and non-residual groups

In this section, we reduce the problem of determining the groups $X \in \mathcal{L}$ with the property that the ring $E(X)$ is local to the case when X is topologically indecomposable, residual and admits a prime number p such that $p1_X$ is not invertible in $E(X)$. We begin by mentioning two lemmas, which will be used in the sequel. The first one is suggested by an observation of A. Orsatti, which he used in [19]. This observation states that if a discrete group $X \in \mathcal{L}$ has local endomorphism ring, then X is indecomposable. Recall that a group $X \in \mathcal{L}$ is said to be topologically indecomposable if it cannot be written in the form $X = A \oplus B$, where A and B are non-zero closed subgroups of X . With the same arguments as in [19], we get:

Lemma 2. *Let $X \in \mathcal{L}$. If $E(X)$ is local, then X is topologically indecomposable.*

The second lemma of which we will make use is from [16, Lemma 3.8].

Lemma 3. *Let $X \in \mathcal{L}$ be a group of finite exponent p^n , where p is a prime and n is a positive integer. If $a \in X$ is an element of order p^n , then $\langle a \rangle$ splits topologically from X . Moreover, the complement of $\langle a \rangle$ can be chosen so as to contain a preassigned open subgroup V of X satisfying $\langle a \rangle \cap V = \{0\}$.*

Recall that a group $X \in \mathcal{L}$ is residual if $d(X) \subseteq k(X)$ and $c(X) \subseteq m(X)$.

Theorem 4. *Let $X \in \mathcal{L}$. If $E(X)$ is local, then either X is topologically isomorphic with one of the groups $\mathbb{R}$, $\mathbb{Q}$, or $\mathbb{Q}^*$, or else X is residual.*

Proof. Assume X is non-residual. We can write $X = V \oplus Y$, where V and Y are closed subgroups of X and V is topologically isomorphic with one of the groups $\mathbb{R}$, $\mathbb{Q}$, or $\mathbb{Q}^*$ [1, Theorem 9.3]. Consequently, if $E(X)$ is local, we must have $X = V$ by Lemma 2. $\square$

In particular, we have the following

Corollary 4. *Let X be a non-residual group in $\mathcal{L}$. The following conditions are equivalent:*

- (i) $E(X)$ is a field.
- (ii) $E(X)$ is local.
- (iii) X is topologically isomorphic with one of the groups $\mathbb{R}$, $\mathbb{Q}$, or $\mathbb{Q}^*$.

Proof. It is clear that (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) $\Rightarrow$ (i). $\square$

The following lemma is inspired by an observation in [13, p. 155].

Lemma 4. *Let $X \in \mathcal{L}$. If $E(X)$ is local, then there is at most one $p \in \mathbb{P}$ such that $p1_X$ is non-invertible in $E(X)$. If this is the case, i.e. if $E(X)$ is local and $p1_X$ is non-invertible for some $p \in \mathbb{P}$, then X is q -divisible for all $q \in \mathbb{P} \setminus \{p\}$ and $t(X) = t_p(X)$.*

Proof. Assume there is $p \in \mathbb{P}$ such that $p1_X$ is non-invertible in $E(X)$. Given any $q \in \mathbb{P} \setminus \{p\}$, we can find $s, t \in \mathbb{Z}$ such that $sp + tq = 1$, whence $sp1_X + tq1_X = 1_X$. Since $p1_X$ is non-invertible, $sp1_X$ is non-invertible as well. It follows that $tq1_X = 1_X - sp1_X$ is invertible because $E(X)$ is local, and hence $q1_X$ is also invertible. We then have

$$qX = \text{im}(q1_X) = X \quad \text{and} \quad X[q] = \ker(q1_X) = \{0\},$$

so X is q -divisible for all $q \in \mathbb{P} \setminus \{p\}$ and $t(X) = t_p(X)$. $\square$

Now, we can determine the groups $X \in \mathcal{L}$ with the property that the ring $E(X)$ is local and $p1_X$ is invertible in $E(X)$ for all $p \in \mathbb{P}$.

Theorem 5. *Let X be a residual group in $\mathcal{L}$. The following conditions are equivalent:*

- (i) $E(X)$ is a field.
- (ii) $E(X)$ is local and $p1_X$ is invertible in $E(X)$ for all $p \in \mathbb{P}$.
- (iii) $E(X)$ is local and X is densely divisible and torsion-free.
- (iv) X is topologically isomorphic with $\mathbb{Q}_p$ for some $p \in \mathbb{P}$.

Proof. Clearly, (i) implies (ii). Assume (ii), and pick an arbitrary $p \in \mathbb{P}$. Since $p1_X$ is invertible in $E(X)$, we have

$$pX = \text{im}(p1_X) = X \quad \text{and} \quad X[p] = \ker(p1_X) = \{0\}.$$

As $p \in \mathbb{P}$ was chosen arbitrarily, it follows that X is divisible and torsion-free [9, p. 98, A) and Theorem 8.4].

Assume (iii). Since X is residual, we have $d(X) \subseteq k(X)$ and $c(X) \subseteq m(X)$. It follows that

$$X = k(X) \quad \text{and} \quad c(X) = \{0\},$$

because X is densely divisible and torsion-free. Since X is also indecomposable by Lemma 2, we conclude from [1, Proposition 6.25] that X is topologically isomorphic with $\mathbb{Q}_p$ for some $p \in \mathbb{P}$.

The fact that (iv) implies (i) is immediate. $\square$

As an application of the preceding, we obtain the description of groups $X \in \mathcal{L}$ for which the ring $E(X)$ is completely primary, i.e. all of its non-invertible elements are nilpotent.

Theorem 6. *Let $X \in \mathcal{L}$. The following conditions are equivalent:*

- (i) $E(X)$ is completely primary and commutative.
- (ii) $E(X)$ is completely primary.
- (iii) X is topologically isomorphic with one of the groups $\mathbb{R}$, $\mathbb{Q}$, $\mathbb{Q}^*$, $\mathbb{Q}_p$, or $\mathbb{Z}(p^n)$, where $p \in \mathbb{P}$ and $n \in \mathbb{N}$.

Proof. Clearly, (i) implies (ii). Assume (ii). If X is non-residual, it follows from Corollary 4 that X is topologically isomorphic with one of the groups $\mathbb{R}$, $\mathbb{Q}$, or $\mathbb{Q}^*$. Assume X is residual. If all of the endomorphisms $p1_X$ with $p \in \mathbb{P}$ are invertible in $E(X)$, we deduce from Theorem 5 that X is topologically isomorphic with $\mathbb{Q}_p$ for some $p \in \mathbb{P}$. Next suppose that there is $p \in \mathbb{P}$ such that $p1_X$ is non-invertible. Then $p1_X$ is nilpotent, so there is $n \in \mathbb{N}$ such that $p^n 1_X = 0$, and hence $p^n X = 0$. Assuming $X \neq \{0\}$ and letting n be the least natural number satisfying $p^n X = 0$, we must have $X \cong \mathbb{Z}(p^n)$. For, otherwise it would follow from Lemma 3 that $X = A \oplus B$, where $A \cong \mathbb{Z}(p^n)$ and $B \neq \{0\}$, which contradicts Lemma 2. The fact that (iii) implies (i) is clear. $\square$

5 Purely topologically indecomposable groups

J. Armstrong showed in [2, Proposition 1] that a discrete torsion-free group $X \in \mathcal{L}$ is isomorphic to a pure subgroup of $\mathbb{J}_p$ for some $p \in \mathbb{P}$ if and only if X is homogeneous with the type of $\mathbb{J}_p$ and is purely indecomposable, i.e., every pure subgroup of X is indecomposable. Consequently, the above mentioned result of

P. Krylov describes the discrete, torsion-free, homogeneous with the type of $\mathbb{J}_p$, purely indecomposable groups in $\mathcal{L}$ having local endomorphism ring.

The objective of this section is to extend P. Krylov's result to a more general class of LCA groups. We begin by introducing the topological analogue of purely indecomposable groups.

Definition 1. A group $X \in \mathcal{L}$ is said to be purely topologically indecomposable if every closed, pure subgroup of X is topologically indecomposable.

We will require the following theorem from [18, Theorem 3].

Theorem 7. Let $X \in \mathcal{L}$ and let D be a closed subgroup of X such that $D \cong \mathbb{Q}_p$ for some $p \in \mathbb{P}$. The following conditions are equivalent:

- (i) D splits topologically from X .
- (ii) $D \not\subseteq (c(X) \cap k(X)) + m(X)$.

For the convenience of the reader, we also note for future use a lemma of G. Grätzer (see [9, Lemma 9.9]).

Lemma 5. Let A be an abelian group, let B be a subgroup of A , and let C be a B -high subgroup of A . The following conditions are equivalent:

- (i) $A = B \oplus C$.
- (ii) For any $p \in \mathbb{P}$ and $a \in A$, if $pa = b + c$ with $b \in B$ and $c \in C$, then there is $b' \in B$ such that $b = pb'$.

We continue by establishing a lemma of independent interest.

Lemma 6. Every non-zero, residual, topologically indecomposable group $X \in \mathcal{L}$ having closed torsion subgroup is totally disconnected.

Proof. Let X be as in the statement of the lemma. Since X is residual, it follows that $c(X) \subseteq k(X)$, because $c(X)$ is divisible [1, P. 27(e)]. Thus, if we had $c(X) = X$, it would follow that X is compact. Then, by a result of G. Itzkowitz (see [11, Theorem 2] or [1, Proposition 6.12]), we could write $X = U \oplus V$, where U and V are closed subgroups of X such that $U = \overline{t(U)}$ and $V \cong (\mathbb{Q}^*)^\mu$ for some cardinal number μ . Now, in this decomposition of X we must have $V = \{0\}$ because

$$X = c(X) \subseteq m(X)$$

by residuality. Since $t(X)$ is closed in X , it follows that

$$X = U = \overline{t(U)} \subseteq t(X),$$

whence we get $X = \{0\}$ because $t(X)$ is totally disconnected [1, Theorem 3.5]. This contradicts the hypothesis that $X \neq \{0\}$. Hence $c(X) \neq X$.

Now, we cannot have $c(X) \neq \{0\}$, because otherwise it would follow that $X = c(X) \oplus Y$ for some non-trivial closed subgroup Y of X [1, Proposition 6.13], in contradiction with the hypothesis that X is topologically indecomposable. Consequently, X must be totally disconnected. $\square$

We go on with the main theorem of this section, which may be viewed as a generalization of Theorem 3.

Theorem 8. *Let X be a residual, purely topologically indecomposable group in $\mathcal{L}$ with the following properties:*

- 1) $t(X)$ is closed in X .
- 2) If $k(X) \neq \{0\}$ and $d(X) = \{0\}$, then $t(X) \neq \{0\}$.

The following conditions are equivalent:

- (i) $E(X)$ is a discrete valuation domain.
- (ii) $E(X)$ is local.
- (iii) X is topologically isomorphic either with a pure subgroup G of $\mathbb{J}_p$ such that $\text{soc}(G) = G$, or with one of the groups $\mathbb{Q}_p$, $\mathbb{Z}(p^\infty)$, or $\mathbb{Z}(p^n)$, where $p \in \mathbb{P}$ and $n \in \mathbb{N}$.

Proof. Clearly, (i) implies (ii). It is also clear that (iii) implies (i) (see Theorem 1(i), Theorem 3, and Theorem 5.) Consequently, it only remains to prove that (ii) implies (iii). So, assume (ii). Since X is then topologically indecomposable by Lemma 2, it follows from Lemma 6 that X is also totally disconnected. In particular, $k(X)$ is open in X .

We first consider the case when $k(X) = \{0\}$. Clearly, X is then discrete, reduced, and torsion-free. It follows from Theorem 5 that the endomorphisms $p1_X$ with $p \in \mathbb{P}$ cannot be all invertible in $E(X)$. Hence, by Lemma 4, there is $p \in \mathbb{P}$ such that $p1_X$ is non-invertible and all the $q1_X$'s with $q \in \mathbb{P} \setminus \{p\}$ are invertible in $E(X)$. In particular, X is q -divisible for all $q \in \mathbb{P} \setminus \{p\}$ and $X \neq pX$ because

$$\ker(p1_X) = X[p] = \{0\}.$$

Thus, considering any non-zero $m \in \mathbb{N}$ and writing $m = p^l m'$, where $l, m' \in \mathbb{N}$ and $(m', p) = 1$, we get $mX = p^l m'X = p^l X$. It follows that

$$\bigcap_{k \in \mathbb{N}} p^k X = \bigcap_{n \in \mathbb{N}} nX = \{0\},$$

because X is reduced and torsion-free [9, §21, Exercise 2]. We then conclude that any non-zero $x \in X$ has finite p -height, and hence X is homogeneous with the type of $\mathbb{J}_p$, i.e., with the type represented by the height sequence $(\infty, \dots, \infty, 0, \infty, \dots)$, where 0 is on the place corresponding to p in the ascending sequence $(p_n)_{n \geq 1}$ of prime numbers. It follows from [2, Proposition 1] that X is isomorphic to a pure subgroup G of $\mathbb{J}_p$, and hence $\text{soc}(G) = G$ by Theorem 3.

In the following, we deal with the remaining case when $k(X) \neq \{0\}$. Since the topological p -primary components of a topological torsion group in $\mathcal{L}$ split topologically from that group [1, Theorem 3.13] and since $k(X)$ is topologically torsion [10,

(24.18)] and pure in X [1, (7.1)(c)], we conclude that there exists $p \in \mathbb{P}$ such that $k(X)$ coincides with its topological p -primary component $k_p(X)$. In particular, $t(X)$ is a p -group.

We first handle the case when $d(X) \neq \{0\}$. If $d(X) \not\subseteq t(X)$, then $d(X)$ contains a copy D of $\mathbb{Q}_p$ [1, Proposition 4.23]. Observe that

$$t(X) = (c(X) \cap k(X)) + m(X).$$

Indeed, $(c(X) \cap k(X)) = \{0\}$ because X is totally disconnected, and $t(X) = m(X)$ because $t(X)$ is closed in X . Now, since

$$D \not\subseteq t(X) = (c(X) \cap k(X)) + m(X),$$

it follows from Theorem 7 that D splits topologically from X , so $X = D$ because X is topologically indecomposable. Further, suppose $d(X) \subseteq t(X)$. As we know from [1, Proposition 4.23], $d(X)$ contains then a copy Δ of $\mathbb{Z}(p^\infty)$. Since $\mathbb{Z}(p^\infty)$ is splitting in the class of totally disconnected LCA groups [1, Proposition 6.21], we conclude that Δ splits topologically from $k(X)$. But $k(X)$ is open in X , so Δ splits topologically from X [1, Corollary 6.9], and hence $X = \Delta$ because X is topologically indecomposable.

Next we consider the remaining case when $d(X) = \{0\}$. By our hypotheses, then $t(X) \neq \{0\}$. We first show that $t(X) \cong \mathbb{Z}(p^n)$ for some $n \in \mathbb{N}_0$. Let $(x_i)_{i \in I}$ be an arbitrary maximal p -independent system of elements of $t(X)$. We must have $\text{card}(I) = 1$. For, if there existed two distinct elements $i_1, i_2 \in I$, we could write

$$\langle x_{i_1}, x_{i_2} \rangle = \langle x_{i_1} \rangle \dot{+} \langle x_{i_2} \rangle.$$

Since $\langle x_{i_1}, x_{i_2} \rangle$ is finite, it would follow that $\langle x_{i_1}, x_{i_2} \rangle$ is topologically decomposable and closed in X . Since $\langle x_{i_1}, x_{i_2} \rangle$ is also p -pure in $t(X)$ [9, Lemma 32.1] and hence pure in X [9, Lemma 26.1], this would contradict our hypothesis that X is purely topologically indecomposable. Consequently, the system $(x_i)_{i \in I}$ must consist of a single element, say x . As $\langle x \rangle$ is pure and finite, we can write $t(X) = \langle x \rangle \dot{+} T$ for some subgroup T of $t(X)$ [9, Theorem 27.5]. Further, since $\langle x \rangle$ is a p -basic subgroup of $t(X)$ [9, Lemma 32.2], $t(X)/\langle x \rangle$ is p -divisible, so divisible because $t(X)$ is a p -group. As T is algebraically isomorphic with $t(X)/\langle x \rangle$ and X is reduced, we conclude that $T = \{0\}$, and hence $t(X) = \langle x \rangle$. This proves that $t(X) \cong \mathbb{Z}(p^n)$ for some $n \in \mathbb{N}_0$.

Now, having established the preceding isomorphism, we shall show that $k(X) = t(X)$. We will achieve this by showing that the relation $k(X) \neq t(X)$ leads to a contradiction. Precisely, we shall show that if $k(X) \neq t(X)$, then X has a closed pure subgroup K , which can be written as a topological direct sum of two non-trivial closed subgroups. So, assume $k(X) \neq t(X)$, and fix a compact open subgroup U of $k(X)$ such that $t(X) \cap U = \{0\}$. Since U is torsion-free, we have $U \cong \mathbb{Z}_p^J$ for some non-empty set J [10, (25.8)]. We first consider the case when $\text{card}(J) > 1$. Then we can write $U = A \oplus B$, where $A \cong \mathbb{Z}_p$ and $B \cong \mathbb{Z}_p^{J \setminus \{j_0\}}$ for some $j_0 \in J$. It follows that

$$t(X) + U = t(X) \oplus A \oplus B.$$

Letting $\pi_B: t(X) \oplus A \oplus B \rightarrow B$ denote the canonical projection and $\eta_B: B \rightarrow D(B)$ the canonical injection, we define $f: t(X) \oplus A \oplus B \rightarrow D(B)$ by setting $f = \eta_B \circ \pi_B$. Clearly, f is continuous. Since $D(B)$ is divisible and $t(X) \oplus A \oplus B$ is open in $k(X)$, f extends to a continuous group homomorphism $\tilde{f}: k(X) \rightarrow D(B)$. Then $\ker(\tilde{f})$ is closed in $k(X)$ and

$$t(X) \oplus A \subseteq \ker(\tilde{f}).$$

Further, since $X/\ker(\tilde{f})$ is algebraically isomorphic to a subgroup of $D(B)$ and $D(B)$ is torsion-free, we conclude that $\ker(\tilde{f})$ is pure in $k(X)$. In the considered case when $\text{card}(J) > 1$, we set $K = \ker(\tilde{f})$. On the other hand, in case $\text{card}(J) = 1$, we take $K = k(X)$. So, in both cases $t(X)$ is a non-trivial subgroup of K . Now, to get a contradiction, we shall show that $t(X)$ splits topologically from K . First notice that $K \cap (t(X) \oplus U)$ is open in K and

$$K \cap (t(X) \oplus U) = t(X) \oplus V,$$

where $V = A$ in case $K = \ker(\tilde{f})$ and $V = U$ in case $K = k(X)$. Consequently, V is open in K because $t(X)$ is finite. Since $t(X)$ is pure in K , we also have

$$p^n K \cap t(X) = p^n t(X) = \{0\},$$

where as above p^n is the order of x . Now, in order to show that $t(X)$ splits from K , pick a $t(X)$ -high subgroup C of K which contains $p^n K$. We assert that $K = t(X) \oplus C$. To prove this, we first show that $t(X)$ splits algebraically from K , i.e. $K = t(X) \dot{+} C$. We will make use of Lemma 5. Fix arbitrary $q \in \mathbb{P}$ and $a \in K$ such that

$$qa = b + c \tag{1}$$

for some $b \in t(X)$ and $c \in C$. If $q \neq p$, then $q \cdot t(X) = t(X)$, so $b = qb'$ for some $b' \in t(X)$. Assume $q = p$. We see from (1) that

$$p^n a = p^{n-1} b + p^{n-1} c,$$

so $p^{n-1} b = 0$ because $p^n a \in p^n K \subseteq C$. Since $t(X) \cong \mathbb{Z}(p^n)$, it follows that there exists $b'' \in t(X)$ such that $b = pb''$. Consequently, Lemma 5 applies and we may conclude that $K = t(X) \dot{+} C$. To show that in fact $K = t(X) \oplus C$, it suffices to show that C is open in K . It is clear from the preceding that

$$p^n K \supseteq p^n (t(X) \oplus V).$$

But V is open in K , so $p^n V$ is open in K because $V \cong \mathbb{Z}_p$, and hence the subgroup

$$p^n (t(X) \oplus V) = p^n V$$

is open in K . Since $p^n K \subseteq C$, it follows that $K = t(X) \oplus C$, which contradicts the assumption that X is purely topologically indecomposable. From this contradiction, we conclude that the equality $k(X) = t(X)$ is true.

To finish, it remains to observe that in fact $X = t(X)$. Indeed, since $k(X)$ is open in X and $t(X)$ is finite, it follows from the above equality that X is discrete. Now, being a cyclic group of order p^n , $t(X)$ splits from X [9, Proposition 27.1], and hence we must have $X = t(X)$ by the indecomposability of X . $\square$

To state the dual of Theorem 8, we introduce the following

Definition 2. A group $X \in \mathcal{L}$ is said to be purely topologically co-indecomposable if for any closed, pure subgroup C of X , the quotient group X/C is topologically indecomposable.

Corollary 5. Let X be a residual, purely topologically co-indecomposable group in $\mathcal{L}$ with the following properties:

- 1) $\overline{rX} / \cap_{n \in \mathbb{N}_0} \overline{nX}$ is compact for some $r \in \mathbb{N}_0$.
- 2) If $c(X) \neq X$ and $m(X) = X$, then $\cap_{n \in \mathbb{N}_0} \overline{nX} \neq X$.

The following conditions are equivalent:

- (i) $E(X)$ is a discrete valuation domain.
- (ii) $E(X)$ is local.
- (iii) X is topologically isomorphic either with a quotient group of $\mathbb{J}_p^*$ by a closed pure subgroup A with $\text{rad}(A) = \{0\}$, or with one of the groups $\mathbb{Q}_p$, $\mathbb{Z}_p$, or $\mathbb{Z}(p^n)$, where $p \in \mathbb{P}$ and $n \in \mathbb{N}$.

Proof. We first show that X satisfies the stated hypotheses if and only if X^* satisfies the hypotheses of Theorem 8. Let X be as in the enounce of our corollary. Then, clearly, X^* is residual and purely topologically indecomposable. Moreover, since

$$\overline{t(X^*)} / X^*[r] \cong (\overline{rX} / \cap_{n \in \mathbb{N}_0} \overline{nX})^*$$

[7, Ex. 3.8.7(b)], we deduce that $\overline{t(X^*)} / X^*[r]$ is discrete, and hence $X^*[r]$ is open in $\overline{t(X^*)}$. As $X^*[r] \subseteq t(X^*)$, it follows that $t(X^*)$ is open in $\overline{t(X^*)}$, whence we deduce that $t(X^*)$ is closed in $\overline{t(X^*)}$, and hence in X^* . Further, if $k(X^*) \neq \{0\}$ and $d(X^*) = \{0\}$, then $c(X) \neq X$ and $m(X) = X$. It follows from 2) that $\cap_{n \in \mathbb{N}_0} \overline{nX} \neq X$, and hence

$$t(X^*) = \overline{t(X^*)} = A(X^*, \cap_{n \in \mathbb{N}_0} \overline{nX}) \neq A(X^*, X) = \{0\}.$$

Conversely, assume X^* satisfies the hypotheses of Theorem 8. Then, clearly, X must be residual and purely topologically co-indecomposable. Moreover, since $t(X^*) = \cup_{n \in \mathbb{N}} X^*[n]$ is locally compact, it follows from [10, (5.28)] that $X^*[r]$ is open in $t(X^*)$ for some $r \in \mathbb{N}_0$, so $\overline{rX} / \cap_{n \in \mathbb{N}_0} \overline{nX}$ is compact. Further, by taking annihilators, it is easy to see that if $c(X) \neq X$ and $m(X) = X$, then $\cap_{n \in \mathbb{N}_0} \overline{nX} \neq X$.

To finish, it remains to observe that a subgroup G of $\mathbb{J}_p$ is pure in $\mathbb{J}_p$ if and only if its annihilator $A(\mathbb{J}_p^*, G)$ is pure in $\mathbb{J}_p^*$ [1, Corollary 7.6], and that $G^* \cong \mathbb{J}_p^* / A(\mathbb{J}_p^*, G)$ [10, (24.5)]. $\square$

6 Topologically indecomposable groups with the finite exchange property

This last section is concerned with extending the result of R.B. Warfield [22, Proposition 1] to the case of groups in $\mathcal{L}$. We first note that the so-called exchange property for modules, which appears in [22, Proposition 1], was introduced by P. Crawley and B. Jónsson in [6]. Later this notion was considered in some more general categories (see for example [20], [14] and [5]). We begin by specializing the corresponding definition to our situation.

Definition 3. Let $\mathcal{S}$ be a subclass of $\mathcal{L}$ closed under taking finitary topological direct products, closed subgroups, and topological isomorphisms. A group X in $\mathcal{S}$ is said to have the finite exchange property in $\mathcal{S}$ if X satisfies the following condition:

For any group Y in $\mathcal{S}$ and any its pair of topological direct sum decompositions

$$Y = X' \oplus G = Y_1 \oplus \cdots \oplus Y_n,$$

where $G, Y_1, \dots, Y_n \in \mathcal{S}$, $n \in \mathbb{N}_+$, and $X' \cong X$, there exist closed subgroups Y'_i and Z_i of Y_i such that $Y_i = Y'_i \oplus Z_i$ for all $i \in \{1, \dots, n\}$, and $Y = X' \oplus Y'_1 \oplus \cdots \oplus Y'_n$.

It is easy to see by use of duality that X has the finite exchange property in $\mathcal{S}$ if and only if X^* has the finite exchange property in the subclass $\mathcal{S}^*$ of $\mathcal{L}$ consisting of those groups $W \in \mathcal{L}$ for which W^* is in $\mathcal{S}$.

Theorem 9 (R.B. Warfield, Jr). *Let $\mathcal{D}$ be the subclass of $\mathcal{L}$ consisting of discrete groups. An indecomposable group $X \in \mathcal{D}$ has local endomorphism ring if and only if X has the finite exchange property in $\mathcal{D}$.*

By use of duality, we obtain:

Corollary 6. *Let $\mathcal{K}$ be the subclass of $\mathcal{L}$ consisting of compact groups. A topologically indecomposable group $X \in \mathcal{K}$ has local endomorphism ring of continuous endomorphisms if and only if X has the finite exchange property in $\mathcal{K}$.*

To continue, we require the following lemma which is a topological version of Lemma 1 in [20], see also [8, Lemma 2.6].

Lemma 7. *Let A be a group in $\mathcal{L}$, let M_1 and M_2 be closed subgroups of A such that $A = M_1 \oplus M_2$, and let $\pi_2: A \rightarrow M_2$ be the canonical projection onto M_2 . For any closed subgroup M of A , the decomposition $A = M_1 \oplus M$ holds if and only if the restricted map $\pi_2|_M: M \rightarrow M_2$ is a topological isomorphism. Moreover, if these two equivalent conditions are satisfied, then the canonical projection $\pi_M: A \rightarrow M$ with respect to decomposition $A = M_1 \oplus M$ is $(\pi_2|_M)^{-1} \circ \pi_2$.*

Proof. Assume $A = M_1 \oplus M$. Then $A = M_1 \dot{+} M$. It follows from [8, Lemma 2.6] that $\pi_2|_M: M \rightarrow M_2$ is an algebraic group isomorphism. Since $\pi_2|_M$ is clearly continuous, it only remains to show that $\pi_2|_M$ is open. To this end, pick an arbitrary open neighbourhood U of zero in M . Then $M_1 + U$ is open in $A = M_1 \oplus M$. Since

$\pi_2: A \rightarrow M_2$ is an open mapping, it follows that $(\pi_2|_M)(U) = \pi_2(M_1 + U)$ is open in M_2 . As the neighbourhood U was chosen arbitrarily, we conclude that $\pi_2|_M$ is open [10, (5.40)(b)].

Conversely, assume $\pi_2|_M: M \rightarrow M_2$ is a topological group isomorphism. By [8, Lemma 2.6] we have $A = M_1 \dot{+} M$ and $\pi_M = (\pi_2|_M)^{-1} \circ \pi_2$. Let ε_1 and ε_M be the idempotents associated with this decomposition of A . If $i_M: M \rightarrow A$ is the canonical injection, then $\varepsilon_M = i_M \circ \pi_M = i_M \circ (\pi_2|_M)^{-1} \circ \pi_2$, so ε_M is continuous, and hence $A = M_1 \oplus M$ by [4, Ch. III, §6, Proposition 2]. $\square$

We now come to the main result of this section concerning the considered problem of determining the groups $X \in \mathcal{L}$ with local ring $E(X)$, which extends the R.B. Warfield's result in Theorem 9 to the case of groups in $\mathcal{L}$. It should be noted that the mentioned result of R.B. Warfield was generalized by C.I. Walker and R.B. Warfield [20, Proposition 3] to additive categories with kernels, and was included in different books (see [14],[8] and [5]).

Using a similar technique to that in ([20],[14],[8] and [5]), we obtain the following

Theorem 10. *For a group $X \in \mathcal{L}$, the following conditions are equivalent:*

- (i) $E(X)$ is local.
- (ii) X is topologically indecomposable and has the finite exchange property in $\mathcal{L}$.

Proof. Clearly, we may consider $X \neq \{0\}$. Assume (i). Then X is topologically indecomposable by Lemma 2. To see that X has the finite exchange property in $\mathcal{L}$, suppose there is $Y \in \mathcal{L}$ such that

$$Y = X' \oplus G \quad \text{and} \quad Y = Y_1 \oplus \cdots \oplus Y_n,$$

where $X', G, Y_1, \dots, Y_n$ ($n \in \mathbb{N}_+$) are closed subgroups of Y and $X' \cong X$. In connection with these two decompositions, let $e: X' \rightarrow Y$ and $e_i: Y_i \rightarrow Y$ denote the corresponding injections, and let $p: Y \rightarrow X'$ and $p_i: Y \rightarrow Y_i$ denote the corresponding projections. It follows that

$$1_{X'} = \sum_{i=1}^n p \circ e_i \circ p_i \circ e.$$

Since $E(X')$ is local, one of the summands, say $\alpha = p \circ e_1 \circ p_1 \circ e$, is invertible in $E(X')$. Consider the endomorphism $\varepsilon \in E(Y)$ defined by $\varepsilon = e_1 \circ p_1 \circ e \circ \alpha^{-1} \circ p$. Then $\varepsilon^2 = \varepsilon$, so

$$Y = \ker(1_Y - \varepsilon) \dot{+} \ker(\varepsilon)$$

[3, Proposition 11.2], and hence

$$Y = \ker(1_Y - \varepsilon) \oplus \ker(\varepsilon) \tag{2}$$

because ε is continuous [4, Ch. III, §6, Proposition 2]. Let $\eta: \ker(1_Y - \varepsilon) \rightarrow Y$ be the canonical injection. Since $(1_Y - \varepsilon) \circ \eta = 0$, we get $\eta = \varepsilon \circ \eta$, whence $p_i \circ \eta = p_i \circ \varepsilon \circ \eta = 0$ for all $i > 1$, and hence $(p_2 + \cdots + p_n) \circ \eta = 0$. It follows that

$$\ker(1_Y - \varepsilon) = \text{im}(\eta) \subseteq \ker(p_2 + \cdots + p_n) = Y_1,$$

so

$$Y_1 = \ker(1_Y - \varepsilon) \dot{+} (Y_1 \cap \ker(\varepsilon)).$$

Now since the idempotents induced by this sum are restrictions of continuous idempotents induced by decomposition (2), we deduce from [4, Ch. III, §6, Proposition 2] that

$$Y_1 = \ker(1_Y - \varepsilon) \oplus (Y_1 \cap \ker(\varepsilon)), \quad (3)$$

and hence

$$Y = \ker(1_Y - \varepsilon) \oplus (Y_1 \cap \ker(\varepsilon)) \oplus Y_2 \oplus \cdots \oplus Y_n. \quad (4)$$

Let $\varphi: Y \rightarrow \ker(1_Y - \varepsilon)$ and $\psi: Y \rightarrow \ker(1_Y - \varepsilon)$ be the projections on $\ker(1_Y - \varepsilon)$ induced by decompositions (2) and (4) respectively, and let $\xi: Y_1 \rightarrow \ker(1_Y - \varepsilon)$ be the projection on $\ker(1_Y - \varepsilon)$ induced by decomposition (3). Then $\varepsilon = \eta \circ \varphi$, $\psi = \varphi \circ e_1 \circ p_1$ and $\xi = \varphi \circ e_1$. It follows that

$$(\varphi \circ e_1 \circ p_1 \circ e) \circ (\alpha^{-1} \circ p \circ \eta) = \varphi \circ \varepsilon \circ \eta = \varphi \circ \eta = 1_{\ker(1_Y - \varepsilon)}$$

and

$$\begin{aligned} (\alpha^{-1} \circ p \circ \eta) \circ (\varphi \circ e_1 \circ p_1 \circ e) &= \alpha^{-1} \circ p \circ \varepsilon \circ e_1 \circ p_1 \circ e \\ &= (\alpha^{-1} \circ p \circ e_1 \circ p_1 \circ e)(\alpha^{-1} \circ p \circ e_1 \circ p_1 \circ e) = 1_{X'}, \end{aligned}$$

proving that $\varphi \circ e_1 \circ p_1 \circ e: X' \rightarrow \ker(1_Y - \varepsilon)$ and $\alpha^{-1} \circ p \circ \eta: \ker(1_Y - \varepsilon) \rightarrow X'$ are mutually inverse isomorphisms. As

$$\psi|_{X'} = \psi \circ e = \varphi \circ e_1 \circ p_1 \circ e,$$

we deduce from Lemma 7 that

$$Y = X' \oplus (Y_1 \cap \ker(\varepsilon)) \oplus Y_2 \oplus \cdots \oplus Y_n.$$

Consequently, X has the finite exchange property in $\mathcal{L}$.

Now assume (ii). We shall show that if $E(X)$ is not a local ring, then X cannot have the finite exchange property in $\mathcal{L}$. So, suppose $E(X)$ is not local. Then there exist two non-invertible endomorphisms $f, g \in E(X)$ such that $1_X = f - g$. Consider the group $Y = Y_1 \oplus Y_2$, where $Y_1 \cong Y_2 \cong X$, and for $i \in \{1, 2\}$ let $\pi_i: Y \rightarrow Y_i$ denote the corresponding projections. Further, let $(f, g): X \rightarrow Y_1 \oplus Y_2$ and $(1_X, 1_X): X \rightarrow Y_1 \oplus Y_2$ be the homomorphisms satisfying

$$\pi_1 \circ (f, g) = f, \quad \pi_2 \circ (f, g) = g, \quad \text{and} \quad \pi_1 \circ (1_X, 1_X) = 1_X = \pi_2 \circ (1_X, 1_X).$$

Clearly, $(1_X, 1_X)$ is injective. Since $f - g = 1_X$, it is also clear that (f, g) is injective as well, and that $f \circ g = g + g \circ g = g \circ f$. Put $p_1 = \pi_1 - \pi_2$ and $p_2 = f \circ \pi_2 - g \circ \pi_1$. Then we have

$$\begin{aligned} p_1 \circ (f, g) &= 1_X, & p_2 \circ (f, g) &= 0, \\ p_1 \circ (1_X, 1_X) &= 0, & p_2 \circ (1_X, 1_X) &= 1_X, \\ \text{and} & & (f, g) \circ p_1 + (1_X, 1_X) \circ p_2 &= 1_Y. \end{aligned} \tag{5}$$

Hence we can write $Y = X' \dot{+} X''$, where $X' = (f, g)(X) = \ker((1_X, 1_X) \circ p_2)$ and $X'' = (1_X, 1_X)(X) = \ker((f, g) \circ p_1)$ [3, Proposition 11.2]. Since the idempotents $(f, g) \circ p_1$ and $(1_X, 1_X) \circ p_2$ are continuous, we conclude that $X', X'' \in \mathcal{L}$ and that $Y = X' \oplus X''$ [4, Ch. III, §6, Proposition 2]. It also follows from (5) that

$$(p_1|_{X'}) \circ (f, g) = 1_{X'} \quad \text{and} \quad (f, g) \circ (p_1|_{X'}) = 1_{X'},$$

so (f, g) induces a topological isomorphism from X onto X' . Analogously, $(1_X, 1_X)$ induces a topological isomorphism from X onto X'' .

Now, if the finite exchange property were to hold for X (and hence for X'), it would follow that there exist topological direct summands A_1 of Y_1 and A_2 of Y_2 such that

$$Y = X' \oplus X'' = X' \oplus A_1 \oplus A_2.$$

Since Y_1, Y_2, X' , and X'' are topologically indecomposable, we would have either $Y = X' \oplus Y_1$ or $Y = X' \oplus Y_2$. In the former case, it would follow from Lemma 7 that $\pi_2|_{X'}: X' \rightarrow Y_2$ is a topological isomorphism, and hence $g = \pi_2 \circ (f, g): X \rightarrow Y_2$ is a topological isomorphism as well. Similarly, in the second case we would get that $f = \pi_1 \circ (f, g)$ is a topological isomorphism. In either case, the conclusion is in contradiction with the assumption that f and g are non-invertible. This proves that X does not have the finite exchange property in $\mathcal{L}$. $\square$

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