BULETINUL ACADEMIEI DE ŞTIINŢE
A REPUBLICII MOLDOVA. MATEMATICA
Numbers 1(104)−2(105), 2024, Pages 44–52
ISSN 1024–7696, E-ISSN 2587–4322

Asymptotic behaviour of non-autonomous Caputo fractional differential equations with a one-sided dissipative vector field

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Abstract. A non-autonomous Caputo fractional differential equation of order $\alpha \in (0,1)$ in $\mathbb{R}^d$ with a driving system $\{\vartheta_t\}_{t\in\mathbb{R}}$ on a compact base space P generates a skew-product flow on $\mathfrak{C}_\alpha \times P$, where $\mathfrak{C}_\alpha$ is the space of continuous functions $f : \mathbb{R}^+ \to \mathbb{R}^d$ with a weighted norm giving uniform convergence on compact time subsets. It was shown by Cui & Kloeden [3] to have an attractor when the vector field of the Caputo FDE satisfies a uniform dissipative vector field. This attractor is closed, bounded and invariant in $\mathfrak{C}_\alpha \times P$ and attracts bounded subsets of $\mathfrak{C}_\alpha$ consisting of constant initial functions. The structure of this attractor is investigated here in detail for an example with a vector field satisfying a stronger one-sided dissipative Lipschitz condition. In particular, the component sets of the attractor are shown to be singleton sets corresponding to a unique entire solution of the skew-product flow. Its evaluation on $\mathbb{R}^d$ is a unique entire solution of the Caputo FDE, which is both pullback and forward attracting.

Mathematics subject classification: 34A08, 34K20, 37B99, 45J05, 45E99.
Keywords and phrases: Non-autonomous Caputo fractional differential equations, skew-product flows, attractor, entire solution,Volterra integral equations.

Dedicated to the memory of Professor B.A. Shcherbakov on the occasion of his 100th anniversary.

1 Introduction

Consider a non-autonomous Caputo fractional differential equation (FDE) of order $\alpha \in (0,1)$ in $\mathbb{R}^d$ with a driving system of the form

$$^{C}D^{\alpha}_{0+}x(t) = g(x(t), \vartheta_t(p)) \qquad \text{for } t \in [0,T]. \tag{1}$$

with a driving system in the vector field, specifically $g(x,p)$, where $\vartheta_t : P \to P$, $t \in \mathbb{R}$, is a group of operators, i.e., an autonomous dynamical system, and P is a suitable metric space.

The solution of the Caputo FDE (1) with initial condition $x(0) = x_0$ and $p_0 \in P$ satisfies the integral equation

$$x(t) = x_0 + \int_0^t a(t,s)g(x(s), \vartheta_s(p_0))ds, \tag{2}$$

DOI: https://doi.org/10.56415/basm.y2024.i1-2.p44

where

$$a(t,s) := \frac{1}{\Gamma(\alpha)}(t-s)^{\alpha-1}, \qquad 0 \le s < t,$$

is a singular but integrable kernel and $\Gamma(\alpha) := \int_0^\infty t^{\alpha-1}e^{-t}dt$ is the Gamma function.

As a motivational example consider the scalar Caputo FDE

$$^{C}D^{\alpha}_{0+}x(t) = -x(t) + \cos t, \tag{3}$$

for which P is the *hull* [11, 13] of the functions $\cos(\cdot)$, i.e.,

$$P = \bigcup_{0\le\tau\le 2\pi} \cos(\tau + \cdot),$$

which is a compact metric space with the metric induced by the supremum norm

$$d_P(p_1, p_2) = \sup_{t\in\mathbb{R}} |p_1(t) - p_2(t)|.$$

In addition, let $\vartheta_t : P \to P$ be the left shift operator $\vartheta_t(\cos(\cdot)) = \cos(t + \cdot)$. This shift operator is continuous in the above metric. Indeed, it is an isometry with

$$d_P(\vartheta_t(p_1), \vartheta_t(p_2)) = d_P(p_1, p_2), \quad p_1, p_2 \in P.$$

The existence and uniqueness of solutions and continuity in initial data holds under the following Assumptions; the second one can be weakened. The proof is similar to that in the autonomous case, i.e., without the driving system, see [5, 7].

Assumption 1. *Let (P, d_P) be a compact metric space and let $\{\vartheta_t\}_{t\in\mathbb{R}}$ be a group of continuous mappings $\vartheta_t : P \to P$, $t \in \mathbb{R}$.*

Assumption 2. *There exists $L > 0$ such that for all $x, y \in \mathbb{R}^d$, $p, q \in P$*

$$\|g(x,p) - g(y,q)\| \le L\|x - y\| + Ld_P(p,q).$$

Unlike for ODEs, these local solutions cannot be patched together to provide a global solution for Caputo FDE. The proof follows by a fixed point argument on the space of continuous functions $C([0,T],\mathbb{R}^d)$ with a Bielecki weighted norm of the form

$$\|x\|_\gamma := \sup_{t\in[0,T]} \frac{\|x(t)\|}{E_\alpha(\gamma t^\alpha)} \qquad \text{for all } x \in C([0,T],\mathbb{R}^d),$$

where $\gamma > 0$ is a suitable constant and the weight function is the Mittag-Leffler function $E_\alpha(\cdot)$ defined as follows:

$$E_\alpha(t) := \sum_{k=0}^{\infty} \frac{t^k}{\Gamma(\alpha k + 1)} \quad \text{for all } t \in \mathbb{R}.$$

More general Mittag-Leffler functions with parameters α, $\beta > 0$ are defined by

$$E_{\alpha,\beta}(t) := \sum_{k=0}^{\infty} \frac{t^k}{\Gamma(k\alpha + \beta)} \quad \text{for all } t \in \mathbb{R}.$$

The Caputo FDE (1) is nonlocal, specifically its solutions depends on their history and not just at the current time. This means, in particular, they cannot generate a semi-group (when g depends only on x, i.e., autonomous case) or two-parameter semi-group (non-autonomous case) on $\mathbb{R}^d$. This issue is of some interest since attractors are usually defined mathematically in terms of some form of dynamical system [10, 11].

Interestingly, Cong & Tuan [1] did show that the solutions of an autonomous Caputo FDE generate a "nonlocal" dynamical system on $\mathbb{R}^d$ for scalar and multi-dimensional triangular vector fields. This follows from the fact [2, Theorem 3.5] that the solutions of such FDE do not intersect in finite time and the solution mappings $x_0 \mapsto S_t(x_0)$ form a bijection on $\mathbb{R}^d$ for each $t \geq 0$.

Later Doan & Kloeden [5] used ideas of Sell [13] for Volterra integral equations to show that an autonomous Caputo FDE generates a semi-group, hence autonomous semi-dynamical system, on the space $\mathfrak{C}$ of continuous functions $f : \mathbb{R}^+ \to \mathbb{R}^d$ is endowed with the topology of uniform convergence on compact subsets. This was extended to a skew-product flow on the space $\mathfrak{C} \times P$ by Cui & Kloeden [3] to the non-autonomous Caputo FDE with a driving system (1).

2 Volterra integral equations

The integral equation (2) is a special case of the (singular) Volterra integral equation [12, 13]

$$x(t) = f(t) + \int_0^t a(t,s) g(x(s), \vartheta_s(p_0)) ds, \tag{4}$$

where $f : \mathbb{R}^+ \to \mathbb{R}^d$ is a continuous function. The topology of uniform convergence on compact subsets of the space $\mathfrak{C}$ of such continuous functions is induced by the metric

$$\rho(f,h) := \sum_{n=1}^{\infty} \frac{1}{2^n} \rho_n(f,h),$$

where

$$\rho_n(f,h) := \frac{\sup_{t\in[0,n]} \|f(t) - h(t)\|}{1 + \sup_{t\in[0,n]} \|f(t) - h(t)\|}.$$

Following Sell [13], define operators $T_t : \mathfrak{C} \times P \to \mathfrak{C}$ by

$$(T_\tau(f,p_0))(\theta) = f(\tau + \theta) + \int_0^\tau a(t+\theta, s) g(x_f(s), \vartheta_s(p_0))) \, ds, \qquad \theta \in \mathbb{R}^+, \tag{5}$$

where x_f is the solution of (4), i.e.,

$$x_f(\tau) = (T_\tau(f, p_0))(0) := f(\tau) + \int_0^\tau a(\tau, s)g(x_f(s), \vartheta_s(p_0))\, ds.$$

Essentially, as in Sell [13, pages 178-179], the operators $T_\tau : \mathfrak{C} \times P \to \mathfrak{C}$, $\tau \geq 0$, satisfy a cocycle property with respect to the driving system $\{\vartheta_\tau\}_{\tau \geq 0}$ on P. This means that

$$\Pi : \mathbb{R}^+ \times \mathfrak{C} \times P \to \mathfrak{C} \times P$$

with

$$\Pi(\tau, f, p_0) := (T_\tau(f, p_0), \vartheta_\tau(p_0))$$

defines a semi-group on $\mathfrak{C} \times P$, which is a *skew-product flow* due to its structure. These results can be used for the Caputo integral equation (2) restricting to constant functions $f = id_{x_0}$, i.e., $f(t) \equiv x_0$, corresponding to initial values $x_0 \in \mathbb{R}^d$.

Theorem 1. *[3, Theorem 1] Suppose that the vector field g satisfies Assumptions 1 and 2. Then the integral equation* (2) *version of the Caputo FDE* (1) *generates a semi-group of continuous operators $\{(T_\tau,\, \vartheta_\tau)\}_{\tau \in \mathbb{R}^+}$ on the space $\mathfrak{C} \times P$, which has the structure of a skew-product flow.*

The proof is given in Cui & Kloeden [3], see also [5,7] for the simpler autonomous case. Later in discussing attractors, the space $\mathfrak{C}$ will be replaced by a Banach subspace $\mathfrak{C}_\alpha$.

3 Dissipative vector fields

Tuan & Trinh [2, Theorem 2] showed that solutions of Caputo FDEs (1), hence also those of (2), satisfy

$$^{C}D^{\alpha}_{0+}\|x(t)\|^2 \leq 2 \left\langle x(t), {}^{C}D^{\alpha}_{0+}x(t) \right\rangle.$$

Hence, if the vector field g of (1) satisfies the uniform dissipativity condition

$$\langle x, g(x, p) \rangle \leq a - b\|x\|^2, \tag{6}$$

where $a, b > 0$ are independent of $p \in P$, then along the solutions of (1)

$$^{C}D^{\alpha}_{0+}\|x(t)\|^2 \leq 2 \langle x(t), g(x(t), \vartheta_t(p)) \rangle \leq 2a - 2b\|x(t)\|^2.$$

It was shown in [9] that these solutions $x(t) = x(t, x_0, p_0)$ satisfy the inequality

$$\|x(t, x_0, p_0)\|^2 \leq \|x_0\|^2 E_\alpha(-2bt^\alpha) + \frac{a}{b}\left(1 - E_\alpha(-2bt^\alpha)\right). \tag{7}$$

It follows from this inequality that $\|x(t, x_0, p_0)\| \leq R$ for all $t \geq 0$ and $p_0 \in P$, when $\|x_0\| \leq R$ and $R^2 \geq 1 + \frac{a}{b}$. Moreover, the set

$$\mathcal{B}^* := \left\{ x \in \mathbb{R}^d \,:\, \|x\|^2 \geq 1 + \frac{a}{b} =: R_*^2 \right\}$$

is a positive invariant absorbing set for the solutions of the Caputo FDE (1). In particular, there exists $T_R \geq 0$ independent of $p_0 \in P$. such that $\|x(t, x_0, p_0)\| \in \mathcal{B}^*$, i.e., $\|x(t, x_0, p_0)\| \leq R^*$ for all $t \geq T_R$, $p_0 \in P$.

Since the absorbing set $\mathcal{B}^*$ is compact in $\mathbb{R}^d$, the corresponding omega limit set

$$\Omega^*_{p_0} = \overline{\{y \in \mathbb{R}^d : \exists \{x_{0,n}\}_{n\in\mathbb{N}} \text{ bnd'd}, t_n \to \infty \text{ such that } \phi(t_n, x_{0,n}, p_0) \to y\}},$$

is a nonempty compact subset of $\mathcal{B}^*$ for each fixed $p_0 \in P$. Moreover, it attracts all of the future dynamics of the Caputo FDE (1). But it is not an attractor of the Caputo FDE (1) in $\mathbb{R}^d$ since the corresponding semi-dynamical system is defined on the product space $\mathfrak{C}_\alpha \times P$ and not on $\mathbb{R}^d \times P$. Nevertheless, $\Omega^*_{p_0}$ represents the *observable* part (in $\mathbb{R}^d$) of the corresponding sector of an attracting set in $\mathfrak{C}_\alpha$ of this Caputo skew-product semi-dynamical system on $\mathfrak{C}_\alpha \times P$ and, essentially, determines it.

4 Caputo skew-product flow attractor

The operators T_t have a double skew-product structure with the solution of the Caputo FDE

$$(T_t(id_{x_0}, p_0)(0) := x(t, x_0, p_0) = x_0 + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} g(x(s, x_0, p_0), \vartheta_s(p_0))\, ds \quad (8)$$

being fed into

$$(T_t id_{x_0}, p_0)(\theta) = x_0 + \frac{1}{\Gamma(\alpha)} \int_0^t (t+\theta-s)^{\alpha-1} g(x(s, x_0, p_0), \vartheta_s(p_0))\, ds \quad (9)$$

when $\theta > 0$, in addition to the driving system ϑ.

A major difficulty in extending such results to the Caputo semi-group $\{(T_t, \vartheta_t)\}_{t\in\mathbb{R}^+}$ is to apply the dissipativity condition (6) to the vector field g inside the integral equations (5) to establish the existence of an absorbing set in the space $\mathfrak{C}$. Restricting to constant initial functions $f(t) \equiv id_{x_0}$ corresponding to initial values $x_0 \in \mathbb{R}^d$ this can be done for the case $\theta = 0$, which corresponds to the Caputo FDE (1) with the initial condition $x(0) = x_0$, $p_0 \in P$, using the inequality (7). It leads to $x(t, x_0, p_0) \in \mathcal{B}^*$ for $t \geq T_R$, $\|x_0\| \leq R$ for all $R \geq R^2_*$ and $p_0 \in P$. Importantly, this and other bounds are uniform in $p_0 \in P$. These bounds can then be used to estimate the integrals for the integral equations (9) with $\theta > 0$.

Due to some technical issues in the compactness part of the proof, the existence of an attractor in the space $\mathfrak{C}$ of uniform convergence on bounded intervals needs to be modified here to a weighted norm on a subspace of $\mathfrak{C}$. It is defined by

$$\|f\|_\alpha := \|f(0)\| + \sum_{N=1}^{\infty} \frac{1}{2^N N^\alpha} \|f\|_N,$$

where

$$\|f\|_N := \sup_{t\in[N^{-1},N]} \|f(t)\|, \quad N = 1, 2, \cdots .$$

Let $\mathfrak{C}_\alpha$ be the subspace of $\mathfrak{C}$ consisting of functions f with $\|f\|_\alpha < \infty$. Then $(\mathfrak{C}_\alpha, \|\cdot\|_\alpha)$ is a Banach space and $\{(T_t, \vartheta_t)\}_{t\in\mathbb{R}^+}$ forms a semi-group on $\mathfrak{C}_\alpha \times P$.

The attractor obtained is somewhat unusual in that it attracts only a restricted class of initial values in $\mathfrak{C}_\alpha$, which is not invariant under the dynamics. This results in some unconventional properties.

Theorem 2. *[3, Theorem 2] Suppose also that Assumption 1 holds and that the vector field g is locally Lipschitz in both variables and satisfies the uniform dissipativity condition* (6). *Then semi-group $\{(T_t, \vartheta_t)\}_{t\in\mathbb{R}^+}$ on the space $\mathfrak{C}_\alpha \times P$ corresponding to the integral equations* (4) *has an attractor $\mathfrak{A} \subset \mathfrak{C}_\alpha \times P$, which attracts bounded subsets of $\mathfrak{C}_\alpha \times P$ consisting of constant initial functions in $\mathfrak{C}_\alpha$ and has the structure*

$$\mathfrak{A} = \bigcup_{p\in P} \mathfrak{A}(p) \times \{p\},$$

where the $\mathfrak{A}(p)$ are closed and bounded subsets of $\mathfrak{C}_\alpha$. Moreover, the sets $\mathfrak{A}(p)$ are positively invariant in the sense that

$$T_t(\mathfrak{A}(p), p) = \mathfrak{A}(\vartheta_t(p)), \qquad t \geq 0, p \in P,$$

and pullback attracting in the sense that

$$\lim_{t\to\infty} \operatorname{dist}_{\mathfrak{C}_\alpha} (T_t(\mathfrak{D}, \vartheta_{-t}(p)), \mathfrak{A}(p)) = 0, \qquad p \in P,$$

for all bounded subsets $\mathfrak{D}$ of $\mathfrak{C}_\alpha$ consisting of constant initial functions.

The proof of the existence of the attractor in Theorem 2 is given in [3, Theorem 4]. In particular, it is shown that the closed and bounded subset $\mathfrak{B}^*$ of $\mathfrak{C}_\alpha$ defined by

$$\mathfrak{B}^* := \left\{ \chi \in \mathfrak{C}_\alpha \,:\, \|\chi\|_\alpha \leq 2R_* + \frac{B^g_{R_*}}{\alpha\Gamma(\alpha)} =: \widehat{R}_* \right\}$$

absorbs under the operators T_t bounded sets of constant initial data functions $\|id_{x_0}\|_\alpha \leq 2\|x_0\| \leq 2R$ in the time $t \geq T_R$. It follows that $\mathfrak{B}^* \times P$ is an absorbing set for the Caputo semi-group $\{(T_t, \vartheta_t)\}_{t\in\mathbb{R}^+}$ in the space $\mathfrak{C}_\alpha \times P$.

Another important property of attractors in general is that they consist of entire solutions. In the context of the Caputo skew-product flow considered here, this means that there are continuous functions $\chi^* : \mathbb{R} \to \mathfrak{C}_\alpha$ and $p^* : \mathbb{R} \to P$ such that $\chi^*(t + s) = T_t(\chi^*(s), p^*(s)))$ for all $s \in \mathbb{R}$ and $t \geq 0$. (Here $p^*(t)) = \vartheta_t(\hat{p})$ for an appropriate $\hat{p} \in P$.) Moreover, $\chi^*(t) \in \mathfrak{A}(p^*(t))$ for all $t \in \mathbb{R}$. Thus the subsets

$$\mathcal{A}(p^*(t)) := \left\{ f(0) \in \mathbb{R}^d \,:\, f \in \mathfrak{A}(p^*(t)) \right\}, \quad t \in \mathbb{R},$$

give the observed asymptotic behaviour in $\mathbb{R}^d$.

5 A strictly contractive example

Consider again the motivational example above, but now writing the scalar Caputo FDE as

$$ {}^{C}D^{\alpha}_{0+}x(t) = -x(t) + p(t, p_0) \tag{10} $$

where $p(t, p_0)$ is given by an autonomous dynamical system, i.e., a group, on a compact interval P in $\mathbb{R}$. Denote the unique solution with the initial value $x(0) = x_0$ by $x(t, x_0, p_0)$ for each $p_0 \in P$.

Firstly, note that this system is strictly contracting. Let $x(t) = x(t, x_0, p_0)$ and $y(t) = y(t, y_0, p_0)$. Then

$$ {}^{C}D^{\alpha}_{0+}z(t) = -z(t), \qquad z(t) := x(t) - y(t). $$

Then $z(t) = E_\alpha(-t^\alpha)z(0)$, which gives the strictly contracting property

$$ |z(t)| = E_\alpha(-t^\alpha)|z(0)|. \tag{11} $$

In particular, $z(t) \to 0$ as $t \to \infty$ at the non-exponential rate $t^{-\alpha}$, see [8]. This issue then is: to what do the individual solutions $x(t)$ and $y(t)$ converge?

Secondly, this linear Caputo equation (10) has an explicit solution given by the variation of constants formula [8, Lemma 1.4]

$$ x(t, x_0, p_0) = E_\alpha(-t^\alpha)x_0 + \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}(-(t-s)^\alpha)p(s, p_0)ds. \tag{12} $$

Let $\tau > 0$ and let $q_{-\tau} = p(-\tau, p_0)$, so $p_0 = p(\tau, q_{-\tau})$. Then replace t by τ and p_0 by $q_{-\tau}$ in (12) to obtain

$$ x(\tau, x_0, q_{-\tau}) = E_\alpha(-\tau^\alpha)x_0 + \int_0^\tau (\tau-s)^{\alpha-1} E_{\alpha,\alpha}(-(\tau-s)^\alpha)p(s, q_{-\tau})ds. $$

Finally, substituting $\nu = s - \tau$ for the integration variable s gives

$$ x(\tau, x_0, q_{-\tau}) = E_\alpha(-\tau^\alpha)x_0 + \int_{-\tau}^0 (-\nu)^{\alpha-1} E_{\alpha,\alpha}(-(-\nu)^\alpha)p(\nu, p_0)d\nu, \tag{13} $$

since $p(\nu + \tau, q_{-\tau}) = p(\nu, p_0)$. Note that the limit (which is, in fact, the pullback limit)

$$ \lim_{\tau\to\infty} x(\tau, x_0, q_{-\tau}) = a(p_0) := \int_{-\infty}^0 (-\nu)^{\alpha-1} E_{\alpha,\alpha}(-(-\nu)^\alpha)p(\nu, p_0)d\nu $$

exists since $E_\alpha(-\tau^\alpha) \to 0$ as $\tau \to \infty$ and the improper integral converges. The latter follows since

$$ \left| \int_{-\tau}^0 (-\nu)^{\alpha-1} E_{\alpha,\alpha}(-(-\nu)^\alpha)p(\nu, p_0)d\nu \right| \le K_P \left| \int_{-\tau}^0 (-\nu)^{\alpha-1} E_{\alpha,\alpha}(-(-\nu)^\alpha))d\nu \right|, $$

where $K_P := \max_{p\in P}|p| < \infty$ and the integral on the right side converges as $\tau < \infty$ because (see, e.g., [9])

$$\begin{aligned}\int_{-\tau}^{0}(-\nu)^{\alpha-1}E_{\alpha,\alpha}(-(-\nu)^{\alpha}))d\nu &= \tau^{\alpha-1}E_{\alpha,\alpha+1}(-\tau^{\alpha}) \\ &= 1-E_{\alpha}(-\tau^{\alpha}) \to 1, \quad \tau\to\infty.\end{aligned}$$

The strictly contractive condition (11) gives

$$|x(\tau,x_0,q_{-\tau})-y(\tau,y_0,q_{-\tau})| = E_{\alpha}(-\tau^{\alpha})|x_0-y_0|,$$

which means that all such solutions converge in the pullback sense to the same limit, i.e., $a(p_0)$.

As mentioned earlier, the Caputo attractor consists of entire solutions. Here this means there are continuous functions $\chi^* : \mathbb{R} \to \mathfrak{C}_\alpha$ and $p^* : \mathbb{R} \to P$ such that $\chi^*(t) \in \mathfrak{A}(p^*(t))$ for all $t \in \mathbb{R}$.

Now, if $\chi \in \mathfrak{A}(p_0)$ in $\mathfrak{C}_\alpha$, then $\chi(0) = a(p_0) \in \mathbb{R}$. Thus $\chi^*(t)(0) = a(p^*(t)) \in \mathbb{R}$ for all $t \in \mathbb{R}$ for an entire solution $\chi^*(t) \in \mathfrak{A}(p^*(t))$ for all $t \in \mathbb{R}$. This corresponds to an entire solution of the Caputo FDE, i.e., $x^*(t) := a(p^*(t))$ in $\mathbb{R}$ for all $t \in \mathbb{R}$, which is both pullback and forwards attracting by the strictly contracting inequality (11).

Since the pullback limit $a(p_0)$ is unique in this example, the entire solution $x^*(t) = a(p^*(t))$ is unique. Hence the $(T_t(id_{a(p_0)}, p_0))(\theta)$ is uniquely determined for all $\theta > 0$, which means that the entire solution $\chi^*_t \in \mathfrak{A}(p^*(t))$ for all $t \in \mathbb{R}$ is unique. Hence the sets $\mathfrak{A}(p_0)$ are singleton sets in $\mathfrak{C}_\alpha$ with $\mathfrak{A}(p_0)(0) = \{\chi^*_{a(p_0)}\}$ in $\mathbb{R}$ with $\chi^*_{a(p_0)}(0) = a(p_0)$.

Finally, note that when the driving system of the Caputo FDE is T-periodic, then $p^*(t)$ and hence $a(p^*(t))$ are T-periodic. In particular, the Caputo FDE (3) has a 2π-periodic solution $x^*(t) := a(p^*(t))$.

Acknowledgement

PK was funded partially by Ministerio de Ciencia e Innovación (Spain) and FEDER (European Community) under grant PID2021-122991NB-C21 and the Chinese NSF grant 12371242. The work of T.S. Doan is funded by Vietnam Academy of Science and Technology under grant number NVCC01.11/24-25.

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Received March 1, 2024

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