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The comparability of motions in dynamical systems and recurrent solutions of (S)PDEs

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Abstract. Shcherbakov's comparability method is very useful to study recurrent solutions of differential equations. In this paper, we extend the method from metric spaces to uniform spaces, which applies well to dynamical systems in infinite-dimensional spaces. This generalized comparability method can be easily used to study recurrent solutions of (stochastic) partial differential equations under weaker conditions than in earlier results. We also show that the distribution of solutions of SDEs naturally generates a semiflow or skew-product semiflow on the space of probability measures, which is interesting in itself. As illustration, we give an application to semilinear stochastic partial differential equations.

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1 Introduction

Recurrence is a core topic in the theory of dynamical systems, which describes the asymptotic behaviors and complexity of a system. By Poincaré recurrence theorem and Birkhoff recurrence theorem we know that recurrence exists widely in dynamical systems. The notion of Poisson stability (also called recurrence in the literature) was first introduced by Poincaré in his famous work [23]; he observed that the orbits of aperiodic solutions are Poisson stable for all bounded Hamiltonian systems. Poisson stable motions contain the following different classes: stationary, periodic, quasi-periodic, almost periodic, almost automorphic, Birkhoff recurrent, Levitan almost periodic, almost recurrent, pseudo-periodic, pseudo-recurrent, etc. It is well known that the complex and meantime interesting phenomenon happens on the Poisson stable set and the dynamics outside of it is simple.

Shcherbakov [25–29] studied systematically the existence of Poisson stable solutions to the equation

$$x' = f(t, x), \quad x \in \mathfrak{B} \tag{1.1}$$

with f being Poisson stable in $t \in \mathbb{R}$ uniformly with respect to x on every compact subset of $\mathfrak{B}$, where $\mathfrak{B}$ is a Banach space. To this end, Shcherbakov developed a method of comparability of functions by character of recurrence. He studied different classes of equations of the form (1.1), and gave conditions for existence of

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solutions with the same character of recurrence as f. He named this type of solutions (uniformly) comparable solutions.

The works of Shcherbakov were extended by many authors, see e.g. Bronshtein [3, ChIV], Caraballo and Cheban [4–7], Cheban [8], Cheban and Liu [10], Cheban and Mammana [13], Cheban and Schmalfuss [14], and others. Very recently, Shcherbakov's comparability method was employed to study recurrent solutions to stochastic differential equations: Cheban and Liu [11,12], Cheng and Liu [15], Cheng, Liu and Röckner [16], Liu and Liu [22], among others.

Shcherbakov's comparability method was established in metric spaces, which applies perfectly to study recurrent solutions to (stochastic) ordinary differential equations, i.e. in finite-dimensional spaces, but it imposes some restrictions for (stochastic) partial ordinary differential equations, i.e. in infinite-dimensional spaces; see [11, Remark 4.2] for details. In this paper, we aim to develop Shcherbakov's comparability method to match well equations of infinite dimension. To this end, we will establish the comparability principle in uniform spaces, which works well for compact-open topology and infinite-dimensional equations.

The paper is organized as follows. In Section 2, we give some preliminaries on dynamical systems, Poisson stable motions, etc. In Section 3, we develop Shcherbakov's comparability method in uniform spaces. In Section 4, we establish that stochastic ODEs and stochastic semilinear PDEs generates skew-product semiflows on the space of probability measures, under rather general conditions. In Section 5, we study the Poisson stability of solutions to stochastic semilinear PDEs by our theoretical results in Sections 3 and 4. In the last Appendix Section, for the convenience of the reader, we collect some preliminaries on the uniform space.

2 Preliminaries

In this section, we introduce some useful preliminaries, including dynamical systems, Poisson stable functions/motions, and a fixed point theorem.

Let $(M,\mathcal{U})$ be a Hausdorff uniform space with uniform neighborhood system $(V; A, \geq)$.

2.1 Dynamical systems and recurrent motions

Firstly, we recall the types of Poisson stable/recurrent functions to be studied in this paper; we refer the reader to [24, 25, 27, 31] for further details and the relations among these types of functions.

Definition 2.1 (Recurrent functions)**.** Let $f : \mathbb{R} \to M$ be a continuous function.

(i) The function f is called *stationary* if $f(t) = f(0)$ for all $t \in \mathbb{R}$.

(ii) For given $\tau \in \mathbb{R}$, the function f is called *periodic* or *τ-periodic* if $f(t+\tau) = f(t)$ for all $t \in \mathbb{R}$.

(iii) The function f is called *quasi-periodic (with base frequencies $\omega_1, \dots, \omega_n$)* if $\omega_1, \dots, \omega_n$ are rationally independent and $f(t) = g(\omega_1 t, \dots, \omega_n t)$ for some continuous function $g : \mathbb{R}^n \to M$ satisfying $g(t_1 + 2\pi, \dots, t_n + 2\pi) = g(t_1, \dots, t_n)$ for all $(t_1, \dots, t_n) \in \mathbb{R}^n$.

(iv) The function f is called *Bohr almost periodic* if for each $a \in A$, the set

$$T(f, a) := \{\tau \in \mathbb{R} : f(t + \tau) \in V_a(f(t)) \text{ for all } t \in \mathbb{R}\}$$

is *relatively dense* in $\mathbb{R}$, i.e. there exists $l = l(a) > 0$ such that $(x, x + l) \cap T(f, a) \neq \emptyset$ for all $x \in \mathbb{R}$. The set $T(f, a)$ is called the *set of a-almost periods of f*.

(v) The function f is called *positively (respectively, negatively) pseudo-periodic* if for any $a \in A$ and $l > 0$, there exists $\tau > l$ (respectively, $\tau < -l$) such that

$$f(t + \tau) \in V_a(f(t)), \quad \text{for all } t \in \mathbb{R}.$$

f is called *pseudo-periodic* (or *uniformly Poisson stable*, see [31]) if it is both positively and negatively pseudo-periodic.

(vi) The function f is called *almost automorphic* if every sequence $\{s'_n\}$ of real numbers admits a subsequence $\{s_n\}$ such that there exists a function $g : \mathbb{R} \to M$ with the property that for every $a \in A$ and every finite interval $[a, b]$ there exist N satisfying

$$f(t + s_n) \in V_a(g(t)) \text{ and } g(t - s_n) \in V_a(f(t)) \quad \text{for } t \in [a, b] \tag{2.1}$$

whenever $n \geq N$.

Remark 2.2. (i) It is well known that the function f is Bohr almost periodic if and only if it is *Bochner almost periodic*: i.e. for every sequence of real numbers $\{s'_n\}$, there exists a subsequence $\{s_n\}$ such that $\lim_{n\to\infty} f(t + t_n)$ uniformly exists on $\mathbb{R}$.

(ii) By the definitions of Bochner almost periodicity and almost automorphy, it is straightforward to check that the range $f(\mathbb{R})$ of f is precompact in M when f is Bohr almost periodic or almost automorphic.

(iii) We note that the above definition of almost automorphy is a little stronger than the Bochner's original definition [2], which only requires that the convergence in (2.1) is point-wise. But when we consider almost automorphic solutions to differential equations, all the almost automorphic solutions in Bochner's original sense will automatically satisfy (2.1) uniformly on compact intervals; see [30] for details.

Definition 2.3. (i) Let $\mathbb{T} = \mathbb{R}$ or $\mathbb{R}^+$. The triple $(M, \mathbb{T}, \varphi)$ is called a *dynamical system* if $\varphi : \mathbb{T} \times M \to M$ is continuous and satisfies:

$$\varphi(0, x) = x,\ \varphi(t + s, x) = \varphi(t, \varphi(s, x)) \quad \text{for all } x \in M \text{ and } t, s \in \mathbb{T}. \tag{2.2}$$

If $\mathbb{T} = \mathbb{R}$, φ is called a *flow* on M; if $\mathbb{T} = \mathbb{R}^+$, φ is called a *semiflow* on M.

(ii) A continuous mapping $t \mapsto \varphi(t, x)$ is called a *motion through* x. If the motion $\pi(\cdot, x)$ through x has property $\mathcal{P}$, we also say that the point x has property $\mathcal{P}$; for example, the motion $\pi(\cdot, x)$ being periodic is equivalent to that x is a periodic point.

(iii) We denote

$$\gamma(x) := \{\varphi(t, x) : t \in \mathbb{T}\} \quad \text{and} \quad H(x) := \overline{\gamma(x)}$$

the *orbit through* x and *hull of* x, respectively. A set $S \subset M$ is called an *invariant set* (respectively, *positively invariant set*) if $\varphi(t, S) = S$ (respectively, $\varphi(t, S) \subset S$) for all $t \in \mathbb{R}^+$. A closed invariant set S is called *minimal* if it contains no proper nonempty closed invariant sets.

Remark 2.4. (i) Note that, for given $x \in M$, the orbit $\gamma(x)$ and hull $H(x)$ are invariant sets when $\mathbb{T} = \mathbb{R}$, while positively invariant sets when $\mathbb{T} = \mathbb{R}^+$.

(ii) Note that when the flow (respectively, semiflow) φ is restricted to an invariant (respectively, positively invariant) set S, φ is also a flow (respectively, semiflow).

Definition 2.5. For a given flow $(M, \mathbb{R}, \varphi)$ and a point $x \in M$, the motion $\varphi(\cdot, x)$ or the point x is called *stationary* (respectively, *periodic, quasi-periodic, Bohr almost periodic, pseudo-periodic, almost automorphic*) if the corresponding motion $\varphi(\cdot, x) : \mathbb{R} \to M$ through x is a function with these properties.

Definition 2.6. Let $(M, \mathbb{R}, \varphi)$ be a flow and $x \in M$.

(i) The motion $\pi(\cdot, x)$ or the point x is called *Lagrange stable* if $H(x)$ is compact.

(ii) The motion $\pi(\cdot, x)$ or the point x is called *Birkhoff recurrent* if it is Lagrange stable and for any $a \in A$ the set

$$\{\tau \in \mathbb{R} : \pi(\tau, x) \in V_a(x)\}$$

is relatively dense in $\mathbb{R}$.

(iii) The motion $\pi(\cdot, x)$ is called *Levitan almost periodic* if there exists a Bohr almost periodic point y with respect to another flow $(Y, \mathbb{R}, \sigma)$ such that $\mathfrak{N}_y \subset \mathfrak{N}_x$, where

$$\mathfrak{N}_x := \{\{t_\alpha\} \subset \mathbb{R} : \{t_\alpha\} \text{ is a net such that} \lim_\alpha \varphi(t_\alpha, x) \to x\}$$

and similarly for $\mathfrak{N}_y$ (with x replaced by y).

Definition 2.7. Let $(M, \mathbb{R}, \varphi)$ be a flow and $x \in M$.

(i) A point x is called *positively Poisson stable* if $x \in \omega(x)$; note that x is positively Poisson stable if and only if for any $a \in A$, $t_0 \in \mathbb{R}$ and $z \in \gamma(x)$, there exists $T = T(a, t_0, z) > 0$ such that

$$\pi(\tau, z) \in V_a(z) \quad \text{for some } \tau \in [t_0, t_0 + T].$$

The point x is *negatively Poisson stable* if $x \in \alpha(x)$ and x is *Poisson stable* if it is both positively and negatively Poisson stable.

(ii) A point x is called *almost recurrent* if it is Poisson stable and the T in (i) is independent of $t_0 \in \mathbb{R}$, i.e. $T = T(a, z)$.

(iii) A point x is called *pseudo-recurrent* if it is Poisson stable and the T in (i) is independent of $z \in \gamma(x)$, i.e. $T = T(a, t_0)$.

Remark 2.8. It is known (see, e.g. [31]) that:

(i) A point x is Birkhoff recurrent if and only if x is Poisson stable and the T in Definition 2.7 is independent of both $t_0 \in \mathbb{R}$ and $z \in \gamma(x)$, i.e. $T = T(a)$. That is, Birkhoff recurrent = almost recurrent + pseudo-recurrent.

(ii) If x is Poisson stable and the T in Definition 2.7 is independent of $a \in A$, i.e. $T = T(t_0, z)$, then x is indeed a periodic point.

Remark 2.9. It is well known that we have the following increasing inclusion relations: stationary, periodic, quasi-periodic, Bohr almost periodic. For the inclusion relations among other recurrent motions, see Figure 1 for details ($A \Rightarrow B$ means "A implies B"); a similar figure can be found in [31].

Remark 2.10. (i) Every almost automorphic point $x \in M$ is Levitan almost periodic, but the converse is not true in general. A point x is almost automorphic if and only if it is Levitan almost periodic and $H(x)$ is compact. See [21, 30] for details.

(ii) By Birkhoff's recurrence theorem, a set $E \subset M$ is a compact minimal set if and only if $E = H(x)$ for some Birkhoff recurrent point x; see [24] for details.

(iii) A point $x \in M$ is Birkhoff recurrent if and only if x is almost recurrent and $H(x)$ is compact.

Let $\mathcal{X}$ and $\mathcal{Y}$ be two Hausdorff uniform spaces. Here, for simplicity, we do not point out explicitly the topology and uniform neighborhood system, which should not cause confusion. To study recurrent solutions to (stochastic) differential equations, we need to define recurrent functions with parameters. Consider the space $C(\mathbb{R} \times \mathcal{X}, \mathcal{Y})$ of all continuous mappings from $\mathbb{R} \times \mathcal{X}$ to $\mathcal{Y}$ which is endowed with the compact-open topology. Denote $f^\tau(t, x) := f(t + \tau, x)$ for $f \in C(\mathbb{R} \times \mathcal{X}, \mathcal{Y})$ and $(t, x) \in \mathbb{R} \times \mathcal{X}$. Then the mapping

$$\theta : \mathbb{R} \times C(\mathbb{R} \times \mathcal{X}, \mathcal{Y}) \to C(\mathbb{R} \times \mathcal{X}, \mathcal{Y}), \quad (t, f) \mapsto f^t$$

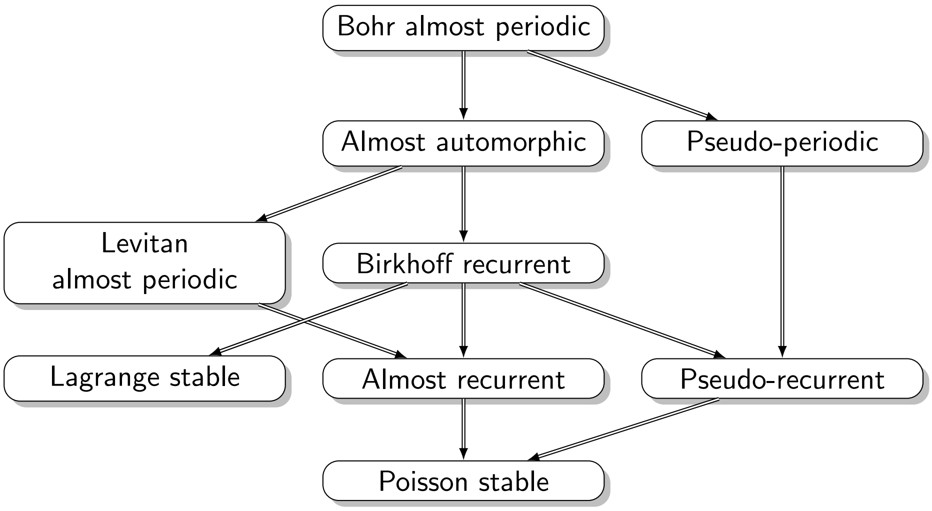

Figure 1. The relations among recurrent functions/motions.

defines a flow (called *shift flow*) on $C(\mathbb{R}\times\mathcal{X},\mathcal{Y})$, i.e. $\theta(0,\cdot)=Id$, $\theta(t+s,\cdot)=\theta(t,\theta(s,\cdot))$ for $t,s\in\mathbb{R}$ and the mapping θ is continuous; see [9, 24] for details. For simplicity, we also denote $\theta(t,f)$ by $\theta_t f$ in what follows.

Definition 2.11. Let $\mathcal{X}$ and $\mathcal{Y}$ be two Hausdorff uniform spaces and θ be the shift flow on $C(\mathbb{R}\times\mathcal{X},\mathcal{Y})$. A function $f\in C(\mathbb{R}\times\mathcal{X},\mathcal{Y})$ is called *stationary* (respectively, *periodic, quasi-periodic, Bohr almost periodic, almost automorphic, Birkhoff recurrent, Lagrange stable, Levitan almost periodic, almost recurrent, pseudo-periodic, pseudo-recurrent, Poisson stable*) *in t uniformly for x on compact sets*, provided the corresponding motion $\theta(\cdot,f)$ through the point f possesses these properties.

Definition 2.12. Let $\mathbb{T}=\mathbb{R}$ or $\mathbb{R}^+$, $\mathcal{X}$ and $\mathcal{Y}$ be two Hausdorff uniform spaces, and $(\mathcal{Y},\mathbb{R},\theta)$ be a flow.

(i) A *cocycle* Φ over θ is a continuous mapping

$$\Phi:\mathbb{T}\times\mathcal{X}\times\mathcal{Y}\to\mathcal{X},\quad (t,x,y)\mapsto\Phi(t,x,y)$$

which satisfies the property:

$$\Phi(0,x,y)=x,\Phi(t+s,x,y)=\Phi(t,\Phi(s,x,y),\theta_s y)\quad\text{for all } t,s\in\mathbb{T},x\in\mathcal{X},y\in\mathcal{Y}.$$

The space $\mathcal{X}$ is called a *fiber space* and $\mathcal{Y}$ a *base space*.

(ii) The associated *skew-product flow (or semiflow)* Π: $\mathbb{T}\times\mathcal{X}\times\mathcal{Y}\to\mathcal{X}\times\mathcal{Y}$ is the flow (or semiflow) $(\mathcal{X}\times\mathcal{Y},\mathbb{T},\Pi)$:

$$\Pi(t,x,y):=(\Phi(t,x,y),\theta_t y). \tag{2.3}$$

Remark 2.13. Note that we always assume that the base flow θ is defined for $t\in\mathbb{R}$, and the cocycle Φ may be defined on $\mathbb{R}$ or $\mathbb{R}^+$.

2.2 Recurrence in distribution

Let $\mathcal{X}$ be a Polish space, i.e. a separable complete metric space. Denote $\mathcal{P}(\mathcal{X})$ be the space of all Borel probability measures on $\mathcal{X}$ endowed with the β metric:

$$\beta(\mu,\nu) := \sup\left\{\left|\int f \mathrm{d}\mu - \int f \mathrm{d}\nu\right| : \|f\|_{BL} \le 1\right\}, \quad \mu,\nu \in \mathcal{P}(\mathcal{X}),$$

where f are Lipschitz continuous real-valued functions on $\mathcal{X}$ with the norms

$$\|f\|_{BL} = \|f\|_L + \|f\|_\infty, \quad \|f\|_L = \sup_{x\neq y} \frac{|f(x)-f(y)|}{d(x,y)}, \quad \|f\|_\infty = \sup_{x\in\mathcal{X}} |f(x)|.$$

Recall that a sequence $\{\mu_n\} \subset \mathcal{P}(\mathcal{X})$ is said to *weakly converge* to μ if

$$\int f \mathrm{d}\mu_n \to \int f \mathrm{d}\mu \quad \text{for all} \quad f \in C_b(\mathcal{X}),$$

the space of all bounded continuous real-valued functions on $\mathcal{X}$. It is well-known that $(\mathcal{P}(\mathcal{X}),\beta)$ is a Polish space and that a sequence $\{\mu_n\}$ weakly converges to μ if and only if $\beta(\mu_n,\mu) \to 0$ as $n \to \infty$. See [18, Chapter 11] for the metric β and its related properties.

Definition 2.14. A sequence of $\mathcal{X}$-valued random variables $\{X_n\}$ is said to *converge in distribution* to the random variable X if the corresponding laws $\{\mu_n\}$ weakly converge to the law μ of X, i.e. $\beta(\mu_n,\mu) \to 0$.

Definition 2.15. (i) A set $\Gamma \subset \mathcal{P}(\mathcal{X})$ is called *relatively compact* if every sequence of elements of Γ contains a weakly convergent subsequence.

(ii) A set $\Gamma \subset \mathcal{P}(\mathcal{X})$ is called *tight* if for every $\epsilon > 0$ there exists a compact set $K \subset \mathcal{X}$ such that $\mu(K) > 1-\epsilon$, for every $\mu \in \Gamma$.

Proposition 2.16. [1, Section 5] *The set $\Gamma \subset \mathcal{P}(\mathcal{X})$ is tight if and only if it is relatively compact.*

Definition 2.17. An $\mathcal{X}$-valued stochastic process $X(t), t \in \mathbb{R}$, is said to be *stationary* (respectively, *periodic, quasi-periodic, Bohr almost periodic, almost automorphic, Birkhoff recurrent, Levitan almost periodic, Poisson stable*) *in distribution* if its law $\mu(\cdot) : \mathbb{R} \to \mathcal{P}(\mathcal{X})$ on $\mathcal{X}$, as a function from $C(\mathbb{R},\mathcal{P}(\mathcal{X}))$, possesses the corresponding property.

2.3 A fixed point theorem depending on a parameter

In what follows, we will need the following basic result:

Theorem 2.18. *Let Λ be a topological space and E be a complete metric space. Let $F : \Lambda \times E \to E$ be a mapping such that there exists a constant $\alpha \in [0,1)$ such that*

$$d(F(\lambda,x),F(\lambda,y)) \le \alpha\, d(x,y) \quad \text{for all } \lambda \in \Lambda \text{ and } x,y \in E. \tag{2.4}$$

Then:

(i) *There exists a unique mapping* $\varphi : \Lambda \to E$ *such that*

$$F(\lambda, \varphi(\lambda)) = \varphi(\lambda) \quad \textit{for all } \lambda \in \Lambda.$$

(ii) *If* $F(\cdot, x) : \Lambda \to \Lambda$ *is continuous for any* $x \in E$*, then the mapping* φ *is continuous.*

(iii) *If in addition* Λ *is a metric space and there exists a constant* $L \geq 0$ *such that*

$$d(F(\lambda, x), F(\mu, x)) \leq L\, d(\lambda, \mu) \quad \textit{for all } x \in E \textit{ and } \lambda, \mu \in \Lambda,$$

then the mapping $\varphi : \Lambda \to E$ *is Lipschitz continuous and*

$$d(\varphi(\lambda), \varphi(\mu)) \leq \frac{L}{1-\alpha}\, d(\lambda, \mu) \quad \textit{for all } \lambda, \mu \in \Lambda.$$

(iv) *Let* $F_n, n = 1, 2, \dots,$ *be a sequence of mappings from* $\Lambda \times E$ *to* E *such that* (2.4) *holds with* F *replaced by* F_n *and* α *independent of* n*, and* $F_n \to F$ *for all points of* $\Lambda \times E$*. Then*

$$\lim_{n\to\infty} \varphi_n(\lambda) = \varphi(\lambda) \quad \textit{for all } \lambda \in \Lambda,$$

where $\varphi_n(\lambda)$ *is the unique fixed point of the mapping* $F_n(\lambda, \cdot)$*.*

Proof. The item (i) follows from Banach fixed point theorem. To see (ii), note that

$$\begin{aligned} d(\varphi(\lambda), \varphi(\mu)) &= d\left(F(\lambda, \varphi(\lambda)), F(\mu, \varphi(\mu))\right) \\ &\leq d(F(\lambda, \varphi(\lambda)), F(\lambda, \varphi(\mu))) + d(F(\lambda, \varphi(\mu)), F(\mu, \varphi(\mu))) \\ &\leq \alpha\, d(\varphi(\lambda), \varphi(\mu)) + d(F(\lambda, \varphi(\mu)), F(\mu, \varphi(\mu))), \end{aligned}$$

so

$$d(\varphi(\lambda), \varphi(\mu)) \leq \frac{1}{1-\alpha}\, d(F(\lambda, \varphi(\mu)), F(\mu, \varphi(\mu))). \tag{2.5}$$

Hence the continuity of φ follows from that of F with respect to $\lambda \in \Lambda$. The item (iii) follows from (2.5) and the Lipschitz continuity of F with respect to $\lambda \in \Lambda$.

For the item (iv) we have

$$\begin{aligned} d(\varphi_n(\lambda), \varphi(\lambda)) &= d(F_n(\lambda, \varphi_n(\lambda)), F(\lambda, \varphi(\lambda))) \\ &\leq d(F_n(\lambda, \varphi_n(\lambda)), F_n(\lambda, \varphi(\lambda))) + d(F_n(\lambda, \varphi(\lambda)), F(\lambda, \varphi(\lambda))) \\ &\leq \alpha\, d(\varphi_n(\lambda), \varphi(\lambda)) + d(F_n(\lambda, \varphi(\lambda)), F(\lambda, \varphi(\lambda))), \end{aligned}$$

so

$$d(\varphi_n(\lambda), \varphi(\lambda)) \leq \frac{1}{1-\alpha} d(F_n(\lambda, \varphi(\lambda)), F(\lambda, \varphi(\lambda))).$$

The result now follows by letting $n \to \infty$. □

3 Shcherbakov's comparability principle for Poisson stable motions in uniform spaces

For a net $\{t_\alpha\}_{\alpha\in\Lambda}$ with the index set Λ being a directed set $(\Lambda,\geq)$, we will simply denote it by $\{t_\alpha\}$, omitting the index set, if it is not necessary to point explicitly out the index set and no confusion would arise; we will write out the index set when it is necessary.

Let $(M,\mathbb{T},\varphi)$ be a flow or semiflow, and $x\in M$. Denote by $\mathfrak{N}_x$ (respectively, $\mathfrak{M}_x$) the set of all nets $\{t_\alpha\}\subset\mathbb{T}$ such that $\varphi(t_\alpha,x)\to x$ (respectively, $\varphi(t_\alpha,x)$ converges) in M, and denote $\mathfrak{M}_{x_1,x_2}$ the set of all the nets $\{t_\alpha\}\subset\mathbb{T}$ such that $\varphi(t_\alpha,x_1)\to x_2$. Clearly we have $\mathfrak{N}_x=\mathfrak{M}_{x,x}$ and $\mathfrak{M}_x=\cup_{z\in M}\mathfrak{M}_{x,z}$.

Let $(M_i,\mathcal{U}_i)$ be Hausdorff uniform spaces with uniform neighborhood systems $(V;A_i,\geq)$, $i=1,2$.

Definition 3.1. Let $(M_i,\mathbb{T},\varphi_i), i=1,2$, be two flows or semiflows.
(i) A point $x\in M_1$ is said to be *comparable* (respectively, *strongly comparable*) *with* $y\in M_2$ *by character of recurrence* if $\mathfrak{N}_y\subset\mathfrak{N}_x$ (respectively, $\mathfrak{M}_y\subset\mathfrak{M}_x$).
(ii) A point $x\in M_1$ is said to be *uniformly comparable with* $y\in M_2$ *by character of recurrence* if for arbitrary $a_1\in A_1$ there exists $a_2\in A_2$ such that

$$\varphi_2(t+\tau,y)\in V_{a_2}(\varphi_2(t,y)) \ \text{ implies } \ \varphi_1(t+\tau,x)\in V_{a_1}(\varphi_1(t,x))$$

for all $t\in\mathbb{T}$, or equivalently

$$\varphi_1(t_2,x)\in V_{a_1}(\varphi_1(t_1,x)) \ \text{ whenever } \ \varphi_2(t_2,y)\in V_{a_2}(\varphi_2(t_1,y)) \tag{3.1}$$

for $t_1,t_2\in\mathbb{T}$.

For brevity, we also simply say that x is (strongly or uniformly) comparable with y if no confusion arises.

Remark 3.2. It is immediate to see that the point x is comparable with y if and only if for any $a_1\in A_1$ there exists $a_2\in A_2$ such that $\varphi_2(\tau,y)\in V_{a_2}(y)$ for some $\tau\in\mathbb{T}$ implies $\varphi_1(\tau,x)\in V_{a_1}(x)$. By this characterization, it is clear that uniform comparability implies comparability. Indeed, it can also be checked that strong comparability implies comparability as stated in the following result:

Proposition 3.3. *Let $(M_i,\mathbb{T},\varphi_i), i=1,2$, be two flows or semiflows. If the point $x\in M_1$ is strongly comparable with $y\in M_2$, then x is comparable with y.*

Proof. Let $\{t_\alpha\}\in\mathfrak{N}_y$. Note that by the definition of $\mathfrak{N}_y$ and $\mathfrak{M}_y$ we have $\mathfrak{N}_y\subset\mathfrak{M}_y$, so $\{t_\alpha\}\in\mathfrak{M}_y\subset\mathfrak{M}_x$ by the strong comparability assumption. That is, $\varphi_1(t_\alpha,x)\to\bar{x}$ for some $\bar{x}\in M_1$. We only need to show that $\bar{x}=x$. We divide the proof into two steps.

Step 1. Take any member t_{α_0} from the net $\{t_\alpha\}$, then by the definition of nets there exists an index $\alpha_0'>\alpha_0$. Take another index β_0 such that $\beta_0>\alpha_0'$, then there exists an index $\beta_0'>\beta_0$. We continue in this way and obtain a collection of pairs $\{(t_{\alpha_\mu},t_{\alpha_\mu'})\}$ with the properties:

(i) $\alpha'_\mu > \alpha_\mu$ for any μ;

(ii) for any pair $(t_{\alpha_\mu}, t_{\alpha'_\mu})$ from this collection, there is another pair $(t_{\alpha_\nu}, t_{\alpha'_\nu})$ from this collection such that $\alpha_\nu > \alpha'_\mu$.

By the axiom of choice, there exists a function f from the index set of this collection to one member of each pair, i.e. $f : \mu \mapsto f(\mu)$ with $f(\mu) \in \{\alpha_\mu, \alpha'_\mu\}$ for each μ. So we obtain a subnet $\{t_{f(\mu)}\}$ of the net $\{t_\alpha\}$. Furthermore, by the construction of this subnet it is immediate to see that the remaining set $\{t_\alpha\} \setminus \{t_{f(\mu)}\}$ is also a subnet.

Step 2. Let $t_{f(\mu)} = 0$ for each member from the subnet $\{t_{f(\mu)}\}$ and other members from the subnet $\{t_\alpha\} \setminus \{t_{f(\mu)}\}$ remain unchanged; we denote by $\{\tilde{t}_\alpha\}$ the new net obtained from the net $\{t_\alpha\}$ in this way. Since $\varphi_2(t_\alpha, y)$ converges to y when t_α varies on the original net $\{t_\alpha\}$, $\varphi_2(t_\alpha, y)$ also converges to y when t_α varies on the subnet $\{t_\alpha\} \setminus \{t_{f(\mu)}\}$. But $\varphi_2(\tilde{t}_\alpha, y)$ remains y for all $\tilde{t}_\alpha$ from the subnet $\{t_{f(\mu)}\}$, so it follows that $\varphi_2(\tilde{t}_\alpha, y)$ is eventually in each neighborhood of y when $\tilde{t}_\alpha$ varies on the new net $\{\tilde{t}_\alpha\}$. That is, $\varphi_2(\tilde{t}_\alpha, y)$ converges to y when $\tilde{t}_\alpha$ varies on the new net $\{\tilde{t}_\alpha\}$.

By the uniform comparability assumption we have $\{\tilde{t}_\alpha\} \in \mathfrak{N}_y \subset \mathfrak{M}_x$, so $\varphi_1(\tilde{t}_\alpha, x)$ converges. But $\varphi_1(\tilde{t}_\alpha, x)$ remains x when $\tilde{t}_\alpha$ varies on the subnet $\{t_{f(\mu)}\}$, so this enforces that $\varphi_1(\tilde{t}_\alpha, x)$ converges to x for the new net $\{\tilde{t}_\alpha\}$. This implies that $\varphi_1(t_\alpha, x)$ converges to x as t_α varies on the subnet $\{t_\alpha\} \setminus \{t_{f(\mu)}\}$ of new net $\{\tilde{t}_\alpha\}$. But $\{t_\alpha\} \setminus \{t_{f(\mu)}\}$ is also a subnet of the original net $\{t_\alpha\}$, so by the assumption $\varphi_1(t_\alpha, x) \to \bar{x}$ for the original net $\{t_\alpha\}$ we get $x = \bar{x}$. The proof is complete. □

If the space $(M_1, \mathcal{U}_1)$ is complete, then uniform comparability implies strong comparability as the following result indicates:

Proposition 3.4. *Let $(M_i, \mathbb{T}, \varphi_i), i = 1, 2$, be two flows or semiflows. If $(M_1, \mathcal{U}_1)$ is complete and $x \in M_1$ is uniformly comparable with $y \in M_2$, then x is strongly comparable with y.*

Proof. Let x be uniformly comparable with y and take $\{t_\alpha\} \in \mathfrak{M}_y$. For given $a_1 \in A_1$, by the uniform comparability there exists $a_2 \in A_2$ such that $\varphi_2(t_2, y) \in V_{a_2}(\varphi_2(t_1, y))$ implies $\varphi_1(t_2, x) \in V_{a_1}(\varphi_1(t_1, x))$ for all $t_1, t_2 \in \mathbb{T}$. Since $\varphi_2(t_\alpha, y)$ converges, there exists α_0 such that $\varphi_2(t_\beta, y) \in V_{a_2}(\varphi_2(t_\alpha, y))$ whenever $\alpha, \beta \geq \alpha_0$, which further implies that $\varphi_1(t_\beta, x) \in V_{a_1}(\varphi_1(t_\alpha, x))$. By the completeness of $(M_1, \mathcal{U}_1)$, $\varphi_1(t_\alpha, x)$ converges and hence $\{t_\alpha\} \in \mathfrak{M}_x$. □

Lemma 3.5. *Let $(M_i, \mathbb{T}, \varphi_i), i = 1, 2$, be two flows or semiflows. If the point $\varphi_1(t, x)$ is comparable with point $\varphi_2(t, y)$ for all $t \in \mathbb{T}$, then $\mathfrak{M}_{y,\varphi_2(t,y)} \subset \mathfrak{M}_{x,\varphi_1(t,x)}$ for $t \in \mathbb{T}$.*

Proof. Fix $t \in \mathbb{T}$ and let $\{t_\alpha\} \in \mathfrak{M}_{y,\varphi_2(t,y)}$. We divide the proof into three cases.

Case 1. When α is large enough we have $t_\alpha \geq t$. In this case, we may assume that $t_\alpha = t + \tau_\alpha$ with $\tau_\alpha \geq 0$. Then we have

$$\varphi_2(\tau_\alpha, \varphi_2(t, y)) = \varphi_2(t_\alpha, y) \to \varphi_2(t, y),$$

i.e. $\{\tau_\alpha\} \in \mathfrak{N}_{\varphi_2(t,y)}$. Since $\varphi_1(t,x)$ is comparable with $\varphi_2(t,y)$, it follows that $\{\tau_\alpha\} \in \mathfrak{N}_{\varphi_1(t,x)}$ and so

$$\varphi_1(t_\alpha, x) = \varphi_1(\tau_\alpha, \varphi_1(t,x)) \to \varphi_1(t,x).$$

That is, $\{t_\alpha\} \in \mathfrak{M}_{x,\varphi_1(t,x)}$.

Case 2. When α is large enough we have $t_\alpha \in [0,t]$. If the subnet $\{t_\beta\}$ of $\{t_\alpha\}$ satisfies $t_\beta \to t_0$ for some $t_0 \in [0,t]$, then t can be written as $t = t_0 + \tau$ for some $\tau \in [0,t]$. Since $\{t_\beta\} \in \mathfrak{M}_{y,\varphi_2(t,y)}$, it follows that

$$\varphi_2(t_0, y) = \lim_\beta \varphi_2(t_\beta, y) = \varphi_2(t,y) = \varphi_2(\tau, \varphi_2(t_0, y)),$$

i.e. the point $\varphi_2(t_0,y)$ is τ-periodic. By the comparability of $\varphi_1(t_0,x)$ with $\varphi_2(t_0,y)$, we can see that $\{t_{\tilde\alpha}\} \in \mathfrak{N}_{\varphi_1(t_0,x)}$ for arbitrary net $\{t_{\tilde\alpha}\}$ with $t_{\tilde\alpha} \to \tau$, so τ is also a period of $\varphi_1(t_0,x)$. Thus, we have

$$\lim_\beta \varphi_1(t_\beta, x) = \varphi_1(t_0, x) = \varphi_1(\tau, \varphi_1(t_0,x)) = \varphi_1(t,x).$$

Since the subnet $\{t_\beta\}$ of net $\{t_\alpha\}$ is arbitrary, we have $\lim_\alpha \varphi_1(t_\alpha, x) = \varphi_1(t,x)$ (see [20, p. 74 (c)]); that is, $\{t_\alpha\} \in \mathfrak{M}_{x,\varphi_1(x,t)}$.

Case 3. If the net $\{t_\alpha\}$ contains two subnets $\{t_\beta\}$ and $\{t_\gamma\}$ such that $t_\beta \ge t$ and $t_\gamma \le t$ when β, γ are large enough, then we have the same conclusion by combining the above two cases. □

We now give a characterization for comparability:

Theorem 3.6. *Let $(M_i, \mathbb{T}, \varphi_i), i = 1, 2$, be two flows or semiflows. Then $\varphi_1(t,x)$ is comparable with $\varphi_2(t,y)$ for all $t \in \mathbb{T}$ if and only if there exists a continuous mapping $h : \gamma(y) \to \gamma(x)$ such that $h(\varphi_2(t,y)) = \varphi_1(t,x)$ for all $t \in \mathbb{T}$.*

Proof. Assume that the continuous mapping $h : \gamma(y) \to \gamma(x)$ satisfies $h(\varphi_2(t,y)) = \varphi_1(t,x)$ for all $t \in \mathbb{T}$. For given $t \in \mathbb{T}$, take $\{t_\alpha\} \in \mathfrak{N}_{\varphi_2(t,y)}$. Then we have

$$\begin{aligned}\lim_\alpha \varphi_1(t_\alpha, \varphi_1(t,x)) &= \lim_\alpha h(\varphi_2(t_\alpha, \varphi_2(t,y))) = h(\lim_\alpha \varphi_2(t_\alpha, \varphi_2(t,y))) \\ &= h(\varphi_2(t,y)) = \varphi_1(t,x).\end{aligned}$$

That is, $\{t_\alpha\} \in \mathfrak{N}_{\varphi_1(t,x)}$ and hence $\mathfrak{N}_{\varphi_2(t,y)} \subset \mathfrak{N}_{\varphi_1(t,x)}$.

Conversely, assume that $\varphi_1(t,x)$ is comparable with $\varphi_2(t,y)$ for all $t \in \mathbb{T}$. We choose a connected subset $\mathbb{T}_0 \subset \mathbb{T}$ such that $0 \in \mathbb{T}_0$ and for any $\tilde y \in \gamma(y)$ there exists a unique $t_{\tilde y} \in \mathbb{T}_0$ satisfying $\tilde y = \varphi_2(t_{\tilde y}, y)$. Define the synchronization mapping

$$h : \gamma(y) \to \gamma(x), \quad \tilde y \mapsto \varphi_1(t_{\tilde y}, x) =: h(\tilde y).$$

In particular, we have $h(y) = x$. For any given $t \in \mathbb{T}$, there exists a unique $\tilde y \in \gamma(y)$ such that $\tilde y = \varphi_2(t,y)$, and in this case we have $\varphi_2(t,y) = \varphi_2(t_{\tilde y}, y)$ with $t_{\tilde y} \in \mathbb{T}_0$.

By Lemma 3.5 we have $\mathfrak{M}_{\varphi_2(t_{\tilde{y}},y),\varphi_2(t,y)} \subset \mathfrak{M}_{\varphi_1(t_{\tilde{y}},x),\varphi_1(t,x)}$; in particular, taking $\{t_\alpha\} \in \mathfrak{M}_{\varphi_2(t_{\tilde{y}},y),\varphi_2(t,y)}$ with $t_\alpha \equiv 0$ we get $\varphi_1(t,x) = \varphi_1(t_{\tilde{y}},x)$. So we have

$$h(\varphi_2(t,y)) = h(\varphi_2(t_{\tilde{y}},y)) = \varphi_1(t_{\tilde{y}},x) = \varphi_1(t,x),$$

as required. Finally let us show that h is continuous. For a net $\{y_\alpha\} \subset \gamma(y)$ which converges to $y_0 \in \gamma(y)$, there exists corresponding $\{t_\alpha\} \subset \mathbb{T}_0$ and $t_0 \in \mathbb{T}_0$ such that $y_\alpha = \varphi_2(t_\alpha,y)$ and $y_0 = \varphi_2(t_0,y)$. Note that by Lemma 3.5 we have $\{t_\alpha\} \in \mathfrak{M}_{y,\varphi_2(t_0,y)} \subset \mathfrak{M}_{x,\varphi_1(t_0,x)}$, so $\varphi_1(t_\alpha,x) \to \varphi_1(t_0,x)$. That is, $h(y_\alpha) \to h(y_0)$. The proof is complete. □

We can also have similar characterizations for uniform comparability which are stated in the following Theorem 3.7 and Corollaries 3.8, 3.9.

Theorem 3.7. *Let $(M_i,\mathbb{T},\varphi_i), i = 1,2,$ be two flows or semiflows. Then $x \in M_1$ is uniformly comparable with $y \in M_2$ if and only if there exists a uniformly continuous function $h : \gamma(y) \to \gamma(x)$ such that $h(\varphi_2(t,y)) = \varphi_1(t,x)$ for all $t \in \mathbb{T}$.*

Proof. Let the uniform continuous function $h : \gamma(y) \to \gamma(x)$ be given. So for any $a_1 \in A_1$ there exists $a_2 \in A_2$ such that $\varphi_1(t_2,x) \in V_{a_1}(\varphi_1(t_1,x))$ whenever $\varphi_2(t_2,y) \in V_{a_2}(\varphi_2(t_1,y))$. But this is exactly the definition of uniform comparability.

Conversely, if x is uniformly comparable with y, then by Theorem 3.6 there exists a continuous function h which sends the orbit of y to that of x. We only need to show that this function h is uniformly continuous. But as shown in the first part of the proof, the uniform comparability of x with y is equivalent to the uniform continuity of h. □

Corollary 3.8. *Let $(M_1,\mathcal{U}_1)$ be complete and $x \in M_1$ be uniformly comparable with $y \in M_2$. Then there exists a uniform continuous function $\tilde{h} : H(y) \to H(x)$ which is a homomorphism, i.e.*

$$\tilde{h}(\varphi_2(t,\tilde{y})) = \varphi_1(t,\tilde{h}(\tilde{y})), \quad \textit{for all } \tilde{y} \in H(y) \textit{ and } t \in \mathbb{T}.$$

Proof. By [20, Theorem 6.26] we know that the function h admits a unique uniform continuous extension $\tilde{h} : H(y) \to H(x)$. We now show that $\tilde{h}$ is a homomorphism. For any given $\tilde{y} \in H(y)$ there exists a net $\{t_\alpha\} \subset \mathbb{T}$ such that $\lim_\alpha \varphi_2(t_\alpha,y) = \tilde{y}$. Since $\{t_\alpha\} \in \mathfrak{M}_y$, it follows from Proposition 3.4 that $\{t_\alpha\} \in \mathfrak{M}_x$ and we denote by $\tilde{x}$ the limit of $\varphi_1(t_\alpha,x)$. Then for given $t \in \mathbb{T}$, by the continuity of $\tilde{h}$ and φ_1,φ_2 we have

$$\begin{aligned}\tilde{h}(\varphi_2(t,\tilde{y})) &= \tilde{h}(\varphi_2(t,\lim_\alpha \varphi_2(t_\alpha,y))) = \lim_\alpha \tilde{h}(\varphi_2(t,\varphi_2(t_\alpha,y)))\\ &= \lim_\alpha \varphi_1(t,\varphi_1(t_\alpha,x)) = \varphi_1(t,\tilde{x}).\end{aligned}$$

On the other hand,

$$\tilde{x} = \lim_\alpha \varphi_1(t_\alpha,x) = \lim_\alpha h(\varphi_2(t_\alpha,y)) = \lim_\alpha \tilde{h}(\varphi_2(t_\alpha,y)) = \tilde{h}(\tilde{y}).$$

Therefore, we get $\tilde{h}(\varphi_2(t,\tilde{y})) = \varphi_1(t,\tilde{h}(\tilde{y}))$. The proof is complete. □

Corollary 3.9. *Let $(M_i, \mathbb{T}, \varphi_i), i = 1, 2$, be two flows or semiflows, and $(M_1, \mathcal{U}_1)$ be complete. Then $x \in M_1$ is uniformly comparable with $y \in M_2$ if and only if there exists a uniformly continuous function $\tilde{h} : H(y) \to H(x)$ which is a homomorphism, i.e.*

$$\tilde{h}(\varphi_2(t, \tilde{y})) = \varphi_1(t, \tilde{h}(\tilde{y})), \quad \text{for all } \tilde{y} \in H(y) \text{ and } t \in \mathbb{T}.$$

We now characterize strong comparability by homomorphism.

Theorem 3.10. *Let $(M_i, \mathbb{T}, \varphi_i), i = 1, 2$, be two flows or semiflows. Then $x \in M_1$ is strongly comparable with $y \in M_2$ if and only if there exists a continuous mapping $h : H(y) \to H(x)$ such that $h(y) = x$ and $h(\varphi_2(t, \tilde{y})) = \varphi_1(t, h(\tilde{y}))$ for all $\tilde{y} \in H(y)$ and $t \in \mathbb{T}$.*

Proof. Sufficiency. Let $h : H(y) \to H(x)$ be the function stated in the theorem. Take $\{t_\alpha\} \in \mathfrak{M}_y$ with $\varphi_2(t_\alpha, y) \to \tilde{y}$. Then by the properties of h we have

$$\varphi_1(t_\alpha, x) = \varphi_1(t_\alpha, h(y)) = h(\varphi_2(t_\alpha, y)) \to h(\tilde{y}) \in H(x),$$

i.e. $\{t_\alpha\} \in \mathfrak{M}_x$. So x is strongly comparable with y.

Necessity. We firstly define the function h. For any $\tilde{y} \in H(y)$, there exists a net $\{t_\alpha\}_{\alpha \in \Sigma_1}$ such that $\varphi_2(t_\alpha, y) \to \tilde{y}$. Then by the strong comparability assumption, the limit $\lim_\alpha \varphi_1(t_\alpha, x)$ exists and we denote it by $\tilde{x}$. Then we define $h(\tilde{y}) := \tilde{x}$. We need to show that the definition is well-defined, i.e. $h(\tilde{y})$ does not depend on the choice of the net $\{t_\alpha\}_{\alpha \in \Sigma_1}$. Indeed, if there exists another net $\{t_\beta\}_{\beta \in \Sigma_2}$ such that we also have $\varphi_2(t_\beta, y) \to \tilde{y}$, then the limit $\lim_\beta \varphi_1(t_\beta, x)$ exists. We consider the sequence of nets $\{t^n_\mu\}_{\mu \in \Sigma_n}$, for $n = 1, 2, \ldots$, with:

$$\{t^n_\mu\}_{\mu \in \Sigma_n} = \{t_\alpha\}_{\alpha \in \Sigma_1} \text{ for } n = 2k - 1 \text{ and } \{t^n_\alpha\}_{\mu \in \Sigma_n} = \{t_\beta\}_{\beta \in \Sigma_2} \text{ for } n = 2k.$$

Then clearly we have $\lim_n \lim_\mu \varphi_2(t^n_\mu, y) = \tilde{y}$ by the construction. Denote by $\mathbb{N}$ the set of natural numbers. Consider the product directed set

$$\Sigma := \mathbb{N} \times \prod \{\Sigma_n : n \in \mathbb{N}\}$$

and define the product order on Σ by $(n, g) \geq (m, f)$ if and only if $n \geq m$ and $g(k) \geq f(k)$ for all $k \in \mathbb{N}$. Then by [20, Theorem 2.4] we have $\lim_{(n,f) \in \Sigma} \varphi_2(t^n_{f(n)}, y) = \tilde{y}$ and hence $\{t^n_{f(n)}\}_{(n,f) \in \Sigma} \in \mathfrak{M}_y \subset \mathfrak{M}_x$; that is, $\varphi_1(t^n_{f(n)}, x)$ converges. But we note that $\{t^{2k-1}_{f(2k-1)}\}$ is a subnet of $\{t_\alpha\}_{\alpha \in \Sigma_1}$ and $\{t^{2k}_{f(2k)}\}$ is a subnet of $\{t_\beta\}_{\beta \in \Sigma_2}$, so this enforces that $\lim_\alpha \varphi_1(t_\alpha, x) = \lim_\beta \varphi_1(t_\beta, x)$, i.e. the mapping h is well-defined.

We now check that h satisfies the homomorphism property: $h(\varphi_2(t, \tilde{y})) = \varphi_1(t, h(\tilde{y}))$ for all $\tilde{y} \in H(y)$ and $t \in \mathbb{T}$. Take $\{t_\alpha\} \in \mathfrak{M}_{y,\tilde{y}} \subset \mathfrak{M}_y$, then by the group property (2.2) we have $\{t_\alpha + t\} \in \mathfrak{M}_{y,\varphi_2(t,\tilde{y})}$. By above definition for h we have $\{t_\alpha\} \in \mathfrak{M}_{x,h(\tilde{y})}$ and hence $\{t_\alpha + t\} \in \mathfrak{M}_{x,\varphi_1(t,h(\tilde{y}))}$, so $h(\varphi_2(t, \tilde{y})) = \varphi_1(t, h(\tilde{y}))$.

Lastly, let us show that h is continuous. Let $\{y^\mu\}_{\mu \in \Lambda} \subset H(y)$ be a net which converges to $\tilde{y} \in H(y)$. We need to show $h(y^\mu) \to h(\tilde{y})$. Indeed, for each y^μ there exists a net $\{t^\mu_\alpha\}_{\alpha\in\Sigma_\mu}$ such that $\varphi_2(t^\mu_\alpha, y) \to y^\mu$. So $\{t^\mu_\alpha\}_{\alpha\in\Sigma_\mu} \in \mathfrak{M}_y \subset \mathfrak{M}_x$ and we denote

$$\lim_{\alpha} \varphi_1(t^\mu_\alpha, x) =: x^\mu; \tag{3.2}$$

by the definition of h we have $h(y^\mu) = x^\mu$. Let

$$\Sigma := \Lambda \times \prod\{\Sigma_\mu : \mu \in \Lambda\}.$$

Define the product order on Σ by

$$(\nu, g) \geq (\mu, f) \text{ if and only if } \nu \geq \mu \text{ and } g(\mu) \geq f(\mu) \text{ for } \mu \in \Lambda.$$

Then by [20, Theorem 2.4] we have

$$\varphi_2(t^\mu_{f(\mu)}, y) \to \tilde{y}, \quad \text{for } (\mu, f) \in \Sigma.$$

So it follows that $\{t^\mu_{f(\mu)}\}_{(\mu,f)\in\Sigma} \in \mathfrak{M}_y \subset \mathfrak{M}_x$ and we denote

$$\lim_{(\mu,f)\in\Sigma} \varphi_1(t^\mu_{f(\mu)}, x) =: \tilde{x}; \tag{3.3}$$

by the definition of h we have $h(\tilde{y}) = \tilde{x}$. On the other hand, since $(M_1, \mathcal{U}_1)$ is a uniform space, for any $a_1 \in A_1$ there exists $b_1 \in A_1$ such that $x_1 \in V_{a_1}(x_3)$ whenever $x_1 \in V_{b_1}(x_2)$ and $x_2 \in V_{b_1}(x_3)$. By (3.2) and (3.3), we have $\varphi_1(t^\mu_{f(\mu)}, x) \in V_{b_1}(x^\mu)$ and $\varphi_1(t^\mu_{f(\mu)}, x) \in V_{b_1}(\tilde{x})$ when (μ, f) is large enough in Σ. This implies $x^\mu \in V_{a_1}(\tilde{x})$, i.e. $h(y^\mu) \in V_{a_1}(h(\tilde{y}))$. The proof is complete. □

We have the following relation between strong comparability and uniform comparability:

Corollary 3.11. *Let $(M_i, \mathbb{T}, \varphi_i), i = 1, 2,$ be two flows or semiflows. Let $(M_1, \mathcal{U}_1)$ be complete and $H(y)$ be compact. Then $x \in M_1$ is strongly comparable with $y \in M_2$ if and only if x is uniformly comparable with y.*

Proof. The necessity follows immediately from Theorems 3.7 and 3.10. The sufficiency follows from Corollary 3.8 and Theorem 3.10. □

Remark 3.12. By Corollary 3.11, to verify the uniform comparability in applications we only need to verify strong comparability (i.e. $\mathfrak{M}_y \subset \mathfrak{M}_x$) if y is Lagrange stable and the state space of x is complete, which is more realistic in practice.

The following result provides another criterion for uniform comparability and comparability.

Theorem 3.13. *Let $\mathbb{T} = \mathbb{R}$ or $\mathbb{R}^+$. Let Φ be a cocycle over θ with fiber space $\mathcal{X}$ and base space $\mathcal{Y}$, and $\nu : \mathcal{Y} \to \mathcal{X}$ be a continuous mapping satisfying that $\nu(\theta_t y) = \Phi(t, \nu(y), y)$ for all $(t, y) \in \mathbb{T} \times \mathcal{Y}$. Then:*

(i) The continuous mapping $\zeta := (\nu, Id_{\mathcal{Y}})$ *from* $(\mathcal{Y}, \mathbb{R}, \theta)$ *to* $(\mathcal{E}, \mathbb{T}, \Pi)$ *is a homomorphism, i.e.* $\zeta(\theta_t y) = \Pi(t, \zeta(y))$ *for all* $(t, y) \in \mathbb{T} \times \mathcal{Y}$*, where* $\mathcal{E} := \mathcal{X} \times \mathcal{Y}$ *and* Π *is the skew-product flow (or semiflow) corresponding to the cocycle* Φ*, i.e.* $\Pi(t, \zeta(y)) = (\Phi(t, \nu(y), y), \theta_t y)$*.*

(ii) The point $\zeta(y) = (\nu(y), y) \in \mathcal{E}$ *is comparable with the point* y *by character of recurrence.*

(iii) If $H(y)$ *is compact, then the point* $\zeta(y) = (\nu(y), y) \in \mathcal{E}$ *is uniformly comparable with the point* y *by character of recurrence.*

Proof. The statement (i) follows immediately from the property $\nu(\theta_t y) = \Phi(t, \nu(y), y)$ and the definition of skew-product flows/semiflows. The statements (ii) and (iii) follow from Theorems 3.6, 3.7 and the fact that ζ is a homomorphism. □

For the relations between (uniform) comparability and Poisson stable motions, we have the following important results:

Theorem 3.14. *Let* $(M_i, \mathbb{R}, \varphi_i), i = 1, 2,$ *be two flows. The following statements hold:*

(i) Let $x \in M_1$ *be comparable with* $y \in M_2$ *by character of recurrence. If* y *is stationary (respectively,* τ*-periodic, Levitan almost periodic, almost recurrent, Poisson stable), then so is* x*.*

(ii) Let $x \in M_1$ *be uniformly comparable with* $y \in M_2$ *by character of recurrence. If* y *is quasi-periodic (respectively, Bohr almost periodic, almost automorphic, Birkhoff recurrent, Lagrange stable, pseudo-periodic, pseudo-recurrent), then so is* x*.*

Proof. Let us prove the conclusion for each class of recurrent motions.

Stationary. If y is stationary then any net $\{t_\alpha\}$ belongs to $\mathfrak{N}_y$ and hence belongs to $\mathfrak{N}_x$ by the comparability assumption. This enforces that x is stationary.

Periodic. Take any net $\{t_\alpha\}$ with $t_\alpha \to \tau$. If y is τ-periodic and x is comparable with y, then we have $\{t_\alpha\} \in \mathfrak{N}_y \subset \mathfrak{N}_x$, which implies that x is τ-periodic.

Levitan almost periodic. If y is Levitan almost periodic, then there exists a Bohr almost periodic point z with respect to another flow such that $\mathfrak{N}_z \subset \mathfrak{N}_y$. But by assumption x is comparable with y, i.e. $\mathfrak{N}_y \subset \mathfrak{N}_x$, so we have $\mathfrak{N}_z \subset \mathfrak{N}_x$. That is, x is Levitan almost periodic.

Almost recurrent. If x is comparable with y, then for any $a_1 \in A_1$ there exists $a_2 \in A_2$ such that $\varphi_1(\tau, x) \in V_{a_1}(x)$ whenever $\varphi_2(\tau, y) \in V_{a_2}(y)$. When y is almost recurrent, by definition, for this chosen $a_2 \in A_2$ there exists $l = l(a_2)$ such that for any interval $[t_0, t_0 + l]$ we can find a $\tau \in [t_0, t_0 + l]$ satisfying $\varphi_2(\tau, y) \in V_{a_2}(y)$. Therefore, x is almost recurrent.

Poisson stable. Since y is Poisson stable, there exists a net $\{t_\alpha\}$ with $t_\alpha \to +\infty$ $(-\infty)$ such that $\{t_\alpha\} \in \mathfrak{N}_y$. But x is comparable with y, i.e. $\mathfrak{N}_y \subset \mathfrak{N}_x$. So we have $\{t_\alpha\} \in \mathfrak{N}_x$. That is, x is Poisson stable.

Quasi-periodic. Note that quasi-periodicity can be characterized as follows: y is quasi-periodic (with base frequencies $\omega_1, \dots, \omega_m$) if for all $t \in \mathbb{T}$ we have $\varphi_2(t, y) = \Phi(\sigma(t, z))$ for some continuous function $\Phi : T^m \to M_2$ and $z \in T^m$, where $\sigma(t, z)$ is a rotation on the m-torus T^m, i.e. $\sigma(t, z) = (z_1 + \omega_1 t, \dots, z_m + \omega_m t)$, and $\Phi(\sigma(0, z)) = \Phi(z) = y$. Now x is uniformly comparable with y, so there exists a uniformly continuous function $h : H(y) \to H(x)$ such that $\varphi_1(t, x) = h(\varphi_2(t, y))$ for any $t \in \mathbb{T}$. So we get $\varphi_1(t, x) = h \circ \Phi(\sigma(t, z))$ with $h \circ \Phi : T^m \to M_1$ being continuous. That is, x is quasi-periodic.

Lagrange stable. If y is Lagrange, then $H(y)$ is compact. It then follows from Theorem 3.7 and Corollary 3.8 that $H(x)$ is compact and hence x is Lagrange stable.

Almost automorphic and Birkhoff recurrent. Note that y is almost automorphic (respectively, Birkhoff recurrent) if and only if y is Levitan almost periodic (respectively, almost recurrent) and is Lagrange stable. But as shown above the uniform comparability of x with y implies that x is Levitan almost periodic (respectively, almost recurrent, Lagrange stable) when so is y. Thus the result follows.

Bohr almost periodic and pseudo-periodic. Since x is uniformly comparable with y, for any $a_1 \in A_1$ there exists $a_2 \in A_2$ such that $\varphi_1(t_2, x) \in V_{a_1}(\varphi_1(t_1, x))$ whenever $\varphi_2(t_2, y) \in V_{a_2}(\varphi_2(t_1, y))$. For this a_2, if $\tau \in \mathcal{T}(y, a_2)$, i.e. is an a_2-almost period of y, then τ is an a_1-almost period of x, i.e. we have $\varphi_1(t + \tau, x) \in V_{a_1}(\varphi_1(t, x))$ for all $t \in \mathbb{R}$.

When y is Bohr almost periodic (respectively, pseudo-periodic), $\mathcal{T}(y, a_2)$ is relatively dense (respectively, unbounded). But we have shown above that $\mathcal{T}(y, a_2) \subset \mathcal{T}(x, a_1)$, so $\mathcal{T}(x, a_1)$ is relatively dense (respectively, unbounded). That is, x is Bohr almost periodic (respectively, pseudo-periodic).

Pseudo-recurrent. By the definition x is pseudo-recurrent if for arbitrary $a_1 \in A_1$ and $t_0 \in \mathbb{R}$ there exists $T = T(a_1, t_0)$ such that $\varphi_1([t_0, t_0 + T], z) \cap V_{a_1}(z) \neq \emptyset$ for any $z \in \gamma(x)$, that is:

$$\varphi_1([t_0, t_0 + T], \varphi_1(t, x)) \cap V_{a_1}(\varphi_1(t, x)) \neq \emptyset, \quad \text{for all } t \in \mathbb{T}. \tag{3.4}$$

Since x is uniformly comparable with y, for the above a_1 there exists $a_2 \in A_2$ such that $\varphi_1(t_2, x) \in V_{a_1}(\varphi_1(t_1, x))$ whenever $\varphi_2(t_2, y) \in V_{a_2}(\varphi_2(t_1, y))$. On the other hand, by the pseudo recurrence of y it follows that for the above a_2 and t_0 there exists $\widetilde{T} = \widetilde{T}(a_2, t_0)$ such that

$$\varphi_2([t_0, t_0 + \widetilde{T}], \varphi_2(t, y)) \cap V_{a_2}(\varphi_2(t, y)) \neq \emptyset, \quad \text{for all } t \in \mathbb{T}.$$

This implies that (3.4) holds with $T = \widetilde{T}(a_2(a_1), t_0)$. That is, x is pseudo-recurrent. □

Remark 3.15. Since uniform comparability implies comparability, the classes of recurrent motions appearing in Theorem 3.14 (i) are trivially included in (ii).

4 Continuous skew-product semiflow associated to SDEs

In this section, we prove that the distribution of solutions of SDEs naturally generates a semiflow or skew-product semiflow on the space of probability measures, which is interesting in itself.

4.1 The stochastic ODE case

Consider the stochastic differential equation on $\mathbb{R}^d$

$$\mathrm{d}X = f(t,X)\mathrm{d}t + g(t,X)\mathrm{d}W. \tag{4.1}$$

where $f(t,x) : \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}^d$, $g(t,x) : \mathbb{R} \times \mathbb{R}^d \to \mathcal{M}^{d\times m}$ are continuous and W is an m-dimensional Brownian motion. Here $\mathcal{M}^{d\times m}$ denotes the set of all $d \times m$-matrices, equipped with norm of $\mathbb{R}^{dm}$.

We equip the spaces $C(\mathbb{R}\times\mathbb{R}^d, \mathbb{R}^d)$ and $C(\mathbb{R}\times\mathbb{R}^d, \mathcal{M}^{d\times m})$ with the compact-open topology, which are metric spaces with the metric given by

$$\rho(f,g) := \sum_{n=1}^{\infty} \frac{1}{2^n} \min\left\{1, \sup_{(t,x)\in I_n\times B_n} d(f(t,x), g(t,x))\right\}$$

for $f, g \in C(\mathbb{R} \times \mathbb{R}^d, \mathbb{R}^d)$ or $f, g \in C(\mathbb{R} \times \mathbb{R}^d, \mathcal{M}^{d\times m})$, where $I_n = [-n, n]$ and B_n is the closed ball in $\mathbb{R}^d$ centered at the origin with radius n. We consider the shift flow on their product space $C(\mathbb{R} \times \mathbb{R}^d, \mathbb{R}^d) \times C(\mathbb{R} \times \mathbb{R}^d, \mathcal{M}^{d\times m})$, i.e. $\theta_t(f, g) = (f^t, g^t)$. Also the hull $H(f, g)$ of the point (f, g) is given by

$$H(f,g) := \{(\bar{f}, \bar{g}) : \text{there exists a sequence } \{s_n\} \subset \mathbb{R} \text{ such that } f^{s_n} \to \bar{f} \text{ and } g^{s_n} \to \bar{g}\},$$

where the convergence is with respect to the compact-open topology.

Define the mapping

$$\Phi : \mathbb{R}^+ \times \mathcal{P}(\mathbb{R}^d) \times H(f,g) \to \mathcal{P}(\mathbb{R}^d), \tag{4.2}$$

with $\Phi(t, \mu_0, (\bar{f}, \bar{g}))$ being the law (or distribution) $\mathcal{L}(X(t))$ on $\mathbb{R}^d$ of the solution $X(\cdot)$ at time t of the equation

$$\mathrm{d}X = \bar{f}(t,X)\mathrm{d}t + \bar{g}(t,X)\mathrm{d}W, \quad X(0) = X_0, \tag{4.3}$$

where $\mathcal{L}(X_0) = \mu_0$ and $\Phi(0, \mu_0, (\bar{f}, \bar{g})) = \mu_0$.

We need the following assumption:

(H1) Let $\{(f_n, g_n)\}_{n=1}^{\infty} \subset H(f, g)$ be a sequence such that $f_n \to \bar{f}$ and $g_n \to \bar{g}$ as $n \to \infty$. Denote by $X_n(\cdot)$ the solutions of the following equations

$$\mathrm{d}X_n = f_n(t,X_n)\mathrm{d}t + g_n(t,X_n)\mathrm{d}W, \quad X_n(0) = X_{n,0}, \tag{4.4}$$

where the initial value satisfies $\mathcal{L}(X_{n,0}) \to \mathcal{L}(X_0)$. Then for any compact interval $[s, T]$

$$\sup_{t\in[s,T]} \beta(\mu_n(t), \mu(t)) \to 0,$$

where $\mu_n(t) = \mathcal{L}(X_n(t))$, $\mu(t) = \mathcal{L}(X(t))$ with $X(\cdot)$ being the solution of (4.3).

We have the following basic result on Φ:

Theorem 4.1. *Assume that (H1) holds. Then the mapping* Φ *given by* (4.2) *is a continuous cocycle with base space* $H(f,g)$ *and fiber space* $\mathcal{P}(\mathbb{R}^d)$*, i.e. the mapping* $\Phi : \mathbb{R}^+ \times \mathcal{P}(\mathbb{R}^d) \times H(f,g) \to \mathcal{P}(\mathbb{R}^d)$ *is continuous and satisfies*

$$\Phi(0,\mu,(\bar{f},\bar{g})) = \mu, \quad \Phi(t+\tau,\mu,(\bar{f},\bar{g})) = \Phi(t,\Phi(\tau,\mu,(\bar{f},\bar{g})),(\bar{f}^\tau,\bar{g}^\tau)) \tag{4.5}$$

for any $t,\tau \geq 0$*,* $(\bar{f},\bar{g}) \in H(f,g)$ *and* $\mu \in \mathcal{P}(\mathbb{R}^d)$*.*

Proof. The mapping Φ satisfies the cocycle property (4.5). Indeed, if $X(\cdot)$ satisfies the equation (4.3), then it is immediate to check that the process $X(\cdot+\tau)$ satisfies the equation

$$\mathrm{d}Y = \bar{f}^\tau(t,Y)\mathrm{d}t + \bar{g}^\tau(t,Y)\mathrm{d}\tilde{W}^\tau, \quad Y(0) = X(\tau)$$

with $\widetilde{W}^\tau(t) := W(t+\tau) - W(\tau)$ for $t \in \mathbb{R}$. Note that $\widetilde{W}$ is a Brownian motion with the same distribution as W. Then by the uniqueness of law on $\mathbb{R}^d$ for solutions of (4.3) (note that (H1) implies uniqueness of law on $\mathbb{R}^d$ holds for (4.3)) and the Kolmogorov-Chapman equality we have

$$\Phi(t+\tau,\mu,(\bar{f},\bar{g})) = \Phi(t,\Phi(\tau,\mu,(\bar{f},\bar{g})),(\bar{f}^\tau,\bar{g}^\tau)),$$

i.e. the cocycle property holds, and $\Phi(0,\mu,(\bar{f},\bar{g})) = \mu$ is obvious by the meaning of the notation Φ.

Next we show that Φ is a continuous mapping. Take sequences $\{t_n\} \subset \mathbb{R}^+$, $\{(f_n,g_n)\} \subset H(f,g)$, and $\{\mu_n\} \subset \mathcal{P}(\mathbb{R}^d)$ such that $t_n \to t_0$, $f_n \to \bar{f}$, $g_n \to \bar{g}$ and $\mu_n \to \mu_0$ as $n \to \infty$. For any given $\epsilon > 0$, when n is large enough we have

$$\beta(\Phi(t_n,\mu_n,(f_n,g_n)),\Phi(t_n,\mu_0,(\bar{f},\bar{g}))) < \frac{\epsilon}{2}, \quad \beta(\Phi(t_n,\mu_0,(\bar{f},\bar{g})),\Phi(t_0,\mu_0,(\bar{f},\bar{g}))) < \frac{\epsilon}{2}$$

by (H1) and the continuity of the law $\mu(\cdot) = \mathcal{L}(X(\cdot))$ in t. So it follows that

$$\begin{aligned} \beta(\Phi(t_n,\mu_n,(f_n,g_n)),\Phi(t_0,\mu_0,(\bar{f},\bar{g}))) &\leq \beta(\Phi(t_n,\mu_n,(f_n,g_n)),\Phi(t_n,\mu_0,(\bar{f},\bar{g}))) \\ &\quad + \beta(\Phi(t_n,\mu_0,(\bar{f},\bar{g})),\Phi(t_0,\mu_0,(\bar{f},\bar{g}))) \\ &< \epsilon \end{aligned}$$

when n is large. The proof is complete. □

Corollary 4.2. *The mapping given by*

$$\Pi : \mathbb{R}^+ \times \mathcal{P}(\mathbb{R}^d) \times H(f,g) \to \mathcal{P}(\mathbb{R}^d) \times H(f,g), \quad \Pi(t,(\bar{f},\bar{g}),\mu) := (\Phi(t,\mu,(\bar{f},\bar{g})),(\bar{f}^t,\bar{g}^t))$$

is a continuous skew-product semiflow.

Remark 4.3. Now a natural question arises: when does the assumption (H1) hold? Indeed, this assumption is satisfied in normal situations. Consider the stochastic differential equation (4.1). If we assume that $f : \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}^d$, $g : \mathbb{R} \times \mathbb{R}^d \to \mathcal{M}^{d\times m}$ are continuous and satisfy Lipschitz and linear growth conditions in $x \in \mathbb{R}^d$ uniformly with respect to $t \in \mathbb{R}$, then it is well known that (H1) holds; see, e.g. [19, p.54, Theorem 3] for details. Indeed, this last theorem states a more stronger fact: the law of X_n converges to that of X in the path space $C(\mathbb{R},\mathbb{R}^d)$.

Indeed, the assumption (H1) holds under very general conditions, see the following Theorem 4.4 for details.

Theorem 4.4. *Let* $\{(f_n, g_n)\}_{n=1}^{\infty}$ *be a sequence such that* $f_n \to \bar{f}$ *and* $g_n \to \bar{g}$ *pointwise as* $n \to \infty$. *Assume that* $f_n, g_n, \bar{f}, \bar{g}$ *are locally Lipschitz in* $x \in \mathbb{R}^d$ *with local Lipschitz constants uniformly bounded by a positive constant independent of* $t \in \mathbb{R}$, *i.e. for any* $r > 0$ *there exists* $L(r) > 0$ *such that*

$$|f_n(t,x) - f_n(t,y)| \le L(r)|x-y| \quad \text{and} \quad |\bar{f}(t,x) - \bar{f}(t,y)| \le L(r)|x-y|$$

whenever $|x|, |y| \le r$, *the similar holding for* g_n *and* $\bar{g}$. *Assume further that the solutions of* (4.4) *and* (4.3) *exist globally in* t. *Then if the initial value satisfies* $\mathcal{L}(X_{n,0}) \to \mathcal{L}(X_0)$, *the solution* $X_n(\cdot)$ *of* (4.4) *converges to the solution* $X(\cdot)$ *of* (4.3) *in distribution on the path space* $C(\mathbb{R}^+, \mathbb{R}^d)$. *In particular, we have for any compact interval* $[0,T]$

$$\sup_{t\in[0,T]} \beta(\mu_n(t), \mu(t)) \to 0,$$

where $\mu_n(t) = \mathcal{L}(X_n(t))$, $\mu(t) = \mathcal{L}(X(t))$ *for* $t \in [0,T]$.

Proof. By the Skorohod representation theorem and the uniqueness of laws of the solutions to (4.4) and (4.3), we may assume that $X_{n,0} \to X_0$ almost surely by possibly extending the underlying probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Note that for any $\epsilon > 0$ we may take a constant $C_1 > 0$ such that

$$\mathbb{P}\{|X_0| \le C_1, |X_{n,0}| \le C_1 \text{ for all } n\} \ge 1 - \epsilon/4.$$

Since the solution of (4.3) has continuous path, there exists a constant $C_2 > C_1$ such that

$$\mathbb{P}\{\sup_{t\in[0,T]} |X(t)| \le C_2\} \ge 1 - \epsilon/2. \tag{4.6}$$

We take globally Lipschitz functions $\tilde{f}_n, \tilde{g}_n, \tilde{f}, \tilde{g}$ such that

$$\tilde{f}_n = f_n,\ \tilde{g}_n = g_n,\ \tilde{f} = \bar{f},\ \tilde{g} = \bar{g}$$

on B_{C_2+1}, i.e. the ball centered at the origin with radius $C_2 + 1$, and let

$$\widetilde{X}_{n,0} = X_{n,0} \cdot \mathcal{I}_{|x|\le C_1},\ \widetilde{X}_0 = X_0 \cdot \mathcal{I}_{|x|\le C_1},$$

where $\mathcal{I}_A$ means the indicator function of the set A. Denote by $\widetilde{X}(\cdot)$ the solution of equation (4.3) with $\bar{f}, \bar{g}, X_0$ replaced by $\tilde{f}, \tilde{g}, \widetilde{X}_0$ respectively and by $\widetilde{X}_n(\cdot)$ the solution of equation (4.4) with $\bar{f}_n, \bar{g}_n, X_{n,0}$ replaced by $\tilde{f}_n, \tilde{g}_n, \widetilde{X}_{n,0}$ respectively. Then it is known (see e.g. [19, p.52, Theorem 2]) that

$$\lim_{n\to\infty} E \sup_{t\in[0,T]} |\widetilde{X}_n(t) - \widetilde{X}(t)|^2 = 0. \tag{4.7}$$

On the other hand, we know (see e.g. [19, p.44, Theorem 2]) that

$$\widetilde{X}_n(t,\omega) = X_n(t,\omega),\ \widetilde{X}(t,\omega) = X(t,\omega) \quad \text{for } t \in [0,T]$$

on the set $\Omega_1 := \{\omega : \sup_{t\in[0,T]} |X(t,\omega)| \le C_2\}$; by (4.6) we have $\mathbb{P}(\Omega_1) \ge 1 - \epsilon/2$. This together with (4.7) implies that there exists $N = N(\epsilon)$ such that

$$\mathbb{P}\{\sup_{t\in[0,T]} |X_n(t) - X(t)| > \epsilon\} < \epsilon \quad \text{whenever } n \ge N,$$

i.e. $X_n(\cdot)$ converges to $X(\cdot)$ uniformly in probability on $[0,T]$. The proof is complete. □

4.2 The stochastic PDE case

Let V and H be separable Hilbert spaces, and $L(V,H)$ be the space of bounded linear operators from V to H. We consider the following semilinear stochastic differential equation for H-valued stochastic process X

$$\mathrm{d}X = AX\mathrm{d}t + f(t,X)\mathrm{d}t + g(t,X)\mathrm{d}W, \tag{4.8}$$

where A is an infinitesimal generator which generates a C^0-semigroup $\{U(t)\}_{t\ge0}$ on H, $f \in C(\mathbb{R}\times H, H)$, $g \in C(\mathbb{R}\times H, L(V,H))$, W is a V-valued Q-Wiener process with covariance operator Q being of trace class, i.e. $\mathrm{Tr}Q < \infty$. As for the detailed definition of Q-Wiener processes and their basic properties as well as stochastic integrals based on them, we refer the reader to the monograph [17] for details.

Like in ODE case, we consider the compact-open topology on the spaces $C(\mathbb{R}\times H, H)$ and $C(\mathbb{R}\times H, L(V,H))$, and we may define the shift flow on them and consider the hull $H(f,g)$. But the difference is that the spaces $C(\mathbb{R}\times H, H)$ and $C(\mathbb{R}\times H, L(V,H))$ are not metrizable because H is of infinite dimension. The compact-open topology on $C(\mathbb{R}\times H, H)$ is generated by the family $\mathcal{D} := \{d_K : K \subset \mathbb{R}\times H \text{ compact}\}$ of pseudo-metrics on $C(\mathbb{R}\times H, H)$:

$$d_K(f_1,f_2) := \sup_{(t,x)\in K} \|f_1(t,x) - f_2(t,x)\|_H \quad \text{for } f_1, f_2 \in C(\mathbb{R}\times H, H).$$

The compact-open topology generated by the above family of pseudo-metrics is actually a uniform topology which makes $C(\mathbb{R}\times H, H)$ a uniform space; see the appendix for details. Completely in the same way, the space $C(\mathbb{R}\times H, L(V,H))$ is also a uniform space with the topology being the compact-open topology generated by a family of pseudo-metrics on $C(\mathbb{R}\times H, L(V,H))$. As in Section 2, we can define the shift flow on $C(\mathbb{R}\times H, H) \times C(\mathbb{R}\times H, L(V,H))$.

Similar to stochastic ODE case, we define the mapping

$$\Phi : \mathbb{R}^+ \times \mathcal{P}(H) \times H(f,g) \to \mathcal{P}(H), \tag{4.9}$$

with $\Phi(t,\mu_0,(\bar f,\bar g))$ being the law (or distribution) $\mathcal{L}(X(t))$ on H of the solution $X(\cdot)$ at time t of the equation

$$\mathrm{d}X = AX\mathrm{d}t + \bar f(t,X)\mathrm{d}t + \bar g(t,X)\mathrm{d}W, \quad X(0) = \xi_0, \tag{4.10}$$

where $\mathcal{L}(\xi_0) = \mu_0$ and $\Phi(0,\mu_0,(\bar f,\bar g)) = \mu_0$.

Similar to (H1), we formulate the following assumption for (4.8):

(H2) Let $\{(f_\alpha, g_\alpha)\} \subset H(f,g)$ be a net such that $f_\alpha \to \bar{f}$ and $g_\alpha \to \bar{g}$ as $\alpha \to \infty$. Denote by $X_\alpha(\cdot)$ the solutions of the following equations

$$\mathrm{d}X_\alpha = AX_\alpha \mathrm{d}t + f_\alpha(t, X_\alpha)\mathrm{d}t + g_\alpha(t, X_\alpha)\mathrm{d}W, \quad X_\alpha(0) = \xi_\alpha, \tag{4.11}$$

where the initial value satisfies $\mathcal{L}(\xi_\alpha) \to \mathcal{L}(\xi_0)$. Then for any compact interval $[s,T]$

$$\sup_{t\in[s,T]} \beta(\mu_\alpha(t), \mu(t)) \to 0 \quad \text{as } \alpha \to \infty,$$

where $\mu_\alpha(t) = \mathcal{L}(X_\alpha(t))$, $\mu(t) = \mathcal{L}(X(t))$ with $X(\cdot)$ being the solution of (4.10).

Then we have the following result on Φ:

Theorem 4.5. *Assume that (H2) holds. Then the mapping Φ given by* (4.9) *is a continuous cocycle with base space $H(f,g)$ and fiber space $\mathcal{P}(H)$, i.e. the mapping $\Phi : \mathbb{R}^+ \times \mathcal{P}(H) \times H(f,g) \to \mathcal{P}(H)$ is continuous and satisfies*

$$\Phi(0, \mu, (\bar{f}, \bar{g})) = \mu, \quad \Phi(t+\tau, \mu, (\bar{f}, \bar{g})) = \Phi(t, \Phi(\tau, \mu, (\bar{f}, \bar{g})), (\bar{f}^\tau, \bar{g}^\tau))$$

for any $t, \tau \geq 0$, $(\bar{f}, \bar{g}) \in H(f,g)$ and $\mu \in \mathcal{P}(H)$.

Proof. The proof is completely similar to Theorem 4.1 if we notice that $X(\cdot + \tau)$ satisfies the equation

$$\mathrm{d}Y = AY\mathrm{d}t + \bar{f}^\tau(t, Y)\mathrm{d}t + \bar{g}^\tau(t, Y)\mathrm{d}\widetilde{W}^\tau, \quad Y(0) = X(\tau)$$

with $\widetilde{W}^\tau$ being a Q-Wiener process with the same distribution as W, provided $X(\cdot)$ satisfies equation (4.10). □

Like Corollary 4.2, the similar result holds for SPDE case:

Corollary 4.6. *The mapping given by*

$$\Pi : \mathbb{R}^+ \times \mathcal{P}(H) \times H(f,g) \to \mathcal{P}(H) \times H(f,g), \quad \Pi(t, (\bar{f}, \bar{g}), \mu) := (\Phi(t, \mu, (\bar{f}, \bar{g})), (\bar{f}^t, \bar{g}^t))$$

is a continuous skew-product semiflow.

The condition (H2) holds under fairly general conditions, see the following Theorem 4.7 for details.

Theorem 4.7. *Let* $\sup_{t\in\mathbb{R}} |f(t,0)| \vee |g(t,0)| \leq C$ *for some constant $C > 0$ and f, g be Lipschitz in $x \in H$ with Lipschitz constants independent of t. Then the solution $X_\alpha(\cdot)$ of* (4.11) *converges to the solution $X(\cdot)$ of* (4.10) *in distribution on the path space $C(\mathbb{R}^+, H)$. In particular, (H2) holds.*

Proof. Note that if $\sup_{t\in\mathbb{R}}|f(t,0)|\vee|g(t,0)|\le C$ and f,g are Lipschitz, then each pair (f_α,g_α) from the hull $H(f,g)$ satisfies the same condition with the same constant C and Lipschitz constants.

(1) Let $\mathcal{H}$ be the Banach space of all the H-valued progressively measurable processes $Y(\cdot)$ defined on the interval $[0,T]$ with the norm

$$\|Y\|_c := \left(E\sup_{t\in[0,T]}|Y(t)|^2\right)^{1/2} < +\infty,$$

by identifying the indistinguishable processes. Denote $L^2 := L^2(\Omega,\mathcal{F}_0,\mathbb{P})$. Let the mapping $\mathcal{S}: L^2\times\mathcal{H}\to\mathcal{H}$ be given by

$$\mathcal{S}(\xi,Y)(t) := U(t)\xi + \int_0^t U(t-s)\bar f(s,Y(s))\mathrm{d}s + \int_0^t U(t-s)\bar g(s,Y(s))\mathrm{d}W(s)$$

for $(\xi,Y)\in L^2\times\mathcal{H}$ and $t\in[0,T]$. Similarly the mapping $\mathcal{S}_\alpha: L^2\times\mathcal{H}\to\mathcal{H}$ is given by

$$\mathcal{S}_\alpha(\xi,Y)(t) := U(t)\xi + \int_0^t U(t-s)f_\alpha(s,Y(s))\mathrm{d}s + \int_0^t U(t-s)g_\alpha(s,Y(s))\mathrm{d}W(s)$$

for $(\xi,Y)\in L^2\times\mathcal{H}$ and $t\in[0,T]$.

Clearly the mapping $\mathcal{S}$ is Lipschitz in $\xi\in L^2$. The mapping $\mathcal{S}$ is also Lipshitz in $X\in\mathcal{H}$; indeed, by Cauchy-Schwarz inequality and martingale inequality we have:

$$\begin{aligned}
&\|\mathcal{S}(\xi,X)-\mathcal{S}(\xi,Y)\|_c^2\\
&= E\sup_{t\in[0,T]}|\mathcal{S}(\xi,X)(t)-\mathcal{S}(\xi,Y)(t)|^2\\
&\le 2E\sup_{t\in[0,T]}\left|\int_0^t U(t-s)(\bar f(s,X(s))-\bar f(s,Y(s))\mathrm{d}s\right|^2\\
&\qquad + 2E\sup_{t\in[0,T]}\left|\int_0^t U(t-s)(\bar g(s,X(s))-\bar g(s,Y(s))\mathrm{d}W(s)\right|^2\\
&\le 2K^2L^2\left(T\cdot E\int_0^T|X(s)-Y(s)|^2\mathrm{d}s + 4E\int_0^T|X(s)-Y(s)|^2\mathrm{d}s\right)\\
&\le 2K^2L^2(T+4)T\cdot E\sup_{t\in[0,T]}|X(s)-Y(s)|^2\\
&= 2K^2L^2(T+4)T\cdot\|X-Y\|_c,
\end{aligned}\tag{4.12}$$

where $K:=\sup_{t\in[0,T]}\|U(t)\|$. When T is appropriately small such that

$$2K^2L^2(T+4)T<1,\tag{4.13}$$

the mapping $\mathcal{S}$ admits a unique fixed point $X(\xi)$ which is the solution of the equation (4.10) with initial value ξ and X is Lipschitz in $\xi\in L^2$ (when ξ varies) by Theorem

2.18. Since f_α and g_α admit the same Lipschitz constants as f and g, the same estimate as (4.12) holds for the mappings $\mathcal{S}_\alpha$ and that the solution X_α of (4.11) is Lipschitz in $\xi \in L^2$ with the same Lipschitz constant as the solution X of (4.10).

(2) By the weak uniqueness of mild solutions to equations (4.10) and (4.11) and the Skorohod representation theorem, we may assume that $\xi_\alpha \to \xi_0$ almost surely by possibly extending the probability space $(\Omega, \mathcal{F}, \mathbb{P})$. As in the first step, denote by $X(\xi_0)$ and $X_\alpha(\xi_\alpha)$ the solutions of (4.10) and (4.11) respectively.

Denote for each α and all t, x

$$\begin{aligned}&\tilde{\xi}_\alpha = \xi_\alpha \mathrm{e}^{-|\xi_\alpha|},\ \tilde{f}_\alpha(t,x) = \mathrm{e}^{-|\xi_\alpha|} f_\alpha(t, \mathrm{e}^{|\xi_\alpha|}x),\ \tilde{g}_\alpha(t,x) = \mathrm{e}^{-|\xi_\alpha|} g_\alpha(t, \mathrm{e}^{|\xi_\alpha|}x),\\ &\tilde{\xi}_0 = \xi_0 \mathrm{e}^{-|\xi_0|},\ \tilde{f}(t,x) = \mathrm{e}^{-|\xi_0|}\bar{f}(t, \mathrm{e}^{|\xi_0|}x),\ \tilde{g}(t,x) = \mathrm{e}^{-|\xi_0|}\bar{g}(t, \mathrm{e}^{|\xi_0|}x),\end{aligned}$$

and

$$\widetilde{X}_\alpha(\tilde{\xi}_\alpha) = \mathrm{e}^{-|\xi_\alpha|} X_\alpha(\xi_\alpha),\ \widetilde{X}(\tilde{\xi}_0) = \mathrm{e}^{-|\xi_0|} X(\xi_0).$$

Then it is immediate to see that $L_{\tilde{f}_\alpha} = L_{f_\alpha}$, $L_{\tilde{g}_\alpha} = L_{g_\alpha}$, $L_{\tilde{f}} = L_{\bar{f}}$, $L_{\tilde{g}} = L_{\bar{g}}$, and that $\widetilde{X}_\alpha(\tilde{\xi}_\alpha)$ and $\widetilde{X}(\tilde{\xi}_0)$ satisfy the equations

$$\begin{aligned}\widetilde{X}_\alpha(\tilde{\xi}_\alpha)(t) = U(t-s)\tilde{\xi}_\alpha &+ \int_0^t U(t-r)\tilde{f}_\alpha(r, \widetilde{X}_\alpha(\tilde{\xi}_\alpha)(r))\mathrm{d}r\\ &+ \int_0^t U(t-r)\tilde{g}_\alpha(r, \widetilde{X}_\alpha(\tilde{\xi}_\alpha)(r))\mathrm{d}W(r)\end{aligned} \tag{4.14}$$

and

$$\widetilde{X}(\tilde{\xi}_0)(t) = U(t-s)\tilde{\xi}_0 + \int_0^t U(t-r)\tilde{f}(r, \widetilde{X}(\tilde{\xi}_0)(r))\mathrm{d}r + \int_0^t U(t-r)\tilde{g}(r, \widetilde{X}(\tilde{\xi}_0)(r))\mathrm{d}W(r), \tag{4.15}$$

respectively. Note that all $\tilde{\xi}_\alpha$ and $\tilde{\xi}_0$ are bounded by a common constant and that $|\tilde{f}(t,0)| \le |\bar{f}(t,0)|$ and $|\tilde{g}(t,0)| \le |\bar{g}(t,0)|$ for all t; the similar holds for $\tilde{f}_\alpha$ and f_α, and $\tilde{g}_\alpha$ and g_α.

Denote the mappings from $L^2 \times \mathcal{H}$ to $\mathcal{H}$ by

$$\begin{aligned}\widetilde{\mathcal{S}}_\alpha(\xi, X)(t) &:= U(t-s)\xi + \int_0^t U(t-r)\tilde{f}_\alpha(r, X(r))\mathrm{d}r + \int_0^t U(t-r)\tilde{g}_\alpha(r, X(r))\mathrm{d}W(r),\\ \widetilde{\mathcal{S}}(\xi, X)(t) &:= U(t-s)\xi + \int_0^t U(t-r)\tilde{f}(r, X(r))\mathrm{d}r + \int_0^t U(t-r)\tilde{g}(r, X(r))\mathrm{d}W(r)\end{aligned}$$

for $(\xi, X) \in L^2 \times \mathcal{H}$ and $t \in [0,T]$. Then for any fixed $(\xi, X) \in L^2 \times \mathcal{H}$ we have by Cauchy-Schwarz inequality and martingale inequality:

$$\begin{aligned}&\|\widetilde{\mathcal{S}}_\alpha(\xi, X) - \widetilde{\mathcal{S}}(\xi, X)\|_c^2\\ &= E \sup_{t\in[0,T]} |\widetilde{\mathcal{S}}_\alpha(\xi, X) - \widetilde{\mathcal{S}}(\xi, X)|^2\\ &\le 2E \sup_{t\in[0,T]} \left(\int_0^t U(t-s)(\tilde{f}_\alpha(s, X(s)) - \tilde{f}(s, X(s)))\mathrm{d}s\right)^2\end{aligned}$$

$$
\begin{aligned}
&+ 2E \sup_{t\in[0,T]} \left(\int_0^t U(t-s)(\tilde{g}_\alpha(s,X(s)) - \tilde{g}(s,X(s)))\mathrm{d}W(s) \right)^2 \\
&\le 2K^2T \cdot E \int_0^T |\tilde{f}_\alpha(s,X(s)) - \tilde{f}(s,X(s))|^2 \mathrm{d}s \\
&+ 8K^2 \cdot E \int_0^T |\tilde{g}_\alpha(s,X(s)) - \tilde{g}(s,X(s))|^2 \mathrm{d}s.
\end{aligned}
$$

So by the dominated convergence theorem[1], we have

$$
\|\widetilde{\mathcal{S}}_\alpha(\xi, X) - \widetilde{\mathcal{S}}(\xi, X)\|_c \to 0 \quad \text{as } \alpha \to \infty. \tag{4.16}
$$

Note that the solutions $\widetilde{X}_\alpha(\tilde{\xi}_\alpha)$ and $\widetilde{X}(\tilde{\xi}_0)$ of (4.14) and (4.15) are the unique fixed points of the mappings $\widetilde{\mathcal{S}}_\alpha(\tilde{\xi}_\alpha, \cdot)$ and $\widetilde{\mathcal{S}}(\tilde{\xi}_0, \cdot)$ respectively. Since $L_{\tilde{f}_\alpha} = L_{\tilde{f}}$ and $L_{\tilde{g}_\alpha} = L_{\tilde{g}}$, it follows from step (1) that there exists a constant λ (independent of α) such that

$$
\|\widetilde{X}_\alpha(\tilde{\xi}_\alpha) - \widetilde{X}_\alpha(\tilde{\xi}_0)\|_c \le \lambda |\tilde{\xi}_\alpha - \tilde{\xi}_0| \quad \text{for all } \alpha,
$$

where the meaning of $\widetilde{X}_\alpha(\tilde{\xi}_0)$ is obvious. Furthermore, by Theorem 2.18 and (4.16) we have

$$
\|\widetilde{X}_\alpha(\tilde{\xi}_0) - \widetilde{X}(\tilde{\xi}_0)\|_c \to 0 \quad \text{as } \alpha \to \infty.
$$

Therefore,

$$
\|\widetilde{X}_\alpha(\tilde{\xi}_\alpha) - \widetilde{X}(\tilde{\xi}_0)\|_c \le \|\widetilde{X}_\alpha(\tilde{\xi}_\alpha) - \widetilde{X}_\alpha(\tilde{\xi}_0)\|_c + \|\widetilde{X}_\alpha(\tilde{\xi}_0) - \widetilde{X}(\tilde{\xi}_0)\|_c \to 0
$$

as α goes to infinity. So we have $\widetilde{X}_\alpha(\tilde{\xi}_\alpha) \to \widetilde{X}(\tilde{\xi}_0)$ in probability on the path space $C(\mathbb{R}^+, H)$. Since $\xi_\alpha \to \xi_0$ almost surely, $X = \mathrm{e}^{|\xi_0|}\widetilde{X}(\tilde{\xi}_0)$ and $X_\alpha = \mathrm{e}^{|\xi_\alpha|}\widetilde{X}_\alpha(\tilde{\xi}_\alpha)$, we have $X_\alpha \to X$ in probability as $\alpha \to \infty$ on the path space $C(\mathbb{R}^+, H)$. This implies that $X_\alpha \to X$ in distribution on the path space $C(\mathbb{R}^+, H)$.

Finally, we have assumed T to satisfy the condition (4.13) in above arguments; for general $T > 0$, we only need to consider the corresponding equations on the intervals $[0, \widetilde{T}], [\widetilde{T}, 2\widetilde{T}], [2\widetilde{T}, 3\widetilde{T}], \dots$ with $\widetilde{T}$ satisfying (4.13). The proof is now complete. □

5 Applications

Theorem 5.1. *Consider the equation* (4.8). *Assume that A generates an exponentially stable C^0-semigroup $\{U(t)\}_{t\ge 0}$ on H, i.e. there are positive constants N, ν such that $\|U(t)\| \le N\mathrm{e}^{-\nu t}$ for $t \ge 0$, that* $\sup_{t\in\mathbb{R}} |f(t,0)| \vee |g(t,0)| \le C$ *for some constant $C > 0$ and that f, g are Lipschitz in $x \in H$ with* $\max\{L_f, L_g\} \le L$ *for some positive constant L. Assume further that*

$$
\theta := 2N^2L^2 \left(\frac{1}{\nu^2} + \frac{1}{2\nu} \right) < 1.
$$

[1]Note that the dominated convergence theorem remains valid for the net case.

Then equation (4.8) *admits a unique* L^2*-bounded solution* ϕ *on* $\mathbb{R}$ *and there is a continuous mapping* $\mathscr{T} : H(f, g) \to \mathcal{P}(H)$ *satisfying*

$$\mathscr{T}(\bar{f}^t, \bar{g}^t) = \Phi(t, \mathscr{T}(\bar{f}, \bar{g}), (\bar{f}, \bar{g})) \tag{5.1}$$

for all $(t, (\bar{f}, \bar{g})) \in \mathbb{R} \times H(f, g)$*, where* Φ *is the cocycle generated by* (4.8)*.*

Proof. (1) Since the semigroup $U(t)$ is exponentially stable, it is immediate to check that $X_0 \in C_b(\mathbb{R}, L^2(\mathbb{P}, H))$ is a mild solution of (4.8) if and only if it satisfies the following integral equation:

$$X_0(t) = \int_{-\infty}^t U(t-s) f(s, X_0(s)) \mathrm{d}s + \int_{-\infty}^t U(t-s) g(s, X_0(s)) \mathrm{d}W(s).$$

Define an operator $\mathcal{T}$ on $C_b(\mathbb{R}, L^2(\mathbb{P}, H))$ by

$$(\mathcal{T}X)(t) := \int_{-\infty}^t U(t-s) f(s, X(s)) \mathrm{d}s + \int_{-\infty}^t U(t-s) g(s, X(s)) \mathrm{d}W(s). \tag{5.2}$$

Since f, g satisfy $\sup_{t\in\mathbb{R}} |f(t, 0)| \vee |g(t, 0)| \le C$ and the Lipschitz condition, it is not hard to check that $\mathcal{T}$ maps $C_b(\mathbb{R}, L^2(\mathbb{P}, H))$ into itself.

Then by Cauchy-Schwarz inequality and Itô's isometry we have for $X, Y \in C_b(\mathbb{R}, L^2(\mathbb{P}, H))$ and $t \in \mathbb{R}$

$$\begin{aligned}
&\mathbb{E}|(\mathcal{T}X)(t) - (\mathcal{T}Y)(t)|^2 \\
&\le 2\Bigg[\mathbb{E}\left|\int_{-\infty}^t U(t-s)(f(s, X(s)) - f(s, Y(s)))\mathrm{d}s\right|^2 \\
&\quad + \mathbb{E}\left|\int_{-\infty}^t U(t-s)(g(s, X(s)) - g(s, Y(s)))\mathrm{d}W(s)\right|^2\Bigg] \\
&\le 2\Bigg[\mathbb{E}\left(\int_{-\infty}^t N\mathrm{e}^{-\nu(t-s)} L|X(s) - Y(s)|\mathrm{d}s\right)^2 \\
&\quad + \mathbb{E}\int_{-\infty}^t N^2\mathrm{e}^{-2\nu(t-s)} L^2|X(s) - Y(s)|^2\mathrm{d}s\Bigg] \\
&\le 2N^2L^2\Bigg[\left(\int_{-\infty}^t \mathrm{e}^{-\nu(t-s)}\mathrm{d}s\right) \cdot \mathbb{E}\int_{-\infty}^t \mathrm{e}^{-\nu(t-s)}|X(s) - Y(s)|^2\mathrm{d}s \\
&\quad + \mathbb{E}\int_{-\infty}^t \mathrm{e}^{-2\nu(t-s)}|X(s) - Y(s)|^2\mathrm{d}s\Bigg] \\
&\le 2N^2L^2\left(\frac{1}{\nu^2} + \frac{1}{2\nu}\right) \cdot \sup_{s\in\mathbb{R}} \mathbb{E}|X(s) - Y(s)|^2.
\end{aligned}$$

Therefore,

$$\sup_{t\in\mathbb{R}} \mathbb{E}|(\mathcal{T}X)(t) - (\mathcal{T}Y)(t)|^2 \le \theta \cdot \sup_{t\in\mathbb{R}} \mathbb{E}|X(t) - Y(t)|^2.$$

That is, the operator $\mathcal{T}$ is a contraction mapping on $C_b(\mathbb{R}, L^2(\mathbb{P}, H))$. Thus there is a unique $\phi \in C_b(\mathbb{R}, L^2(\mathbb{P}, H))$ satisfying $\mathcal{T}\phi = \phi$, which is the unique L^2-bounded solution of (4.8).

(2) We now show that for fixed $X_0 \in C_b(\mathbb{R}, L^2(\mathbb{P}, H))$, the mapping $\mathcal{T}$ depends continuously on the coefficients from $H(f, g)$. Take a net $\{(f_\alpha, g_\alpha)\} \subset H(f, g)$ with $(f_\alpha, g_\alpha) \to (\bar{f}, \bar{g})$ as $\alpha \to \infty$. If (f, g) in (5.2) is replaced by (f_α, g_α), we denote the corresponding operator by $\mathcal{T}_\alpha$; similarly we denote it by $\bar{\mathcal{T}}$ for $(\bar{f}, \bar{g})$. Then we have for any given $t \in \mathbb{R}$

$$\begin{aligned}
&\mathbb{E}|(\mathcal{T}_\alpha X_0)(t) - (\bar{\mathcal{T}} X_0)(t)|^2 \\
&\le 2\Bigg[\mathbb{E}\left|\int_{-\infty}^t U(t-s)(f_\alpha(s, X_0(s)) - \bar{f}(s, X_0(s)))\mathrm{d}s\right|^2 \\
&\quad + \mathbb{E}\left|\int_{-\infty}^t U(t-s)(g_\alpha(s, X_0(s)) - \bar{g}(s, X_0(s)))\mathrm{d}W(s)\right|^2\Bigg] \\
&\le 2N^2\Bigg[\frac{1}{\nu}\mathbb{E}\int_{-\infty}^t \mathrm{e}^{-\nu(t-s)}|f_\alpha(s, X_0(s)) - \bar{f}(s, X_0(s))|^2\mathrm{d}s \\
&\quad + \mathbb{E}\int_{-\infty}^t \mathrm{e}^{-2\nu(t-s)}|g_\alpha(s, X_0(s)) - \bar{g}(s, X_0(s))|^2\mathrm{d}s\Bigg].
\end{aligned}$$

By the condition $\sup_{t\in\mathbb{R}} |f(t, 0)| \vee |g(t, 0)| \le C$, the Lipschitz condition and dominated convergence theorem, we get

$$\lim_{\alpha\to\infty} \mathbb{E}|(\mathcal{T}_\alpha X_0)(t) - (\bar{\mathcal{T}} X_0)(t)|^2 = 0 \quad \text{for any } t \in \mathbb{R}. \tag{5.3}$$

(3) Denote by $\phi_{(\bar{f},\bar{g})}$ the unique L^2-bounded solution of the equation

$$\mathrm{d}X = AX\mathrm{d}t + \bar{f}(t, X)\mathrm{d}t + \bar{g}(t, X)\mathrm{d}W.$$

Define the mapping

$$\mathscr{T} : H(f, g) \to \mathcal{P}(H), \quad (\bar{f}, \bar{g}) \mapsto \mathcal{L}(\phi_{(\bar{f},\bar{g})}(0)),$$

the law of the bounded solution $\phi_{(\bar{f},\bar{g})}$ at the "time" 0. It follows from (5.3) and Theorem 2.18 that the unique L^2-bounded ϕ depends continuously on $(\bar{f}, \bar{g}) \in H(f, g)$ and hence the mapping $\mathscr{T}$ is continuous. We now show that

$$\mathscr{T}(\theta_t\bar{f}, \theta_t\bar{g}) = \mathcal{L}(\phi_{(\bar{f},\bar{g})}(t)) \quad \text{for } t \in \mathbb{R}, \tag{5.4}$$

i.e. (5.1) holds.

Fix $t \in \mathbb{R}$. Note that

$$\begin{aligned}
&\phi_{(\bar{f},\bar{g})}(t) \\
&= \int_{-\infty}^t U(t-s)\bar{f}(s, \phi_{(\bar{f},\bar{g})}(s))\mathrm{d}s + \int_{-\infty}^t U(t-s)\bar{g}(s, \phi_{(\bar{f},\bar{g})}(s))\mathrm{d}W(s)
\end{aligned}$$

$$= \int_{-\infty}^{0} U(-\tilde{s})\bar{f}(\tilde{s}+t, \phi_{(\bar{f},\bar{g})}(\tilde{s}+t))\mathrm{d}\tilde{s} + \int_{-\infty}^{0} U(-\tilde{s})\bar{g}(\tilde{s}+t, \phi_{(\bar{f},\bar{g})}(\tilde{s}+t))\mathrm{d}\tilde{W}(\tilde{s})$$
$$= \int_{-\infty}^{0} U(-s)\bar{f}_t(s, \phi_{(\bar{f},\bar{g})}(s+t))\mathrm{d}s + \int_{-\infty}^{0} U(-s)\bar{g}_t(s, \phi_{(\bar{f},\bar{g})}(s+t))\mathrm{d}\tilde{W}(s), \quad (5.5)$$

where in the 2nd equality we let $\tilde{s} = s - t$ and in the 3rd equality we let $s = \tilde{s}$ and $\widetilde{W}(s) = W(s+t) - W(t)$ for $s \in \mathbb{R}$. Note that $\widetilde{W}(\cdot)$ is a Brownian motion which has the same law as $W(\cdot)$. On the other hand, denote by $\tilde{\phi}$ be the unique L^2-bounded solution of the equation

$$\mathrm{d}X = AX\mathrm{d}s + \bar{f}_t(s, X)\mathrm{d}s + \bar{g}_t(s, X)\mathrm{d}W,$$

then we have

$$\tilde{\phi}(0) = \int_{-\infty}^{0} U(-s)\bar{f}_t(s, \tilde{\phi}(s))\mathrm{d}s + \int_{-\infty}^{0} U(-s)\bar{g}_t(s, \tilde{\phi}(s))\mathrm{d}W(s).$$

This together with (5.5) and the uniqueness of the L^2-bounded solutions yields that

$$\mathcal{L}(\tilde{\phi}(0)) = \mathcal{L}(\phi_{(\bar{f},\bar{g})}(t)).$$

That is, (5.4) holds. The proof is complete. □

Theorem 5.2. *Consider the equation* (4.8)*. Assume that the conditions of Theorem 5.1 hold.*

- *(i) If f and g are jointly stationary (respectively, τ–periodic, quasi-periodic with the spectrum of frequencies $\nu_1, \nu_2, \ldots, \nu_k$, Bohr almost periodic, Bohr almost automorphic, Birkhoff recurrent, Lagrange stable, Levitan almost periodic, almost recurrent, Poisson stable) in $t \in \mathbb{R}$ uniformly with respect to $x \in H$ on every compact subset, then so is the unique bounded solution ϕ of equation* (4.8) *in distribution.*
- *(ii) If f and g are jointly pseudo-periodic (respectively, pseudo-recurrent) and f and g are jointly Lagrange stable, in $t \in \mathbb{R}$ uniformly with respect to $x \in H$ on every compact subset, then the unique bounded solution ϕ of* (4.8) *is pseudo-periodic (respectively, pseudo-recurrent) in distribution.*

Proof. The result follows immediately from Theorems 5.1, 3.13 and 3.14. □

Remark 5.3. Note that the conditions of Theorems 5.1 and 5.2 are much weaker than that in our earlier work [11, 15]; in particular, we do not need the condition (C3) in [11], which is crucial there and sometimes not easy to verify or even not satisfied at all.

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6 Appendix

In this section, we review some notions and facts about uniform spaces; the reader can refer to [20] for details.

Definition 6.1 (Uniform neighborhood system)**.** Let M be a set, $(A, \geq)$ a directed set, and $V_a(x)$ a subset of M for each $a \in A$ and $x \in M$. Then the triple $(V; A, \geq)$ is called *uniform neighborhood system (UNS)* for M if the following conditions are satisfied:

(i) $x \in V_a(x)$;

(ii) $V_a(x) \subset V_b(x)$ whenever $a \geq b$ for all $x \in M$;

(iii) [symmetric condition] $y \in V_a(x)$ if and only if $x \in V_a(y)$;

(iv) [uniform condition] for each $a \in A$ there exists $b \in A$ such that $z \in V_a(x)$ whenever $z \in V_b(y)$ and $y \in V_b(x)$.

Definition 6.2 (Uniform space)**.** For given UNS $(V; A, \geq)$ for M, a set $U \subset M$ is called an *open set* if for each $x \in U$ there exists a $V_a(x) \subset U$. A set $U \subset M$ is called a *closed set* if its complement U^c is open. The topology $\mathcal{U}$ generated by these open sets (or equivalently by V and A) is called a *uniform topology* and the resulting topological space $(M, \mathcal{U})$ is called a *uniform space*.

Remark 6.3. (i) We may assume without loss of generality that each $V_a(x)$ is open (in the uniform topology). Indeed, if this is not true, we can use the largest open set $V_a^0(x)$ in $V_a(x)$ to replace $V_a(x)$, then it can be shown that $(V^0; A, \geq)$ is a UNS and that V^0 and A generate the same topology as V and A. So we will always assume that $V_a(x)$ is open in this paper.

(ii) We will also assume that the uniform space $(M, \mathcal{U})$ is *Hausdorff*, i.e. every net in M converges to at most one limit point.

(iii) A net $\{x_\alpha\}$ (with range in M) is *Cauchy* if for any $a \in A$ there is an index α_0 such that $x_{\alpha'} \in V_a(x_{\alpha''})$ whenever $\alpha', \alpha'' \geq \alpha_0$. A uniform space $(M, \mathcal{U})$ is *complete* if each Cauchy net in M is convergent. We note that each (Hausdorff) uniform space has a (Hausdorff) completion.

Definition 6.4. A function $d : M \times M \to \mathbb{R}_+$ is called *pseudo-metric* on M if the following conditions are fulfilled:

(i) $d(x, y) = d(y, x)$;

(ii) $d(x, z) \le d(x, y) + d(y, z)$;

(iii) $x = y$ implies $d(x, y) = 0$.

If the converse implication in (iii) is also true, then d becomes a *metric*.

A family of pseudo-metrics can naturally generate a uniform topology. Indeed, for a given family $\mathscr{D}$ of pseudo-metrics on M, define a directed set $(A, \ge)$ as follows: $(d_1, r_1) \ge (d_2, r_2)$ if and only if $r_1 \le r_2$ and $d_1(x, y) \ge d_2(x, y)$ for all $x, y \in M$, where $d_i \in \mathscr{D}$ and r_i are positive numbers. For simplicity, we denote $d_1 \ge d_2$ if $d_1(x, y) \ge d_2(x, y)$ for all $x, y \in M$. Denote $V_{d,r}(x) := \{y \in M : d(x, y) < r\}$. Then it can be shown that $(V; A, \ge)$ is a UNS for M. A set $U \subset M$ is an open set if for each $x \in U$ there exist finite number of $d_i, r_i, i = 1, \dots, n$, such that

$$\bigcap_{i=1}^{n} V_{d_i,r_i}(x) \subset U.$$

These open sets then generate a uniform topology on M. Therefore, the family $\mathscr{D}$ of pseudo-metrics generates a uniform space $(M, \mathcal{U})$. It can be shown that this uniform space $(M, \mathcal{U})$ is Hausdorff if and only if for any two distinct points $x, y \in M$ there is a $d \in \mathscr{D}$ such that $d(x, y) > 0$. In particular, when $\mathscr{D}$ is a singleton set, the resulting topology is called a *pseudo-metric topology* and the space M is *pseudo-metrizable*.

Note that if ρ and σ are two pseudo-metrics on M, then $\rho \vee \sigma := \max\{\rho, \sigma\}$ is also a pseudo-metric on M and we have $\rho \vee \sigma \ge \rho$ and $\rho \vee \sigma \ge \sigma$. For given family $\mathscr{D}$ of pseudo-metrics on M, let $\mathscr{D}'$ denote the smallest family of pseudo-metrics on M such that: (i) $\mathscr{D} \subset \mathscr{D}'$; (ii) for any ρ and σ in $\mathscr{D}'$ we have $\rho \vee \sigma \in \mathscr{D}'$. It is immediate to show that both $\mathscr{D}$ and $\mathscr{D}'$ generate the same uniform topology on M, so we may assume without loss of generality that the family $\mathscr{D}$ of pseudo-metrics is closed under the operation "$\vee$". In particular, if $\mathscr{D}$ is closed under $\vee$ then for any σ and σ in $\mathscr{D}$ there is $\lambda \in \mathscr{D}$ such that $\lambda \ge \rho$ and $\lambda \ge \sigma$.

Conversely, for a given uniform space $(M, \mathcal{U})$, it is known that (cf. [20, Theorem 6.15]) the uniform topology $\mathcal{U}$ is generated by the family of all pseudo-metrics which are uniformly continuous on $M \times M$; it is also known that a uniform space $(M, \mathcal{U})$ is pseudo-metrizable if and only if $\mathcal{U}$ has a countable base.

Therefore, it is equivalent to describe uniform spaces in the framework of UNS or in the framework of (family of) pseudo-metrics.

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