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The family of cubic differential systems with two real and two complex distinct infinite singularities and invariant straight lines of the type (2,2,2)

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Abstract. We denote by $\mathbf{CSL_7}$ the family of cubic differential systems possessing invariant straight lines, finite and infinite, of total multiplicity exactly seven. In a sequence of papers the study of the subfamily of cubic systems belonging to $\mathbf{CSL_7}$ with 4 real distinct singular points at infinity was reached.

The goal of this article is to continue the study of the geometric configurations of invariant lines of $\mathbf{CSL_7}$ with two real and two complex distinct infinite singularities and invariant lines in the configuration of the type $(2, 2, 2)$. We proved that there exists only one configuration of invariant straight lines belonging to the class mentioned above. In addition, we construct invariant affine criteria for the realization of the obtained configuration.

Mathematics subject classification: 34C23, 34A34.
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1 Introduction

We consider here real polynomial differential systems

$$\frac{dx}{dt} = P(x, y), \qquad \frac{dy}{dt} = Q(x, y), \tag{1}$$

where $P,\ Q$ are polynomials in $x,\ y$ with real coefficients, i.e. $P,\ Q \in R[x, y]$. We call degree of a system (1) the integer $n = \max(\deg(P), \deg(Q))$. In particular we call cubic a differential system (1) with degree $n = 3$.

We are interested in polynomial systems (1) possessing algebraic invariant curves. An algebraic curve $f(x, y) = 0$ with $f(x, y) \in \mathbb{C}[x, y]$ is an invariant curve of a system of the form (1) where $P(x, y), Q(x, y) \in \mathbb{C}[x, y]$ if and only if there exists $K[x, y] \in \mathbb{C}[x, y]$ such that

$$X(f) = P(x, y)\frac{\partial f}{\partial x} + Q(x, y)\frac{\partial f}{\partial y} = f(x, y)K(x, y)$$

is an identity in $\mathbb{C}[x, y]$. Since $\mathbb{R} \subset \mathbb{C}$, any system (1) over $\mathbb{R}$ generates a system of differential equations over $\mathbb{C}$. Using the embedding $\mathbb{C}^2 \hookrightarrow \mathbb{P}_2(\mathbb{C})$, $(x, y) \mapsto [x : y :$

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$1] = [X : Y : Z]$ ($x = X/Z, y = Y/Z$ and $Z \neq 0$), we can compactify the differential equation $Q(x, y)dy - P(x, y)dx = 0$ to an associated differential equation over the complex projective plane.

In this work we consider a particular case of invariant algebraic curves, namely invariant straight lines of systems (1). A straight line over $\mathbb{C}$ is the locus $\{(x, y) \in \mathbb{C}^2 | f(x, y) = 0\}$ of an equation $f(x, y) = ux + vy + w = 0$ with $(u, v) \neq (0, 0)$ and $(u, v, w) \in \mathbb{C}^3$.

In view of the above definition of an invariant algebraic curve of a system (1), a line $f(x, y) = ux + vy + w = 0$ over $\mathbb{C}$ is an invariant line if and only if there exists $K(x, y) \in \mathbb{C}[x, y]$ which satisfies the following identity in $\mathbb{C}[x, y]$:

$$X(f) = uP(x, y) + vQ(x, y) = (ux + vy + w)K(x, y).$$

Notation 1. *Let us denote:*

- $CS = \{\ S \mid S$ *is a system* (1) *such that* $gcd(P, Q) = 1$ *and* $max(deg(P, Q)) = 3\}$;
- $CSL = \{\ S \in CS \mid S$ *possesses at least one invariant affine line or the line at infinity with multiplicity at least two* $\}$.

The set CS of cubic differential systems depends on 20 parameters and hence people began by studying particular subclasses of CS. Some of these subclasses are cubic systems having invariant straight lines. We associate to each system in CSL its configuration of invariant lines, i.e. the set of its invariant lines together with the real singular points of the system located on the union of these lines. The notion of *configuration of invariant lines* for a polynomial differential system was first introduced in [23].

Definition 1. [26] Consider a real planar polynomial differential system (1). The *configuration of invariant straight lines* of this system is the set of (complex) invariant straight lines (which may have real coefficients), including the line at infinity of the system, each endowed with its own multiplicity and together with all the real singular points of this system located on these invariant straight lines, each one endowed with its own multiplicity.

In analogous manner to as we view the phase portraits of systems on the Poincaré disc (see e.g. [10]), we can also view the configurations of real lines on the disc. To help imagining the full configurations, we complete the picture by drawing dashed lines whenever these are complex. On the class of CS the group of affine transformations and time rescaling acts. Since cubic systems depend on 20 parameters and since this group depends on 13 parameters, the class of cubic systems modulo this group action, actually depends on five parameters. It is clear that the configuration of invariant lines of a system is an affine invariant.

We mention here some papers on polynomial differential systems in CSL: [2, 3, 5–10, 13–16, 19–21] and [22].

The maximum number of invariant straight lines (including the line at infinity) for cubic systems with a finite number of infinite singularities is 9. In [14] the authors classified all cubic systems possessing the maximum number of invariant straight lines taking into account their multiplicities according to their configurations of invariant lines.

The existence of sufficiently many invariant straight lines of planar polynomial systems could be used for proving the integrability of such systems. During the past 15 years several articles were published on this theme (see for example [24, 25]).

Notation 2. *We shall denote by* $CSL_7^{2r2c\infty}$ *the class of cubic systems with four distinct singularities at infinity: two real and two complex, and invariant lines of total multiplicity seven.*

As we have two real and two complex infinite singularities and the total multiplicity of invariant lines (including the line at infinity) must be 7, then a cubic systems in $\mathbf{CSL}_7^{2r2c\infty}$ could have only one of the following four possible types of configurations of invariant lines:

$$(i)\ \mathfrak{T}=(3,3);\quad (ii)\ \mathfrak{T}=(3,1,1,1);\quad (iii)\ \mathfrak{T}=(2,2,2);\quad (iv)\ \mathfrak{T}=(2,2,1,1).\quad (2)$$

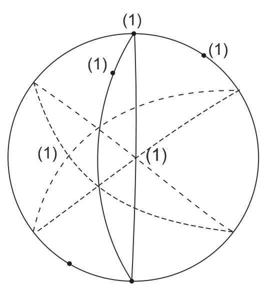
We remark that the cubic systems in $\mathbf{CSL}_7^{2r2c\infty}$ possessing the configurations of invariant lines of the type $\mathfrak{T}=(3,3)$ have already been investigated in [4], where the existence of 14 distinct configurations *Config. 7.1a – Config. 7.14a* of this type is determined. In addition the class of cubic systems in $\mathbf{CSL}_7^{2r2c\infty}$ possessing the configurations of invariant lines of the type $\mathfrak{T}=(3,1,1,1)$ was considered in [11]. For this subfamily of systems the existence of 42 distinct configurations *Config. 7.1b – Config. 7.42b* was proved.

Here we will focused on systems $CSL_7^{2r2c\infty}$ possessing the type of configuration $(2,2,2)$ and we denote such class of systems by $CSL_{(2,2,2)}^{2r2c\infty}$.

The problem which we solve in this article is the following: to construct all possible configurations of invariant straight lines, including the infinite one, for the class $CSL_{(2,2,2)}^{2r2c\infty}$.

Main Theorem. *A non-degenerate system* (S) *from the family* (1) *belongs to the class* $CSL_{(2,2,2)}^{2r2c\infty}$ *if an only if* $\mathcal{D}_1<0$, $\mathcal{D}_4\mathcal{K}_4\neq 0$, $\mathcal{V}_3=\mathcal{J}_6=\mathcal{J}_7=0$. *Moreover such a system (S) could be brought via an affine transformation and time rescaling to the canonical form*

$$\begin{aligned}\dot{x}=&2x(1-x)(1+x-sy),\quad s(s^2-9)\neq 0\\ \dot{y}=&-s+sx-y+sx^2-sy^2-sx^3-3x^2y\\ &+sxy^2-y^3.\end{aligned}\qquad (3)$$



Config. 7.1c

and it possesses the unique configuration Config. 7.1.c.

2 Some preliminary results

Consider real cubic systems, i.e. systems of the form:

$$\begin{aligned}\dot{x} &= p_0 + p_1(x,y) + p_2(x,y) + p_3(x,y) \equiv P(a,x,y),\\ \dot{y} &= q_0 + q_1(x,y) + q_2(x,y) + q_3(x,y) \equiv Q(a,x,y)\end{aligned} \tag{4}$$

with variables x and y and real coefficients. The polynomials p_i and q_i $(i = 0,1,2,3)$ are homogeneous polynomials of degree i in x and y:

$$\begin{aligned}&p_0 = a_{00}, \quad p_3(x,y) = a_{30}x^3 + 3a_{21}x^2y + 3a_{12}xy^2 + a_{03}y^3,\\ &p_1(x,y) = a_{10}x + a_{01}y, \quad p_2(x,y) = a_{20}x^2 + 2a_{11}xy + a_{02}y^2,\\ &q_0 = b_{00}, \quad q_3(x,y) = b_{30}x^3 + 3b_{21}x^2y + 3b_{12}xy^2 + b_{03}y^3,\\ &q_1(x,y) = b_{10}x + b_{01}y, \quad q_2(x,y) = b_{20}x^2 + 2b_{11}xy + b_{02}y^2.\end{aligned}$$

Let $a = (a_{00}, a_{10}, a_{01}, \dots, a_{03}, b_{00}, b_{10}, b_{01}, \dots, b_{03})$ be the 20-tuple of the coefficients of systems (4) and denote $R[a,x,y] = R[a_{00}, a_{10}, a_{01}, \dots, a_{03}, b_{00},$
$b_{10}, b_{01}, \dots, b_{03}, x, y]$.

It is known that on the set of polynomial systems (1), in particular on the set CS of all cubic differential systems (4), the group $\mathit{Aff}(2,\mathbb{R})$ of affine transformations of the plane acts [26]. For every subgroup $G \subseteq \mathit{Aff}(2,\mathbb{R})$ we have an induced action of G on CS. We can identify the set CS of systems (4) with a subset of R^{20} via the map $CS \longrightarrow \mathbb{R}^{20}$ which associates to each system (4) the 20-tuple $a = (a_{00}, a_{10}, a_{01}, \dots, a_{03}, b_{00}, b_{10}, b_{01}, \dots, b_{03})$ of its coefficients.

For the definitions of an affine or GL-comitant or invariant as well as for the definition of a T-comitant and CT-comitant we refer the reader to [23] (see also [1]). Here we shall only construct the necessary invariant polynomials (T-comitants) which are needed to detect the existence of invariant lines for the class of cubic systems with two distinct real and two complex infinite singularities and with exactly seven invariant straight lines including the line at infinity and counting multiplicities.

Let us consider the polynomials

$$\begin{aligned}C_i(a,x,y) &= yp_i(a,x,y) - xq_i(a,x,y) \in R[a,x,y],\ i = 0,1,2,3,\\ D_i(a,x,y) &= \frac{\partial}{\partial x}p_i(a,x,y) + \frac{\partial}{\partial y}q_i(a,x,y) \in R[a,x,y],\ i = 1,2,3.\end{aligned}$$

As it was shown in [27] the polynomials

$$\big\{C_i(a,x,y),\ D_1(a),\ D_2(a,x,y),\ D_3(a,x,y), i = 1,2,3\big\} \tag{5}$$

of degree one in the coefficients of systems (4) are GL-comitants of these systems.

Notation 3. Let $f,\, g \in R[a,x,y]$ and

$$(f,g)^{(k)} = \sum_{h=0}^{k} (-1)^h \binom{k}{h} \frac{\partial^k f}{\partial x^{k-h}\partial y^h}\, \frac{\partial^k g}{\partial x^h \partial y^{k-h}}.$$

$(f,g)^{(k)} \in R[a,x,y]$ is called *the transvectant of index k* of (f,g) (cf.[12, 17]).

Let us apply a translation $x = x' + x_0$, $y = y' + y_0$ to the polynomials $P(a,x,y)$ and $Q(a,x,y)$. We obtain $\widetilde{P}(\tilde{a}(a,x_0,y_0),x',y') = P(a,x'+x_0,y'+y_0)$, $\widetilde{Q}(\tilde{a}(a,x_0,y_0),x',y') = Q(a,x'+x_0,y'+y_0)$. We construct the following polynomials

$$\Omega_i(a,x_0,y_0) \equiv \operatorname{Res}_{x'}\Big(C_i\big(\tilde{a}(a,x_0,y_0),x',y'\big), C_0\big(\tilde{a}(a,x_0,y_0),x',y'\big)\Big)/(y')^{i+1},$$
$$\Omega_i(a,x_0,y_0) \in \mathbf{R}[a,x_0,y_0],\ (i=1,2,3)$$

and we denote

$$\widetilde{\mathcal{G}}_i(a,x,y) = \Omega_i(a,x_0,y_0)|_{\{x_0=x,\ y_0=y\}} \in \mathbf{R}[a,x,y] \quad (i=1,2,3).$$

Remark 1. We note that the constructed polynomials $\tilde{\mathcal{G}}_1(a,x,y)$, $\tilde{\mathcal{G}}_2(a,x,y)$ and $\tilde{\mathcal{G}}_3(a,x,y)$ are affine comitants of systems (4) and are homogeneous polynomials in the coefficients $a_{00},\dots,b_{03}$ and non-homogeneous in x,y and

$$\begin{array}{lll} \deg_a \widetilde{\mathcal{G}}_1 = 3, & \deg_a \widetilde{\mathcal{G}}_2 = 4, & \deg_a \widetilde{\mathcal{G}}_3 = 5, \\ \deg_{(x,y)} \widetilde{\mathcal{G}}_1 = 8, & \deg_{(x,y)} \widetilde{\mathcal{G}}_2 = 10, & \deg_{(x,y)} \widetilde{\mathcal{G}}_3 = 12. \end{array}$$

Notation 4. *Let $\mathcal{G}_i(a,X,Y,Z)$ $(i=1,2,3)$ be the homogenization of $\tilde{\mathcal{G}}_i(a,x,y)$,* i.e.

$$\mathcal{G}_1(a,X,Y,Z) = Z^8\widetilde{\mathcal{G}}_1(a,X/Z,Y/Z),$$
$$\mathcal{G}_2(a,X,Y,Z) = Z^{10}\widetilde{\mathcal{G}}_2(a,X/Z,Y/Z),$$
$$\mathcal{G}_3(a,X,Y,Z) = Z^{12}\widetilde{\mathcal{G}}_3(a,X/Z,Y/Z),$$

and $\mathcal{H}(a,X,Y,Z) = \gcd\Big(\mathcal{G}_1(a,X,Y,Z),\ \mathcal{G}_2(a,X,Y,Z),\ \mathcal{G}_3(a,X,Y,Z)\Big)$ in $\mathbb{R}[a,X,Y,Z]$.

The geometrical meaning of these affine comitants is given by the two following lemmas (see [14]):

Lemma 1. *The straight line $\mathcal{L}(x,y) \equiv ux+vy+w=0$, $u,v,w \in \mathbb{C}$, $(u,v) \neq (0,0)$ is an invariant line for a cubic system* (4) *if and only if the polynomial $\mathcal{L}(x,y)$ is a common factor of the polynomials $\widetilde{\mathcal{G}}_1(x,y)$, $\widetilde{\mathcal{G}}_2(x,y)$ and $\widetilde{\mathcal{G}}_3(x,y)$ over $\mathbb{C}$, i.e.*

$$\widetilde{\mathcal{G}}_i(x,y) = (ux+vy+w)\widetilde{W}_i(x,y) \quad (i=1,2,3),$$

where $\widetilde{W}_i(x,y) \in \mathbb{C}[x,y]$.

Lemma 2. *Consider a cubic system* (4) *and let $a \in \mathbb{R}^{20}$ be its 20-tuple of coefficients.*

1) If $\mathcal{L}(x,y) \equiv ux+vy+w=0$, $u,v,w \in \mathbb{C}$, $(u,v) \neq (0,0)$ is an invariant straight line of multiplicity k for this system then $[\mathcal{L}(x,y)]^k \mid \gcd(\tilde{\mathcal{G}}_1,\tilde{\mathcal{G}}_2,\tilde{\mathcal{G}}_3)$ in $\mathbb{C}[x,y]$, i.e. there exist $W_i(a,x,y) \in \mathbb{C}[x,y]$ $(i=1,2,3)$ such that

$$\widetilde{\mathcal{G}}_i(a,x,y) = (ux+vy+w)^k W_i(a,x,y), \quad i=1,2,3.$$

2) If the line $l_\infty : Z=0$ is of multiplicity $k>1$, then $Z^{k-1} \mid \gcd(\mathcal{G}_1,\mathcal{G}_2,\mathcal{G}_3)$, i.e. we have $Z^{k-1} \mid H(a,X,Y,Z)$.

In order to define the invariant polynomials we need, we first construct the following comitants of second degree with respect to the coefficients of the initial systems (4):

$$
\begin{array}{lll}
S_1 = (C_0, C_1)^{(1)}, & S_{10} = (C_1, C_3)^{(1)}, & S_{19} = (C_2, D_3)^{(1)}, \\
S_2 = (C_0, C_2)^{(1)}, & S_{11} = (C_1, C_3)^{(2)}, & S_{20} = (C_2, D_3)^{(2)}, \\
S_3 = (C_0, D_2)^{(1)}, & S_{12} = (C_1, D_3)^{(1)}, & S_{21} = (D_2, C_3)^{(1)}, \\
S_4 = (C_0, C_3)^{(1)}, & S_{13} = (C_1, D_3)^{(2)}, & S_{22} = (D_2, D_3)^{(1)}, \\
S_5 = (C_0, D_3)^{(1)}, & S_{14} = (C_2, C_2)^{(2)}, & S_{23} = (C_3, C_3)^{(2)}, \\
S_6 = (C_1, C_1)^{(2)}, & S_{15} = (C_2, D_2)^{(1)}, & S_{24} = (C_3, C_3)^{(4)}, \\
S_7 = (C_1, C_2)^{(1)}, & S_{16} = (C_2, C_3)^{(1)}, & S_{25} = (C_3, D_3)^{(1)}, \\
S_8 = (C_1, C_2)^{(2)}, & S_{17} = (C_2, C_3)^{(2)}, & S_{26} = (C_3, D_3)^{(2)}, \\
S_9 = (C_1, D_2)^{(1)}, & S_{18} = (C_2, C_3)^{(3)}, & S_{27} = (D_3, D_3)^{(2)}.
\end{array}
$$

In order to determine the necessary conditions for the existence and the numbers of couples of parallel invariant straight lines which a cubic system could have (see Theorem 1) we use here the following invariant polynomials constructed in [14] and [5]:

$$
\begin{aligned}
\mathcal{V}_1(a,x,y) =& S_{23} + 2D_3^2, \quad \mathcal{V}_2(a,x,y) = S_{26}, \quad \mathcal{V}_3(a,x,y) = 6S_{25} - 3S_{23} - 2D_3^2, \\
\mathcal{V}_4(a,x,y) =& C_3\left[(C_3, S_{23})^{(4)} + 36\,(D_3, S_{26})^{(2)}\right], \ \mathcal{V}_5(a,x,y) = 6C_3(9A_5 - 7A_6) + \\
& 2D_3(4T_{16} - T_{17}) - 3T_3(3A_1 + 5A_2) + 3A_2T_4 + 36T_5^2 - 3T_{44}, \\
\mathcal{U}_1(a,x,y) =& S_{24} - 4S_{27}, \\
\mathcal{U}_2(a,x,y) =& 6(S_{23} - 3S_{25}, S_{26})^{(2)} - 3S_{23}(S_{24} - 8S_{27}) - 24S_{26}^2 + 2C_3(C_3, S_{23})^{(4)} + \\
& 24D_3(D_3, S_{26})^{(1)} + 24D_3^2S_{27}.
\end{aligned}
$$

In this article we shall use the following new polynomials :

$$
\begin{aligned}
&\mathcal{J}_6 = T_8; \\
&\mathcal{J}_7 = 24A_3T_1^2T_{11} + 6T_5T_{136} + 3T_5T_{137} + 16T_2T_{11}T_{25} - 48T_{19}T_{47} + 3T_{16}T_{74}.
\end{aligned}
$$

Here the polynomials

$A_1 = S_{24}/288$, $A_2 = S_{27}/72$, $A_3 = (72D_1A_2 + (S_{22}, D_2)^{(1)})/24$ are affine invariants and

$$
\begin{aligned}
T_1 =& C_3, T_2 = D_3, T_3 = S_{23}/18, \quad T_4 = S_{25}/6, \quad T_5 = S_{26}/72, \\
T_6 =& (3C_1D_3^2 - 27C_1T_3 + 54C_1T_4 + 4C_3D_2^2 - 2C_3S_{14} + \\
& + 16C_3S_{14} - 4C_2D_2D_3 + 2C_2S_{17} + 12C_2S_{21} - 4C_2S_{19})/2^4/3^2, \\
T_{11} =& (D_3^2, C_2)^{(2)} - 9(T_3, C_2)^{(2)} + 18(T_4, C_2)^{(2)} - 6(D_3^2, D_2)^{(1)} + \\
& + 54(T_3, D_2)^{(1)} - 108(T_4, D_2)^{(1)} + 12D_2S_{26} - 12(S_{26}, C_2)^{(1)} + \\
& + 432C_2A_1 - 2160C_2A_2)/2^7/3^4,
\end{aligned}
$$

$$T_{16} = (S_{23}, D_3)^{(2)}/2^6 3^3, T_{17} = (S_{26}, D_3)^{(1)}/2^5/3^3,$$
$$\begin{aligned} T_{25} =&(15552A_2C_1C_3 + D_3^2D_2^2 - 81D_2^2T_3 - 54D_2^2T_4 + 12D_3D_2S_{17} + 8D_3D_2S_{19}+ \\ &+ 16[(C_2, D_3)^{(1)}]^2 - 5184C_1D_3T_5 + 2592C_2D_2T_5 - 72C_3D_2S_{20})/2^6/3^4, \end{aligned}$$
$$\begin{aligned} T_{74} =&(2187T_3^2C_0 + 8748T_4^2C_0 + 20736T_{11}C_2^2 - 62208T_{11}C_1C_3+ \\ &+ 108C_3D_1D_2D_3^2 - 8C_2D_2^2D_3^2 - 54C_2D_1D_3^3 + 6C_1D_2D_3^3+ \\ &+ 27C_0D_3^4 - 54C_3D_3^2S_8 + 108C_3D_3^2S_9 + 27C_2D_3^2S_{11} - 27C_2D_3^2S_{12}+ \\ &+ 4C_2D_3^2S_{14} - 32C_2D_3^2S_{15} + 54D_1D_3^2S_{16} - 3C_1D_3^2S_{17} + 6C_1D_3^2S_{19}- \\ &- 9T_3(54C_0(18T_4 + D_3^2) + 54C_3(2D_1D_2 - S_8 + 2S_9) - C_2(8D_2^2+ \\ &+ 54D_1D_3 - 27S_{11} + 27S_{12} - 4S_{14} + 32S_{15}) + 54D_1S_{16} + 3C_1(2D_2D_3- \\ &- S_{17}+2S_{19}-6S_{21}))-576T_6(2D_2D_3-S_{17}+2S_{19}-6S_{21})-18C_1D_3^2S_{21}+ \\ &+ 18T_4(6C_1D_2D_3 + 54C_0D_3^2 + 54C_3(2D_1D_2 - S_8 + 2S_9) - C_2(8D_2^2+ \\ &+ 54D_1D_3 - 27S_{11} + 27S_{12} - 4S_{14} + 32S_{15}) + 54D_1S_{16} - 3C_1S_{17}+ \\ &+ 6C_1S_{19} - 18C_1S_{21}))/2^8/3^4, \quad T_{19} = (T_6, C_3)^{(1)}/2, \end{aligned}$$
$$T_{44} = ((S_{23}, C_3)^{(1)}, D_3)^{(2)}, T_{47} = (T_{11}, C_3)^{(2)},$$
$$T_{136} = \big(T_{74}, C_3\big)^{(2)}/24, \quad T_{137} = \big(T_{74}, D_3\big)^{(1)}/6.$$

are T-comitants of cubic systems (4) (see [23] for the definition of a T-comitant). We note that in the above invariant polynomials we keep the notations introduced in [5].

In [14] all the possible configurations of invariant lines are determined in the case, when the total multiplicity of these lines (including the line at infinity) equals nine. All possible configurations of invariant lines of total multiplicity eight (including the line at infinity) are determined in [3, 5–7, 9].

In the above mentioned articles several lemmas concerning the number of triplets and/or couples of parallel invariant straight lines which a cubic system could have are proved. Using these lemmas the following theorem is obtained:

Theorem 1. *If a cubic system* (4) *possesses a given number of triplets or/and couples of invariant parallel lines real or/and complex, then the following conditions are satisfied, respectively:*

- *(i)* *two triplets* $\Rightarrow \mathcal{V}_1 = \mathcal{V}_2 = \mathcal{U}_1 = 0$;
- *(ii)* *one triplet and one couple* $\Rightarrow \mathcal{V}_4 = \mathcal{V}_5 = \mathcal{U}_2 = 0$;
- *(iii)* *one triplet* $\Rightarrow \mathcal{V}_4 = \mathcal{U}_2 = 0$;
- *(iv)* *3 couples* $\Rightarrow \mathcal{V}_3 = 0$;
- *(v)* *2 couples* $\Rightarrow \mathcal{V}_5 = 0$.

Remark 2. The above conditions depend only on the coefficients of the cubic homogeneous parts of systems (4).

We rewrite systems (4) using different notations:

$$\begin{aligned} \dot{x} &= a + cx + dy + gx^2 + 2hxy + ky^2 + px^3 + 3qx^2y + 3rxy^2 + sy^3 \equiv P(x, y), \\ \dot{y} &= b + ex + fy + lx^2 + 2mxy + ny^2 + tx^3 + 3ux^2y + 3vxy^2 + wy^3 \equiv Q(x, y). \end{aligned} \tag{6}$$

Let $L(x,y) = Ux + Vy + W = 0$ be an invariant straight line of this family of cubic systems. Then, we have

$$Up(x,y) + Vq(x,y) = (Ux + Vy + W)(Ax^2 + 2Bxy + Cy^2 + Dx + Ey + F),$$

and this identity provides the following 10 equations:

$$\begin{array}{ll} Eq_1 = (p - A)U + tV = 0, & Eq_6 = (2h-E)U+(2m-D)V-2BW=0, \\ Eq_2 = (3q - 2B)U + (3u - A)V = 0, & Eq_7 = kU + (n - E)V - CW = 0, \\ Eq_3 = (3r - C)U + (3v - 2B)V = 0, & Eq_8 = (c - F)U + eV - DW = 0 \\ Eq_4 = (s - C)U + Vw = 0, & Eq_9 = dU + (f - F)V - EW = 0, \\ Eq_5 = (g - D)U + lV - AW = 0, & Eq_{10} = aU + bV - FW = 0. \end{array} \tag{7}$$

It is well known that the infinite singularities (real or complex) of systems (6) are determined by the linear factors of the polynomial

$$C_3 = yp_3(x,y) - xq_3(x,y).$$

Remark 3. Let $C_3 = \prod_{i=1}^{4}(\alpha_i x + \beta_i y)$, $i = 1,2,3,4$. Then $[\beta_i : -\alpha_i : 0]$ are singular points at infinity. Hence invariant affine lines must be of the form $Ux + Vy + W = 0$ with (U,V) among (α_i, β_i). In this case, for any fixed (α_i, β_i), for a specific value of W, six equations among (7) become linear with respect to the parameters $\{A, B, C, D, E, F\}$ (with the corresponding non-zero determinant) and we can determine their values, which annihilate some of the equations (7). So in what follows, for each direction given by (α_i, β_i), we will examine only non-zero equations containing the last parameter W.

For the proof of the Main Theorem we will consider the following homogeneous cubic systems associated to systems (6):

$$\dot{x} = p_3(x,y), \quad \dot{y} = q_3(x,y). \tag{8}$$

Clearly in the case of two real and two complex distinct infinite singularities the polynomial $C_3(x,y)$ has four distinct linear factors over C: two of them being real and two complex.

According to [14] (see also [18]) we have the following result.

Lemma 3. *If a cubic system* (6) *has 2 real and 2 complex distinct infinite singularities, then its associated homogeneous cubic systems could be brought via a linear transformation to the canonical form*

$$\begin{cases} x' = (u+1)x^3 + (s+v)x^2y + rxy^2, & C_3 = x(sx - y)(x^2 + y^2), \\ y' = -sx^3 + ux^2y + vxy^2 + (r-1)y^3, & rs(r+s) \neq 0 \end{cases} \tag{9}$$

for which the necessary invariant condition $\mathcal{D}_1 < 0$ *is satisfied.*

3 The proof of the Main Theorem

Assuming that cubic systems in the family (6) possess four distinct infinite singularities, two real and two complex, according to Lemma 3 via a linear transformation they could be brought to the family of systems

$$\begin{aligned}\dot{x} =& a + cx + dy + gx^2 + 2hxy + ky^2 + (u+1)x^3 + (s+v)x^2y + rxy^2,\\ \dot{y} =& b + ex + fy + lx^2 + 2mxy + ny^2 - sx^3 + ux^2y + vxy^2 + (r-1)y^3\end{aligned} \tag{10}$$

with $C_3 = x(sx+y)(x^2+y^2)$ and the condition $\mathcal{D}_1 < 0$ holds.

Since systems with the configuration of the type $\mathfrak{T} = (2,2,2)$ could only possess three couples of parallel invariant lines, according to Theorem 1 the condition $\mathcal{V}_3 = 0$ is necessary for systems (10). Taking the corresponding associated homogeneous systems (9) we force the condition $\mathcal{V}_3 = 0$ to be satisfied.

On the other hand in paper [5] the canonical form of (9) subject to the condition $\mathcal{V}_3 = 0$ was constructed . This canonical form is the following ([5, systems (77)]):

$$\begin{aligned}\dot{x} =& -2x^3 + 2sx^2y,\\ \dot{y} =& -sx^3 - 3x^2y + sxy^2 - y^3.\end{aligned} \tag{11}$$

Taking into consideration (11) and systems (6), applying an affine transformation and a time rescaling, these systems can be brought to systems belonging to the following family ($g = n = 0$):

$$\begin{aligned}\dot{x} =& a + cx + dy + 2hxy + ky^2 - 2x^3 + 2sx^2y,\\ \dot{y} =& b + ex + fy + lx^2 + 2mxy - sx^3 - 3x^2y + sxy^2 - y^3.\end{aligned} \tag{12}$$

According to [10, Remark 2.13] we have the following

Remark 4. If the perturbed systems have a couple (respectively a triplet) of parallel lines in the direction $Ux + Vy = 0$, then the respective cubic homogeneous systems (8) associated to systems (6) necessarily have the invariant line $Ux + Vy = 0$ of the multiplicity two (respectively three).

3.1 Construction of the cubic systems possessing configuration or potential configuration (2,2,2)

For homogeneous cubic systems (11) we have

$$H(\tilde{a}, X, Y, Z) = \gcd(G_1, G_2, G_3) = 2X^2(sX+Y)(X^2+Y^2)^2. \tag{13}$$

So the above systems possess three couples of invariant straight lines and by Remark 4 systems (12) could possesses three couples of invariant lines only in the directions $x = 0$ and $y = \pm ix$, and namely: two real lines in the direction $x = 0$ and two complex invariant lines in the directions $y = \pm ix$. In what follows we will examine each one of the above mentioned directions.

(a) The direction $x = 0$. In this case, according to (7) and Remark 3, we obtain:

$$U = 1, V = 0, A = -2, B = s, C = 0, D = 2W, E = 2(h - sW), F = c - 2W^2,$$
$$Eq_7 = k,\ Eq_9 = d - 2hW + 2sW^2, Eq_{10} = a - cW + 2W^3.$$

So, to have exactly two parallel invariant affine lines in this direction (i.e. to have exactly two solutions of W) it is necessary and sufficient that $Eq_7 = 0, s \neq 0$ and

$$R_W^{(0)}(Eq_9, Eq_{10}) = R_W^{(1)}(Eq_9, Eq_{10}) = 0.$$

So $k = 0$ and we calculate $R_W^{(1)}(Eq_9, Eq_{10}) = 4(2h^2 - ds - cs^2) = 0$. This yields $d = \dfrac{2h^2 - cs^2}{s}$ and we get $R_W^{(0)}(Eq_9, Eq_{10}) = 8(chs^2 - 2h^3 + as^3)^2/s^3 = 0$ which implies $a = \dfrac{h(2h^2 - cs^2)}{s^3}$. In this case we obtain

$$Eq_9 = -\frac{1}{s}(cs^2 - 2h^2 + 2hsW - 2s^2W^2), \quad Eq_{10} = \frac{h + sW}{s^2} Eq_9$$

and since $s \neq 0$ we conclude that the equations $Eq_9 = 0$ and $Eq_{10} = 0$ have two common solutions. These solutions could be either real or complex or coinciding depending on the value of the expression $2cs^2 - 3h^2$ because we have

$$\text{Discrim}[cs^2 - 2h^2 + 2hsW - 2s^2W^2, W] = 4s^2(2cs^2 - 3h^2).$$

Thus we conclude that if for systems (9) the following conditions

$$s \neq 0, \quad k = 0, \quad d = \frac{2h^2 - cs^2}{s}, \quad a = \frac{h(2h^2 - cs^2)}{s^3} \tag{14}$$

hold, then these systems possess in the direction $x = 0$ two invariant lines which could be either real or complex or coinciding.

(b) The directions $x \pm iy = 0$. Considering the equations (7), Remark 3 and the conditions (14), for $U = 1$ and $V = \pm i$ we obtain:

$$U = 1, V = \pm i, A = -2 \mp is, B = (s \mp i)/2, C = -1, D = 2W \pm i(sW + l),$$
$$E = l + 2h \pm i(2m - W), F = c - 2W^2 \pm i(e - lW - sW^2),$$
$$Eq_7 = 2m \mp i(l + 2h),$$
$$Eq_9 = \frac{2h^2}{s} - cs + e \pm i(f - c) - 2(l + h \pm im)W + (\pm 3i - s)W^2,$$
$$Eq_{10} = (2h^3 - chs^2)/s^3 \pm ib - (c \pm ie)W \pm ilW^2 + (2 \pm is)W^3.$$

As the parameters of cubic systems are real, clearly $Eq_7 = 0$ implies $m = 0$, $l = -2h$. Considering the relations (determined at this moment)

$$s \neq 0,\ k = 0,\ d = \frac{2h^2 - cs^2}{s},\ a = \frac{h(2h^2 - cs^2)}{s^3},\ m = 0,\ l = -2h \tag{15}$$

we calculate $R_W^{(0)}(Eq_9, Eq_{10})$ and $R_W^{(1)}(Eq_9, Eq_{10})$. So we have $R_W^{(1)} = (U_1 + iU_2)/s$, where

$$U_1 = 3cs + 6fs + 6h^2s - es^2 - 5cs^3 - fs^3,$$
$$U_2 = -12h^2 + 3es + 7cs^2 + 5fs^2 + 2h^2s^2 - cs^4.$$

As it was mentioned earlier, in order to have exactly two parallel lines in the directions $y = \pm ix$ it is necessary and sufficient that $R_W^{(0)}(Eq_9, Eq_{10}) = R_W^{(1)}(Eq_9, Eq_{10}) = 0$. Therefore $R_W^{(1)} = 0 \Leftrightarrow U_1 = U_2 = 0$ and we obtain:

$$e = -\frac{-72h^2 + 27cs^2 - 6h^2s^2 + 12cs^4 - 2h^2s^4 + cs^6}{2s(9+s^2)},$$
$$f = \frac{-9c - 6h^2 + 8cs^2 - 2h^2s^2 + cs^4)}{2(9+s^2)}.$$

Taking into consideration the above conditions we calculate:

$$R_W^{(0)} = (U_3 + iU_4)/(-s^6(s-3i)^3(s+3i)^2),$$

where

$$\begin{aligned}U_3 =& -162h^3 + 81chs^2 - 108h^3s^2 + 54bs^4 + 45chs^4 - 8h^3s^4 + 6bs^6 + \\ & 13chs^6 - 2h^3s^6 + chs^8,\\ U_4 =& -108h^3s - 81bs^3 + 81chs^3 - 6h^3s^3 + 2h^3s^5 + bs^7 - chs^7.\end{aligned}$$

So $R_W^{(0)} = 0 \Leftrightarrow U_3 = U_4 = 0$ and we calculate

$$Res_b(U_3, U_4) = -h(s^2-9)s^3(1+s^2)(9+s^2)\Psi, \text{ where}$$
$$\Psi = cs^2(9+s^2)^2 - 2h^2(81 + 9s^2 + s^4).$$

We claim that for $h = 0$ or $s^2 - 9 = 0$ we arrive at the systems possessing invariant lines of total multiplicity 8. Indeed, assuming $h = 0$ we obtain

$$U_3 = 6bs^4(9+s^2), \quad U_4 = b(s^2-9)s^3(9+s^2)$$

and clearly the condition $U_3 = U_4 = 0$ implies $b = 0$. Then considering the above determined conditions we arrive at the following systems

$$\begin{aligned}\dot{x} =& (c - 2x^2)(x - sy),\\ \dot{y} =& -cs(3+s^2)x/2 + c(s^2-1)y/2 - sx^3 - 3x^2y + sxy^2 - y^3\end{aligned}$$

which possess the following 7 invariant affine lines:

$$c - 2x^2 = 0,\ sx + y = 0,\ c(s+i)^2 + 2(x+iy)^2 = 0,\ c(s-i)^2 + 2(x-iy)^2 = 0.$$

So considering the line at infinity we get 8 invariant lines and our claim is proved in the case $h = 0$.

Next we consider the case $s = \pm 3$. Since we may assume $s > 0$ (due to the rescaling $y \to -y$) we examine the case $s = 3$. Then we obtain $U_4 = 0$ and $U_3 = 324(27b + 63ch - 10h^3)$ and therefore the condition $U_4 = 0$ implies $b = -h(63c - 10h^2)/27$. Then we arrive at the following systems

$$\begin{aligned}\dot{x} =&(2h^2 - 9c + 6hx + 18x^2)(h - 3x + 9y)/27,\\ \dot{y} =& - h(63c - 10h^2)/27 + (27c - 4h^2)x/3+\\ &(6c - h^2)y/3 - 2hx^2 + 3x^3 - 3x^2y - 3xy^2 - y^3\end{aligned}$$

which besides the invariant line at infinity, possess the following 7 invariant affine lines:

$$\begin{gathered}2h - 9x + 3y = 0,\ 9c - 2h^2 + 6hx - 18x^2 = 0,\\ (36 - 27i)c - (6 - 4i)h^2 + (3 - 3i)h(x - iy) + 9(x - iy)^2 = 0,\\ (36 + 27i)c - (6 + 4i)h^2 + (3 + 3i)h(x + iy) + 9(x + iy)^2.\end{gathered}$$

Therefore our claim is completely proved.

So it remains to examine the case $\Psi = 0$ which gives us

$$c = \frac{2h^2(81 + 9s^2 + s^4)}{s^2(9 + s^2)^2}$$

and then we have

$$U_3 = \frac{bs^2(9 + s^2)^2 + 6h^3(2s^2 - 9)}{9 + s^2}, \quad U_4 = \frac{s(s^2 - 9)}{6s^2}U_3.$$

Evidently the condition $U_3 = U_4 = 0$ yields $b = -\dfrac{6h^3(2s^2 - 9)}{s^2(9 + s^2)^2}$ and as a result we arrive at the 2-parameter family of systems

$$\begin{aligned}\dot{x} =&2\Big(x + \frac{hs}{9 + s^2}\Big)\Big(x + \frac{9h}{s(9 + s^2)}\Big)\Big(\frac{h}{s} - x + sy\Big),\\ \dot{y} =& - \frac{6h^3(2s^2 - 9)}{s^2(9 + s^2)^2} - \frac{9h^2(5s^2 - 9)}{s(9 + s^2)^2}x + \frac{h^2(s^2 - 9)(4s^2 - 9)}{s^2(9 + s^2)^2}y - 2hx^2-\\ &sx^3 - 3x^2y + sxy^2 - y^3.\end{aligned} \tag{16}$$

We observe that for these systems the condition $h(s^2 - 9) \neq 0$ must hold, otherwise we get either homogeneous cubic systems (for $h = 0$) or systems possessing invariant lines of total multiplicity 8 (for $s = \pm 3$).

Therefore we can apply to systems (16) the following transformation

$$x_1 = \alpha x + \frac{s^2}{s^2 - 9}, \quad y_1 = \alpha y + \frac{3s}{s^2 - 9}, \quad t_1 = \frac{1}{\alpha^2}, \quad \alpha = \frac{s(9 + s^2)}{h(s^2 - 9)}$$

and we arrive at the one-parameter family of systems

$$\begin{aligned}\dot{x} =&2x(1 - x)(1 + x - sy),\\ \dot{y} =& - s + sx - y + sx^2 - sy^2 - sx^3 - 3x^2y + sxy^2 - y^3,\end{aligned} \tag{17}$$

which coincide with systems (3) given in the Main Theorem.

So we proved the following lemma.

Lemma 4. *A system* (12) *possesses invariant lines in the configuration* $(2,2,2)$ *if and only if the following conditions are satisfied:*

$$\begin{gathered} sh(s^2-9)\neq 0,\ k=m=0,\ d=\frac{18h^2s}{(9+s^2)^2},\ l=-2h,\ e=\frac{9h^2(9-5s^2)}{s(9+s^2)^2},\\ f=-\frac{h^2(81-45s^2+4s^4)}{s^2(9+s^2)^2},\ c=\frac{2h^2(81+9s^2+s^4))}{s^2(9+s^2)^2},\\ b=\frac{6h^3(9-2s^2)}{s^2(9+s^2)^2},\ a=\frac{18h^3}{s(9+s^2)^2}. \end{gathered} \tag{18}$$

Next we construct the invariant conditions corresponding to (18).

First of all providing that for systems (12) the conditions (18) are satisfied we calculate

$$\mathcal{D}_4 = 2304s(9+s^2),\quad \mathcal{K}_4=\frac{2}{9}h(s^2-9)x(x^2+y^2)$$

and we deduce that the condition $sh(s^2-9)\neq 0$ is equivalent to $\mathcal{D}_4\mathcal{K}_4\neq 0$.

Next we evaluate the invariant polynomial $\mathcal{J}_6$:

$$\mathcal{J}_6=\frac{1}{3}\big[(3+s^2)x-2sy\big]\big[2mx^4-2(l+2h)x^3y-(3k+2m)x^2y^2-ky^4\big].$$

We observe that the condition $\mathcal{J}_6=0$ forces the second factor to vanish and evidently this is equivalent to $k=m=0$ and $l=-2h$, i.e. we get the conditions provided in (18) for these three parameters.

Taking into consideration the conditions $\mathcal{D}_4\mathcal{K}_4\neq 0$ and $\mathcal{J}_6=0$ for system (12) we calculate

$$\mathcal{J}_7=\sum_{k=0}^{7}\widetilde{C}_k(a,b,c,d,e,f,h,s)x^{7-k}y^k.$$

The condition $\mathcal{J}_7=0$ implies the system $\widetilde{C}_k(a,b,c,d,e,f,h,s)=0$ $(k=0,\dots,7)$ of eight polynomial equations. It is not too difficult to determine that this system of equations gives us the unique solution (a,b,c,d,e,f) for these parameters and the obtained expressions coincide with the corresponding expressions provided by Lemma 4.

Thus we arrive at the invariant conditions provided by Main Theorem.

Next we prove that systems (17) possess a single configuration given in Figure 1. Indeed, we observe that systems (17) possess the following six invariant affine lines:

$$L_1: x=0,\quad L_2: x=1,\quad L_{3,4}: -1+x\mp iy=0,\quad L_{5,6}: 1+x\mp iy=0.$$

Moreover, these systems have 9 singular points:

$$M_{1,2}=(\mp 1,0),\ M_{3,4}=(0,\mp i),\ M_5=(0,-s),\ M_{6,7}=(1,\mp 2i),\ M_{8,9}=\left(\frac{s\mp i}{s\pm i},\frac{2}{s\pm i}\right).$$

We observe that systems (17) have only three real finite singular points: M_1 (respectively M_2) at the intersection of L_3 and L_4 (respectively L_2, L_5 and L_6) and M_5 is located on the invariant line L_1.

In the article [11, page 8] the Convention concerning three invariant lines intersecting at the same singular point is given: two complex conjugate lines (L and $\bar{L}$) and one real (L'). We have the next remark.

Remark 5. Considering the examination of the position of L' with respect to L and $\bar{L}$ given in [11, Convention] it is not too difficult to show that this Convention is true independently whether L' is an invariant line or not of the corresponding system. Only the fact that L' passes through the point of intersection of L and $\bar{L}$ is essential.

Next we observe that the singular point M_5 is located on the non-invariant line $y = s(x-1)$ passing through M_2 which is the intersection point of the invariant lines $y = \pm i(x-1)$. According to [11, Convention] the common projection of these two complex lines is $y = 0$. Since $s \neq 0$ we conclude that the line $y = s(x-1)$ could not coincide with the projection $y = 0$. Therefore assuming $s < 0$ (due to the rescaling $y \to -y$) we arrive at the configuration *Config. 7.1c*. This completes the proof of the Main Theorem. □

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