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Metrically piecewise continuous ρ-almost periodic functions and applications

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Abstract. In this paper, we consider various classes of metrically piecewise continuous ρ-almost periodic functions. We analyze the basic structural results for the introduced classes of functions, providing also certain applications of our results to the abstract Volterra integro-differential equations in Banach spaces.

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1 Introduction and preliminaries

Suppose that $(X, \|\cdot\|)$ is a complex Banach space, $I = [0, \infty)$ or $I = \mathbb{R}$, and $f : I \to X$ is a continuous function. If $\epsilon > 0$, then a number $\tau > 0$ is called an ϵ-period for $f(\cdot)$ if and only if

$$\|f(t+\tau) - f(t)\| \leq \epsilon, \quad t \in I;$$

the set consisted of all ϵ-periods for $f(\cdot)$ is denoted by $\vartheta(f, \epsilon)$. It is said that $f(\cdot)$ is almost periodic if and only if for each $\epsilon > 0$ the set $\vartheta(f, \epsilon)$ is relatively dense in $[0, \infty)$, i.e., there exists $l > 0$ such that any subinterval of $[0, \infty)$ of length l meets $\vartheta(f, \epsilon)$.

Any almost periodic function $f : I \to X$ is uniformly recurrent, which means that $f(\cdot)$ is continuous and there exists a strictly increasing sequence (α_k) of positive real numbers such that $\lim_{k\to+\infty} \alpha_k = +\infty$ and

$$\lim_{k\to+\infty} \sup_{t\in\mathbb{R}} \big\| f(t+\alpha_k) - f(t) \big\| = 0.$$

Suppose now that $p > 0$. Then it is said that a function $f \in L^p_{loc}(I : X)$ is Stepanov p-bounded if and only if

$$\|f\|_{S^p} := \sup_{t\in I} \int_t^{t+1} \|f(s)\|^p \, ds < +\infty$$

and that $f(\cdot)$ is Stepanov p-almost periodic if and only if its Bochner transform $\hat{f} : I \to L^p([0,1] : X)$, defined by $\hat{f}(t)(s) := f(t+s)$, $t \in I$, $s \in [0,1]$, is almost

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periodic. Any Stepanov p-almost periodic function is Stepanov p-bounded, while the converse statament is not true, in general (cf. [1–5, 9, 10, 12, 16, 17, 21] and references quoted therein for more details about the above-mentioned classes of almost periodic type functions and their applications). For further information concerning the multi-dimensional almost periodic type functions and their applications, which will not be considered within the scope of this paper, we refer the reader to the research monographs [13] and [14]. Before proceeding any further, we would like to emphasize that the notion of metrical almost periodicity has been introduced and thoroughly analyzed in the research monograph [14].

On the other hand, the piecewise continuous almost periodic functions and their applications to the (abstract) Volterra impulsive integro-differential equations have been analyzed by numerous authors so far (see the research monographs [11] by A. Halanay, D. Wexler, [18] by A. M. Samoilenko, N. A. Perestyuk, [19] by G. Tr. Stamov and the list of references quoted in our recent research article [7] for some results established in this direction). Concerning this problematic, we will only recall the following well-known notion:

Definition 1. Suppose that the function $f : \mathbb{R} \to X$ $[f : [0, \infty) \to X]$ is piecewise continuous with the possible first kind discontinuities at the points of a fixed sequence $(t_k)_{k\in\mathbb{Z}}$ $[(t_k)_{k\in\mathbb{N}}]$. Then $f(\cdot)$ is said to be (t_k)-piecewise continuous almost periodic if and only if the following holds:

(i) The family of sequences $(t_k^j)_{k\in\mathbb{Z}}$ $[(t_k^j)_{k\in\mathbb{N}}]$, $j \in \mathbb{Z}$ $[j \in \mathbb{N}]$ is equipotentially almost periodic $[(t_k)$ is a Wexler sequence, equivalently], which means that, for every $\epsilon > 0$, there exists a relatively dense set Q_ϵ in $\mathbb{R}$ [in $[0, \infty)$] such that for each $\tau \in Q_\epsilon$ there exists an integer $q \in \mathbb{Z}$ $[q \in \mathbb{N}]$ such that $|t_{i+q} - t_i - \tau| < \epsilon$ for all $i \in \mathbb{Z}$ $[i \in \mathbb{N}]$.

(ii) For every $\epsilon > 0$, there exists $\delta > 0$ such that, if the points t_1 and t_2 belong to (t_i, t_{i+1}) for some $i \in \mathbb{Z}$ $[i \in \mathbb{N}_0;\ t_0 \equiv 0]$ and $|t_1 - t_2| < \delta$, then $\|f(t_1) - f(t_2)\| < \epsilon$.

(iii) For every $\epsilon > 0$, there exists a relatively dense set S in $\mathbb{R}$ [in $[0, \infty)$] such that, if $\tau \in S$, then $\|f(t + \tau) - f(t)\| < \epsilon$ for all $t \in \mathbb{R}$ such that $|t - t_k| > \epsilon$, $k \in \mathbb{Z}$ $[k \in \mathbb{N}]$. Such a point τ is called an ϵ-almost period of $f(\cdot)$.

The space of all (t_k)-piecewise continuous almost periodic functions will be denoted by $PCAP_{(t_k)}(\mathbb{R} : X)$ $[PCAP_{(t_k)}([0, \infty) : X)]$.

As mentioned in the abstract, the main aim of this paper is to introduce and analyze several new classes of metrically piecewise continuous ρ-almost periodic functions. We investigate the basic structural results of such functions, providing also certain applications to the abstract Volterra integro-differential equations in Banach spaces.

The organization of paper can be briefly described as follows. After explaining the notation and terminology used throughout the paper, we introduce and analyze various classes of metrically piecewise continuous ρ-almost periodic type functions in Definition 2. We continue our exposition in Section 2 by stating Proposition 1; an illustrative example is given after that. Subsection 2.1 investigates the relations between metrically piecewise continuous ρ-almost periodic type functions and metrically Stepanov ρ-almost periodic type functions. The main results of this subsection are Theorem 1 and Theorem 2, which slightly generalize the corresponding results established in our recent research article [16]. In Section 3, we analyze some applications to the abstract Volterra integro-differential equations. In the final section of paper, we present some conclusions and final remarks about the introduced classes of metrically piecewise continuous ρ-almost periodic type functions.

Before starting our work, we need to explain the basic notation and terminology used henceforth:

Notation and terminology. We will always assume that $(X, \|\cdot\|)$ and $(Y, \|\cdot\|_Y)$ are complex Banach spaces; I denotes the identity operator on Y, $L(Y)$ denotes the Banach space of all bounded linear operators from Y into Y, $\lfloor s \rfloor := \sup\{k \in \mathbb{Z} : s \geq k\}$ and $\lceil s \rceil := \inf\{k \in \mathbb{Z} : s \leq k\}$ $(s \in \mathbb{R})$. By $\mathcal{B}$ we denote a certain collection of non-empty subsets of X which satisfies that for each $x \in X$ there exists $B \in \mathcal{B}$ such that $x \in B$. The vector space $C_b(I : Y)$, where $\emptyset \neq I \subseteq \mathbb{R}$, consists of all continuous functions $u : I \to Y$ satisfying that $\sup_{t \in I} \|u(t)\|_Y < +\infty$; equipped with the sup-norm $\|\cdot\|_\infty := \sup_{t \in I} \|\cdot(t)\|_Y$, $C_b(I : Y)$ is a Banach space. If $\nu : I \to [0, \infty)$ is an arbitrary function, then $C_{b,\nu}(I : Y)$ consists of all continuous functions $u : I \to Y$ satisfying that $\sup_{t \in I}[\|u(t)\|_Y \nu(t)] < +\infty$; equipped with the pseudodistance $d(\cdot, \cdot\cdot) := \sup_{t \in I}[\|\cdot(t) - \cdot\cdot(t)\|_Y \nu(t)]$, $C_{b,\nu}(I : Y)$ is a complete pseudometric space. For more details about the space $L^p(\Omega : Y)$, where $\emptyset \neq \Omega \subseteq \mathbb{R}$ is a Lebesgue measurable set and $p > 0$, we refer the reader to the recent research article [16] and references quoted therein. We will deal henceforth with the space $L^p_\nu(\Omega : Y) := \{u : \Omega \to Y \ ; \ u(\cdot) \text{ is Lebesgue measurable and } ||u||_p < \infty\}$, where $p > 0$, $\|\cdot\|_p := \|\nu(t) \cdot (t)\|_{L^p(\Omega:Y)}$ and $\nu : \Omega \to (0, \infty)$ is a Lebesgue measurable function.

The space of bounded piecewise continuous functions $f : [0, \infty) \to X$, denoted by $PC([0, \infty) : X)$, is formed of all bounded continuous functions and those bounded functions $f : [0, \infty) \to X$ which are continuous at the point zero and for which there exists a strictly increasing sequence (t_k) of positive real numbers without accumulation points, or a finite sequence (t_k) of positive real numbers, such that $f \in C((t_k, t_{k+1}] : X)$, $f(t_k^-) := f(t_k)$ and $f(t_k^+)$ exist for any $k \in \mathbb{N}$, where the symbols $f(t_k^-)$ and $f(t_k^+)$ denote the left limit and the right limit of the function $f(t)$ at the point $t = t_k$, $k \in \mathbb{N}$, respectively. The vector space $PC([0, \infty) : X)$ is a Banach space when endowed with the sup-norm; we similarly define the Banach spaces $PC(\mathbb{R} : X)$ and $PC([0, \omega] : X)$, where $\omega > 0$ is a finite real number.

2 Metrically piecewise continuous ρ-almost periodic type functions

The main aim of this section is to introduce and analyze several new classes of metrically piecewise continuous ρ-almost periodic type functions.

First of all, if $0 < \epsilon < \delta_0/2$ $[0 < \epsilon < \delta_0/2$ and $0 < \epsilon < t_1]$, then we set $A_\epsilon := I \setminus \bigcup_{k\in\mathbb{Z}} L(t_k, \epsilon)$ $[A_\epsilon := I \setminus \bigcup_{k\in\mathbb{N}} L(t_k, \epsilon)]$ and assume that $P_\epsilon \subseteq Y^{A_\epsilon}$, $0 \in P_\epsilon$ and (P_ϵ, d_ϵ) is a pseudometric space. Set $\|f\|_\epsilon := d_\epsilon(f, 0)$, $f \in P_\epsilon$. If we take $d_\epsilon(f, g) = \sup_{t\in A_\epsilon} \|f(t) - g(t)\|$, then the following notion generalizes the notion introduced in Definition 1 and [7, Definition 6] (this notion also generalizes the notion introduced in [8, Definition 2.1] and [14, Definition 4.1.19], provided that $I = [0, \infty)$ or $I = \mathbb{R}$, $I' = I$, the pseudometric spaces P_ϵ satisfy certain extra assumptions and (t_k) is an arbitrary sequence obeying our general requirements):

Definition 2. Suppose that ρ is a binary relation on Y and the function $F : \mathbb{R}\times X \to Y$ $[F : [0, \infty) \times X \to Y]$ satisfies that, for every $x \in X$, the function $t \mapsto F(t; x)$ is piecewise continuous with the possible first kind discontinuities at the points of a fixed sequence $(t_k)_{k\in\mathbb{Z}}$ $[(t_k)_{k\in\mathbb{N}}]$. Then we say that the function $F(\cdot)$ is:

(i) pre-$(\mathcal{B}, \rho, (t_k), \mathcal{P})$-piecewise continuous almost periodic if and only if, for every $\epsilon > 0$ such that $0 < \epsilon < \delta_0/2$ $[0 < \epsilon < \delta_0/2$ and $0 < \epsilon < t_1]$, $\sigma > 0$ and $B \in \mathcal{B}$, there exists a relatively dense set S in $\mathbb{R}$ [in $[0, \infty)$] such that, if $\tau \in S$, $x \in B$ and $t \in \mathbb{R}$ $(t \geq 0)$ satisfies $|t - t_k| > \epsilon$ for all $k \in \mathbb{Z}$ $[k \in \mathbb{N}]$, then there exists $y_{t,x} \in \rho(F(t; x))$ such that $F(\cdot + \tau; x) - y_{\cdot,x} \in P_\epsilon$ and $\|F(\cdot + \tau; x) - y_{\cdot,x}\|_\epsilon < \sigma$.

(ii) $(\mathcal{B}, \rho, (t_k), \mathcal{P})$-piecewise continuous almost periodic if and only if $F(\cdot; \cdot)$ is pre-$(\mathcal{B}, \rho, (t_k), \mathcal{P})$-piecewise continuous almost periodic, (t_k) is a Wexler sequence and (QUC) holds, where:

(QUC) For every $\epsilon > 0$ and $B \in \mathcal{B}$, there exists $\delta > 0$ such that, if $x \in B$ and the points t_1 and t_2 belong to (t_i, t_{i+1}) for some $i \in \mathbb{Z}$ $[i \in \mathbb{N}_0;\ t_0 \equiv 0]$ and $|t_1 - t_2| < \delta$, then $\|F(t_1; x) - F(t_2; x)\|_Y < \epsilon$.

(iii) pre-$(\mathcal{B}, \rho, (t_k), \mathcal{P})$-piecewise continuous uniformly recurrent if and only if there exists a strictly increasing sequence (α_l) of positive real numbers tending to plus infinity and satisfying that, for every $\epsilon > 0$ with $0 < \epsilon < \delta_0/2$ $[0 < \epsilon < \delta_0/2$ and $0 < \epsilon < t_1]$, $\sigma > 0$ and $B \in \mathcal{B}$, there exists an integer $l_0 \in \mathbb{N}$ such that, if $x \in B$, $l \geq l_0$ and $t \in \mathbb{R}$ satisfies $|t - t_k| > \epsilon$ for all $k \in \mathbb{Z}$ $[k \in \mathbb{N}]$, then there exists $y_{t,x} \in \rho(F(t; x))$ such that $F(\cdot + \alpha_l; x) - y_{\cdot,x} \in P_\epsilon$ and $\|F(\cdot + \alpha_l; x) - y_{\cdot,x}\|_\epsilon < \sigma$.

(iv) $(\mathcal{B}, \rho, (t_k), \mathcal{P})$-piecewise continuous uniformly recurrent if and only if $F(\cdot; \cdot)$ is pre-$(\mathcal{B}, \rho, (t_k), \mathcal{P})$-piecewise continuous uniformly recurrent and the condition (QUC) holds.

We say that the function $F(\cdot; \cdot)$ is (pre-)$(\mathcal{B}, \rho, \mathcal{P})$-piecewise continuous almost periodic [(pre-)$(\mathcal{B}, \rho, \mathcal{P})$-piecewise continuous uniformly recurrent] if and only if $F(\cdot; \cdot)$ is (pre-)$(\mathcal{B}, \rho, (t_k), \mathcal{P})$-piecewise continuous almost periodic [(pre-)$(\mathcal{B}, \rho, (t_k), \mathcal{P})$-piecewise continuous uniformly recurrent] for a certain sequence $(t_k)_{k\in\mathbb{Z}}$ $[(t_k)_{k\in\mathbb{N}}]$ obeying our general requirements. If $\rho = c\mathrm{I}$ for some $c \in \mathbb{C}$, then we also say that $F(\cdot; \cdot)$ is (pre-)piecewise continuous $(c, \mathcal{P})$-almost periodic [(pre-)piecewise continuous $(c, \mathcal{P})$-uniformly recurrent]; furthermore, if $c = -1$, then we also say that $F(\cdot; \cdot)$

is (pre-)piecewise continuous $\mathcal{P}$-almost anti-periodic [(pre-)piecewise continuous $\mathcal{P}$-uniformly anti-recurrent]. We omit the term "$\mathcal{B}$" from the notation if $X = \{0\}$ and we omit the term "c" from the notation if $c = 1$.

As already marked in [7, Remark 2(i)], the condition (QUC) is primarily intended for the analysis of case in which $\rho = \mathrm{I}$; the interested reader may try to formulate an analogue of [7, Proposition 1] for metrically piecewise continuous ρ-almost periodic type functions. The following simple result, which discuss the use of two different parameters $\epsilon > 0$ and $\sigma > 0$ in Definition 2, can be formulated for all other classes of functions introduced therein (cf. also [7, Remark 1]); in particular, this holds provided that $P_\epsilon = C_{b,\nu}(A_\epsilon : Y)$, where $\nu : I \to [0, \infty)$ is an arbitrary function:

Proposition 1. *Suppose that the general requirements from* Definition 2 *and the following condition hold:*

(C) If $0 < \epsilon' < \epsilon$, *then* $\| \cdot \|_\epsilon \leq \| \cdot \|_{\epsilon'}$.

Then $F(\cdot)$ *is pre-*$(\mathcal{B}, \rho, (t_k), \mathcal{P})$*-piecewise continuous almost periodic if and only if, for every* $\epsilon > 0$ *such that* $0 < \epsilon < \delta_0/2$ *[*$0 < \epsilon < \delta_0/2$ *and* $0 < \epsilon < t_1$*] and* $B \in \mathcal{B}$, *there exists a relatively dense set* S *in* $\mathbb{R}$ *[in* $[0, \infty)$*] such that, if* $\tau \in S$, $x \in B$ *and* $t \in \mathbb{R}$ *(*$t \geq 0$*) satisfies* $|t - t_k| > \epsilon$ *for all* $k \in \mathbb{Z}$ *[*$k \in \mathbb{N}$*], then there exists* $y_{t,x} \in \rho(F(t; x))$ *such that* $F(\cdot + \tau; x) - y_{\cdot,x} \in P_\epsilon$ *and* $\|F(\cdot + \tau; x) - y_{\cdot,x}\|_\epsilon < \epsilon$.

We continue with the following simple example:

Example. Suppose that $p : \mathbb{R} \to \mathbb{R}$ is a periodic trigonometric polynomial, $\nu : \mathbb{R} \to [0, \infty)$ is an arbitrary function and $P_\epsilon = C_{b,\nu}(A_\epsilon : Y)$ for every sufficiently small number $\epsilon > 0$. Then there exists a unique $\mathcal{P}$-piecewise continuous almost periodic function $f(\cdot)$ such that $f(t) = \mathrm{sign}(p(t))$ for all real values of t which are not zeros of $p(\cdot)$.

If the choice of pseudometric spaces P_ϵ is the same as in the last example, then we extend the function spaces introduced in [7, Definition 6]; on the other hand, if $P_\epsilon = C_{b,\nu}(A_\epsilon : Y)$ for all sufficiently small numbers $\epsilon > 0$, with some function $\nu : I \to (0, \infty)$ such that $(1/\nu)(\cdot)$ is a bounded function, then the function spaces introduced in Definition 2 are subspaces of the corresponding spaces introduced in [7, Definition 6]. The statements [7, Proposition 3, Proposition 4, Proposition 8] can be reformulated in the metrical framework; we leave details to the interested readers.

2.1 Relations with metrically Stepanov ρ-almost periodic type functions

In this subsection, we will briefly analyze the relations between metrically ρ-piecewise continuous almost periodic type functions and metrically Stepanov ρ-almost periodic type functions. The following result is a metrical extension of [16, Theorem 1]:

Theorem 1. *Suppose that $\rho = T \in L(Y)$, $p > 0$, $F : \Lambda \times X \to Y$ is pre-$(\mathcal{B}, T, (t_k), \mathcal{P})$-piecewise continuous almost periodic, where $\Lambda = \mathbb{R}$ or $\Lambda = [0, \infty)$, and for every $B \in \mathcal{B}$, we have $\|F\|_{\infty,B} \equiv \sup_{t\in\Lambda, x\in B} \|F(t; x)\| < +\infty$. Suppose, further, that for every sufficiently small $\epsilon > 0$, we have $P_\epsilon = C_{b,\eta}(\Lambda : Y)$ with some positive function $\eta : \mathbb{R} \to (0, \infty)$, $\nu : \mathbb{R} \to (0, \infty)$, the function $(\nu/\eta)(\cdot)$ is Stepanov-p-bounded and the following condition holds:*

(LQ) For every $\epsilon > 0$, there exist $d > 0$ and $\epsilon_0 > 0$ such that, for every $x \in \mathbb{R}$ and for every Lebesgue measurable set $\Omega \subseteq [x, x + 1]$ such that $m(\Omega) < \epsilon_0$, we have $\int_d^{+\infty} y^{p-1} m(\{x \in \Omega : \nu(x) > y\})\, dy < \epsilon$.

Then $F \in S^{(\mathbb{F},T,\mathcal{P}_{\mathbf{t}},\mathcal{P})}_{\Omega,\Lambda'}(\Lambda : Y)$, where $\Lambda = \Lambda' = \mathbb{R}$, $\mathbb{F}(\cdot\cdot) \equiv 1$, $\Omega = [0, 1]$, $P = C_b(\mathbb{R} : \mathbb{C})$ and $P_t = L^p_\nu(t + [0, 1] : \mathbb{C})$ for all $t \in \mathbb{R}$, i.e., for each $\epsilon > 0$ and $B \in \mathcal{B}$ there exists a relatively dense set S' in $\mathbb{R}$ such that for each $\tau \in S'$ we have

$$\int_x^{x+1} \big\|F(t + \tau; b) - cF(t; b)\big\|^p \nu^p(t)\, dt \le \epsilon, \quad x \in \mathbb{R},\ b \in B.$$

Proof. We will only outline the main details of the proof, with $\Lambda = \mathbb{R}$ and $T = c\mathrm{I}$ for some $c \in \mathbb{C}$. Let a number $\epsilon > 0$ and a set $B \in \mathcal{B}$ be given. Suppose $x \in \mathbb{R}$ and the interval $[x, x + 1]$ contains the possible first kind discontinuities of functions $F(\cdot; b)$ at the points $\{t_m, ..., t_{m+k}\} \subseteq [x, x + 1]$ $(b \in X)$; then $k \le \lceil 1/\delta_0 \rceil$. Let the numbers $d > 0$ and $\epsilon_0 > 0$ satisfy (LQ), with the number ϵ replaced therein with the number $\epsilon/(2\big((1 + |c|)\|F\|_{\infty,B}\big)^p p)$. We know that there exists a relatively dense set S in $\mathbb{R}$ such that, if $\tau \in S$ and $b \in B$, then $\|F(t + \tau; x) - cF(t; x)\|\eta(t) < \epsilon_1$ for all $t \in \mathbb{R}$ such that $|t - t_k| > \epsilon_1$, $k \in \mathbb{Z}$, where the number $\epsilon_1 \in (0, \epsilon_0/2\lceil 1/\delta_0 \rceil)$ is defined in the same way as in the proof of [16, Theorem 1], with the function $\nu(\cdot)$ replaced by the function $(\nu/\eta)(\cdot)$ in the last line of the proof.

The function $t \mapsto F(t + \tau; b) - cF(t; b)$, $t \in [x, x + 1]$ is not greater than $\epsilon_1/\eta(t)$ if $t \in A_x := [x, t_m - \epsilon_1] \cup (t_m + \epsilon_1, t_{m+1} - \epsilon_1] \cup ... \cup (t_{m+k}, x + 1]$; otherwise, $\|F(t + \tau; b) - cF(t; b)\| \le (1 + |c|)\|F\|_{\infty,B}$. Keeping in mind [16, Lemma 1], this implies

$$\begin{aligned}
&\int_x^{x+1} \big\|F(t + \tau; b) - cF(t; b)\big\|^p \nu^p(t)\, dt \\
&\le \epsilon_1^p \int_{A_x} \nu^p(t)\eta^{-p}(t)\, dt + \big((1 + |c|)\|F\|_{\infty,B}\big)^p \int_{[x,x+1]\setminus A_x} \nu^p(t)\, dt \\
&\le \epsilon_1^p \int_x^{x+1} \frac{\nu^p(t)}{\eta^p(t)}\, dt + \big((1 + |c|)\|F\|_{\infty,B}\big)^p p \int_0^\infty y^{p-1} m\big(\{s \in [x, x + 1] \setminus A_X : \nu(s) > y\}\big)\, dy \\
&\le \epsilon_1^p \|\nu/\eta\|_{S^p} + 2\big((1 + |c|)\|F\|_{\infty,B}\big)^p d^p \lceil 1/\delta_0 \rceil \epsilon_1 \\
&+ \big((1 + |c|)\|F\|_{\infty,B}\big)^p p \int_d^\infty y^{p-1} m\big(\{s \in [x, x + 1] \setminus A_X : \nu(s) > y\}\big)\, dy, \quad b \in B,
\end{aligned}$$

which simply completes the proof of result. □

The following result is a metrical analogue of the first part of [16, Theorem 2]:

Theorem 2. *Suppose that $\Lambda = \mathbb{R}$ or $\Lambda = [0,\infty)$, $\nu \in PC(\Lambda : [0,\infty))$ is bounded and has the possible first kind discontinuities at the points of the sequence (t_k) and satisfies the condition $(\mathrm{QUC})_\nu$, where:*

(QUC)$_\nu$ For every $\epsilon > 0$, there exists $\delta > 0$ such that, if the points t_1 and t_2 belong to (t_i, t_{i+1}) for some $i \in \mathbb{Z}$ [$i \in \mathbb{N}_0$; $t_0 \equiv 0$] and $|t_1 - t_2| < \delta$, then $|\nu(t_1) - \nu(t_2)| < \epsilon$.

Suppose, further, that $\rho = T \in L(Y)$, $p > 0$ and $F \in S^{(\mathbb{F},T,\mathcal{P}_{\mathbf{t}},\mathcal{P})}_{\Omega,\Lambda'}(\Lambda : Y)$, where $\Lambda = \Lambda'$, $\mathbb{F}(\cdot\cdot) \equiv 1$, $\Omega = [0,1]$, $P = C_b(\Lambda : \mathbb{C})$ and $P_t = L^p_\nu(t + [0,1] : \mathbb{C})$ for all $t \in \Lambda$. If, for every $B \in \mathcal{B}$, we have $\|F\|_{\infty,B} < +\infty$, and $F(\cdot;\cdot)$ satisfies that, for every $x \in X$, the function $t \mapsto F(t;x)$, $t \in \Lambda$ is piecewise continuous with the possible first kind discontinuities at the points of the sequence (t_k) and the condition (QUC) *holds, then $F(\cdot;\cdot)$ is pre-$(\mathcal{B},T,(t_k),\mathcal{P})$-piecewise continuous almost periodic with $P_\epsilon = C_{b,\nu}(A_\epsilon : Y)$ for each $\epsilon > 0$ sufficiently small.*

Proof. We will provide all relevant details of the proof, which basically follows from the argumentation contained in the proof of [7, Theorem 2] and the prescribed assumptions on the function $\nu(\cdot)$. For simplicity, we will assume that $\Lambda = \mathbb{R}$, $T = c\mathrm{I}$ for some $c \in S_1$ and $X = \{0\}$. Let $\epsilon > 0$ be fixed; then there exists $\delta \in (0, \min\{\epsilon/2, \delta_0/4\})$ such that, if the points t_1 and t_2 belong to the same interval (t_i, t_{i+1}) of the continuity of functions $F(\cdot)$ and $\nu(\cdot)$ and $|t_1 - t_2| < \delta$, then $\|F(t_1) - F(t_2)\| + |\nu(t_1) - \nu(t_2)| < \epsilon/8(1 + \|v\|_\infty + \|F\|_\infty)$. Suppose that $\eta_k \in (0, \epsilon\delta^{1/p}/4)$ for all $k \in \mathbb{N}$ and $\lim_{k\to+\infty} \eta_k = 0$. We will prove that there exists $k_0 \in \mathbb{N}$ such that, for every $\tau \in \mathbb{R}$ with $\int_t^{t+1} \|F(s+\tau) - cF(s)\|^p \nu^p(s)\, ds \le \eta^p_{k_0}$, $t \in \mathbb{R}$, we have $\|F(t+\tau) - cF(t)\|\nu(t) \le \epsilon$ for all $t \notin \bigcup_{l\in\mathbb{Z}}(t_l - \epsilon, t_l + \epsilon)$. If we assume the contrary, then for each $k \in \mathbb{N}$ there exist points $s_k \notin \bigcup_{l\in\mathbb{Z}}(t_l - \epsilon, t_l + \epsilon)$ and $\tau_k \in \mathbb{R}$ such that $\int_t^{t+1} \|F(s+\tau_k) - cF(s)\|^p \nu^p(s)\, ds \le \eta^p_k$, $t \in \mathbb{R}$ and $\|F(s_k + \tau_k) - cF(s_k)\|\nu(s_k) > \epsilon$. Since the functions $F(\cdot)$ and $\nu(\cdot)$ are continuous from the left side, for each $k \in \mathbb{N}$ there exist points $s'_k \notin \bigcup_{l\in\mathbb{Z}}(t_l - (3\epsilon/4), t_l + (3\epsilon/4))$ and $\tau_k \in \mathbb{R}$ such that $\int_t^{t+1} \|F(s + \tau_k) - cF(s)\|^p \nu^p(s)\, ds \le \eta^p_k$, $t \in \mathbb{R}$, $\|F(s'_k + \tau_k) - cF(s'_k)\|\nu(s'_k) > 3\epsilon/4$ and $s'_k + \tau_k \notin \{t_l : l \in \mathbb{Z}\}$. Since $\delta < \epsilon/2$, it follows that, for every $k \in \mathbb{N}$, the interval $(s'_k - \delta, s'_k + \delta)$ belongs to the same interval (t_j, t_{j+1}) of continuity of function $f(\cdot)$, for some $j \in \mathbb{Z}$. On the other hand, at least one of the intervals $(s'_k + \tau_k, s'_k + \tau_k + \delta)$ and $(s'_k + \tau_k - \delta, s'_k + \tau_k)$ belongs to the same interval (t_p, t_{p+1}) of continuity of function $F(\cdot)$, for some $p \in \mathbb{Z}$. For fixed $k \in \mathbb{N}$, we may assume w.l.o.g. that the above holds for the interval $(s'_k + \tau_k, s'_k + \tau_k + \delta)$; since $|c| = 1$, this readily implies:

$$\begin{aligned}
&\Big\|\nu(s + s'_k)\big[F(s + s'_k + \tau_k) - cF(s + s'_k)\big] - \nu(s'_k)\big[F(s'_k + \tau_k) - cF(s'_k)\big]\Big\| \\
&\le \big\|\nu(s + s'_k)F(s + s'_k + \tau_k) - \nu(s'_k)F(s'_k + \tau_k)\big\| + \big\|\nu(s + s'_k)F(s + s'_k) - \nu(s'_k)F(s'_k)\big\| \\
&\le \nu(s + s'_k)\big\|F(s + s'_k + \tau_k) - F(s'_k + \tau_k)\big\| + \big\|F(s'_k + \tau_k)\big\| \cdot \big|\nu(s + s'_k) - \nu(s'_k)\big| \\
&+ \nu(s + s'_k)\big\|F(s + s'_k) - F(s'_k)\big\| + \big\|F(s'_k)\big\| \cdot \big|\nu(s + s'_k) - \nu(s'_k)\big| \le \epsilon/2, \quad \text{a.e. } s \in [0,\delta].
\end{aligned}$$

Since $v(\cdot)$ and $F(\cdot)$ are bounded, the above implies

$$\big\|F(s + s'_k + \tau_k) - cF(s + s'_k)\big\|\nu(s + s'_k) \ge \epsilon/4, \quad \text{a.e. } s \in [0,\delta],\ k \in \mathbb{N}$$

and

$$\eta_k^p \geq \int_{t_k}^{t_k+1} \big\|F(s+s_k'+\tau_k)-cF(s+s_k')\big\|^p \nu^p(s+s_k')\,ds$$
$$\geq \int_{t_k}^{t_k+\delta} \big\|F(s+s_k'+\tau_k)-cF(s+s_k')\big\|^p \nu^p(s+s_k')\,ds \geq (\epsilon/4)^p\delta,\ k\in\mathbb{N}.$$

This is a contradiction and the proof of theorem is therefore completed. □

We continue with the following useful observation:

Remark 1. Suppose that $p:\mathbb{R}\rightarrow\mathbb{R}$ is a non-periodic trigonometric polynomial, $\nu:\mathbb{R}\rightarrow(0,\infty)$ is a Lebesgue measurable function and the condition (LT) holds, where:

(LT) For every $\epsilon>0$, there exists a sufficiently large integer $d>0$ such that $\sup_{t\in\mathbb{R}}\int_d^{+\infty} y^{p-1}m(\{x\in[t,t+1]:\nu(x)>y\})\,dy<\epsilon$, where $m(\cdot)$ denotes the Lebesgue measure.

Then we have shown, in [16, Example 3], that the function $F(t):=\text{sign}(p(t))$, $t\in\mathbb{R}$ belongs to the class $S_{\Omega,\Lambda'}^{(\mathbb{F},\rho,\mathcal{P}_t,\mathcal{P})}(\Lambda:Y)$ with $\Lambda=\Lambda'=\mathbb{R}$, $\mathbb{F}(\cdot\cdot)\equiv 1$, $\rho=\mathrm{I}$, $\Omega=[0,1]$, $P=C_b(\mathbb{R}:\mathbb{C})$ and $P_t=L_\nu^p(t+[0,1]:\mathbb{C})$ for all $t\in\mathbb{R}$, where $p>0$ is arbitrary. Unfortunately, the set consisting of all zeroes of $p(\cdot)$ can have infinitely many accumulation points in $\mathbb{R}$ (cf. also [7, Remark 5(ii)]).

We know that the pre-$(\mathcal{B},(t_k))$-piecewise continuous almost periodic functions form, under certain logical assumptions, the vector space with the usual operations; see, e.g., [7, Theorem 4]. It is not clear how we can extend the above-mentioned result to the metrically pre-$(\mathcal{B},(t_k))$-piecewise continuous almost periodic functions.

3 Applications to the abstract Volterra integro-differential equations

In this section, we will present some applications of our results to the abstract Volterra integro-differential equations in Banach spaces.

1. The results obtained in [7] and [16], which have been slightly extended in Subsection 2.1, and the composition principles for Stepanov p-almost periodic functions (cf. [12] for more details) can be successfully applied in the analysis of the existence and uniqueness of (t_k)-piecewise continuous almost periodic solutions of the following semilinear Volterra integral equation

$$u(t)=g(t)+\int_{-\infty}^{t} a(t-s)F(s;u(s))\,ds,\quad t\in\mathbb{R}. \tag{1}$$

Let us assume that the following conditions hold:

(i) $1\leq p<\infty$, $1/p+1/q=1$ and $\mathcal{B}$ denotes the collection of all sets in X with relatively compact range;

(ii) $F(\cdot;\cdot)$ is pre-$(\mathcal{B},(t_k))$-piecewise continuous almost periodic and, for every $B \in \mathcal{B}$, we have $\|F\|_{\infty,B} < +\infty$;

(iii) There exists a finite real constant $L > 0$ such that $\|F(t;x)-F(t;y)\| \leq L\|x-y\|$ for all $t \in \mathbb{R}$ and $x,\ y \in X$;

(iv) $g \in PCAP_{(t_k)}(\mathbb{R} : X)$;

(v) $\sum_{k=0}^{\infty} \|a(\cdot)\|_{L^q[k,k+1]} < +\infty$ and $L\int_0^{\infty} |a(t)|\, dt < 1$.

Since $PCAP_{(t_k)}(\mathbb{R} : X)$ is a Banach space equipped with the sup-norm, any function $f \in PCAP_{(t_k)}(\mathbb{R} : X)$ has relatively compact range (see [7, Proposition 2]) and the conditions (i)-(v) are satisfied, we can apply Theorem 1 with $\nu \equiv \eta \equiv 1$ and [12, Proposition 2.6.11, Theorem 2.7.2] in order to see that the mapping $\Psi : u \mapsto g(\cdot) + \int_{-\infty}^{\cdot} a(\cdot - s)F(s; u(s))\, ds$, $u \in PCAP_{(t_k)}(\mathbb{R} : X)$ is a well-defined contraction. Using the Banach contraction principle, it readily follows that there exists a unique (t_k)-piecewise continuous almost periodic solution of (1).

2. It would be very difficult to say something more about the existence and uniqueness of metrically (t_k)-piecewise continuous almost periodic solutions of (1); cf. also [7, Example 3] and [15, Theorem 2.1, Proposition 2.2, Proposition 2.3]. On the other hand, Theorem 1 can be simply reformulated for the corresponding classes of piecewise continuous T-uniformly recurrent functions. Keeping this observation in mind, we can similarly provide, as in part 1., some applications to the abstract degenerate semilinear fractional differential equations considered in [15, Section 3]; see, especially, [15, Theorem 3.1]. Details can be left to the interested readers.

3. Suppose that $c \in \mathbb{C}$, $|c| = 1$, $1 \leq p < +\infty$, $1/p + 1/q = 1$, $f : \mathbb{R} \to X$ is bounded and pre-$(c,(t_k),\mathcal{P})$-piecewise continuous almost periodic, where for every sufficiently small $\epsilon > 0$, we have $P_\epsilon = C_{b,\eta}(\mathbb{R} : X)$ with some positive function $\eta : \mathbb{R} \to (0,\infty)$ such that the function $(\nu/\eta)(\cdot)$ is Stepanov p-bounded for some function $\nu : \mathbb{R} \to (0,\infty)$ satisfying (LQ). Due to Theorem 1, for each $\epsilon > 0$ there exists a relatively dense set S' in $\mathbb{R}$ such that for each $\tau \in S'$ we have

$$\int_x^{x+1} \big\|f(t+\tau) - cf(t)\big\|^p \nu^p(t)\, dt \leq \epsilon^p, \quad x \in \mathbb{R}. \tag{2}$$

Suppose now that $(R(t))_{t>0} \subseteq L(X,Y)$ is a strongly continuous operator family such that $\sum_{k=0}^{\infty} \|R(\cdot)\|_{L^q_\nu[k,k+1]} < \infty$ and the function $(1/\nu)(\cdot)$ is bounded. Applying (2), we get

$$\begin{aligned} &\int_{x-k-1}^{x-k} \big\|f(t+\tau) - cf(t)\big\|^p \nu^{-p}(x-t)\, dt \\ &\leq \int_{x-k-1}^{x-k} \big\|f(t+\tau) - cf(t)\big\|^p \nu^p(t)\nu^{-p}(x-t)\nu^{-p}(t)\, dt \\ &\leq \int_{x-k-1}^{x-k} \big\|f(t+\tau) - cf(t)\big\|^p \nu^p(t)\, dt \cdot \|1/\nu(\cdot)\|_\infty^{2p} \leq \epsilon^p \|1/\nu(\cdot)\|_\infty^{2p}, \quad x \in \mathbb{R},\ k \in \mathbb{Z}. \end{aligned}$$

Now we can apply [15, Proposition 2.3], with $\sigma \equiv 1$ and $\Lambda = \Lambda' = \mathbb{R}$, in order to see that the function

$$t \mapsto F(t) := \int_{-\infty}^{t} R(t-s)f(s)\,ds, \quad t \in \mathbb{R}$$

is c-almost periodic in the usual sense. This can be applied in the analysis of the existence and uniqueness of c-almost periodic solutions for a large class of the abstract (fractional) Volterra integro-differential inclusions without initial conditions; see [12] for many applications of this type.

At the end of this section, we would like to emphasize that it would be curious to find some new applications of metrically piecewise continuous ρ-almost periodic functions to the abstract impulsive Volterra integro-differential equations (cf. also [6, 7] and references quoted therein for more details on the subject).

4 Conclusions and final remarks

In this paper, we have introduced and analyzed several new classes of metrically piecewise continuous ρ-almost periodic functions. We have clarified the most important structural results about the introduced classes of piecewise continuous ρ-almost periodic type functions and their relations with metrically Stepanov ρ-almost periodic type functions. We have also presented some applications of our results to the abstract Volterra integro-differential equations in Banach spaces. The metrically piecewise continuous almost automorphic functions and their relations with metrically Stepanov almost automorphic type functions will be investigated somewhere else (cf. also [20]).

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