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Poisson Stable Solutions of Semi-Linear Differential Equations

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Abstract. We study the problem of existence of Poisson stable (in particular, almost periodic, almost automorphic, recurrent) solutions to the semi-linear differential equation

$$x' = (A_0 + A(t))x + F(t, x)$$

with unbounded closed linear operator A_0, bounded operators $A(t)$ and Poisson stable functions $A(t)$ and $F(t, x)$. Under some conditions we prove that there exists a unique (at least one) solution which possesses the same recurrence property as the coefficients.

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Dedicated to the memory of Professor B. A. Shcherbakov on the 100th birthday

1 Introduction

Let Y be a complete metric space, $(Y, \mathbb{R}, \sigma)$ be a dynamical system on Y, $(\mathfrak{B}, |\cdot|)$ be a Banach space and $\mathcal{L}(\mathfrak{B})$ be the space of all linear bounded operators acting on the space $\mathfrak{B}$. Denote by $[\mathfrak{B}]$ the linear space $\mathcal{L}(\mathfrak{B})$ equipped with the operator norm $\|A\| := \sup\limits_{|x|\le 1} |Ax|$ and by $[\mathfrak{B}]_s$ the space $\mathcal{L}(\mathfrak{B})$ equipped with the topology of strong convergence. We denote by $C(Y, \mathfrak{B})$ (respectively, $C(Y \times \mathfrak{B}, \mathfrak{B})$) the space of all continuous mappings $\varphi : Y \to \mathfrak{B}$ (respectively, $f : Y \times \mathfrak{B} \to \mathfrak{B}$) and by $C_b(Y, \mathfrak{B})$ (respectively, $C_b(Y \times \mathfrak{B}, \mathfrak{B})$) the space of all bounded mappings from $C(Y \times \mathfrak{B}, \mathfrak{B})$ (respectively, the space of all functions $f \in C(Y \times \mathfrak{B}, \mathfrak{B})$ satisfying the conditions: $\sup\limits_{y\in Y} |f(y, 0)| < +\infty$ and $f(y, u_1) - f(y, u_2)| \le L|u_1 - u_2|$ for some positive constant L and all $(y, u_i) \in Y \times \mathfrak{B}$ $(i = 1, 2)$).

The problem of the existence of Poisson stable (in particular, periodic, quasi-periodic, Bohr almost periodic, almost automorphic, recurrent in the sense of Birkhoff, Levitan almost periodic, almost recurrent in the sense of Bebutov, pseudo-recurrent, pseudo-periodic in the sense of Bohr) solutions of semi-linear differential equations of the form

$$x' = (A_0 + A(\sigma(t, y)))x + F(\sigma(t, y), x), \quad (y \in Y) \tag{1}$$

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where $A_0 : D(A_0) \to \mathfrak{B}$ is an infinitesimal generator of C_0-semigroup $\{U(t)\}_{t\geq 0}$ acting on the Banach space $\mathfrak{B}$, $A \in C(Y, [\mathfrak{B}]_s)$ and $F \in C(Y \times \mathfrak{B}, \mathfrak{B})$.

Earlier this problem was studied in the works of Shcherbakov B. A. [17, Ch.IV], Shcherbakov B. A. and Koreneva L. V. [22] and Bronshtein I.U. [1, Ch.IV]. Namely, Shcherbakov B. A. [17, Ch.IV] and Shcherbakov B. A. and Koreneva L. V. [22] have studied this problem for the equations of the form (1) in the case when the "linear part" of the equation (1) is stationary and bounded, i.e., $A_0 + A(y) = A_0$ for any $y \in Y$ and a linear operator $A_0 \in \mathcal{L}(\mathfrak{B})$.

Bronshtein I. U. in [1, Ch.IV] studied this problem for the equations (1) with non-stationary linear part, but in the case when A_0 is equal to zero, i.e., for the equations of the form

$$x' = A(\sigma(t, y))x + F(\sigma(t, y), x) \quad (y \in Y) \tag{2}$$

with compact Y and $A \in C(Y, [\mathfrak{B}])$ and $F \in C(Y \times \mathfrak{B}, \mathfrak{B})$.

This paper is organized as follows. In the second section we collect some known notions and facts about Poisson stable motions of dynamical systems. Namely we present the construction of shift dynamical systems, definitions and basic properties of Poisson stable motions and Shcherbakov's principle of comparability for Poisson stable motions by their character of recurrence. The third section is dedicated to the study of the problem of existence of a unique Poisson stable solution for linear nonhomogeneous differential equations $x' = (A_0 + A(\sigma(t, y)))x + f(\sigma(t, y))$ with unbounded linear operators $A_0 + A(y)$ $(y \in Y)$. In the fourth section we study the problem of Poisson stability of solutions for nonlinear equations (2) with Lipschitz nonlinear perturbations (both global and local Lipschitzian F). We give also an example which illustrates our results for infinite-dimensional differential equations (1) with unbounded and non-stationary "linear part".

2 Preliminaries

2.1 Poisson stable motions of dynamical systems

Let $\mathbb{R} := (-\infty, \infty)$, $\mathbb{R}_+ := \{t \in \mathbb{R} |\ t \geq 0\}$ (respectively, $\mathbb{R}_- := \{t \in \mathbb{R} |\ t \leq 0\}$), $\mathbb{T} \in \{\mathbb{R}, \mathbb{R}_+\}$ and $(X, \mathbb{T}, \pi)$ be a dynamical system on the space X.

Recall the classes of Poisson stable motions we study in this paper, see [14, 17, 21, 23] for details.

Definition 1. A point $x \in X$ is called *stationary* (respectively, *τ-periodic*) if $\pi(t, x) = x$ (respectively, $\pi(t + \tau, x) = \pi(t, x)$) for all $t \in \mathbb{T}$.

Definition 2. A point $x \in X$ is called *quasi-periodic* with the base of frequency $\nu = (\nu_1, \nu_2, \ldots, \nu_k)$ if the associated function $f(\cdot) := \pi(\cdot, x) : \mathbb{R} \to X$ satisfies the following conditions:

1. the numbers $\nu_1, \nu_2, \ldots, \nu_k$ are rationally independent;

2. there exists a continuous function $\Phi : \mathbb{R}^k \to X$ such that

$$\Phi(t_1 + 2\pi, t_2 + 2\pi, \dots, t_k + 2\pi) = \Phi(t_1, t_2, \dots, t_k)$$

for all $(t_1, t_2, \dots, t_k) \in \mathbb{R}^k$;

3. $f(t) = \Phi(\nu_1 t, \nu_2 t, \dots, \nu_k t)$ for $t \in \mathbb{R}$.

Definition 3. For given $\varepsilon > 0$, a number $\tau \in \mathbb{T}$ is called an *ε-shift of x* (respectively, *ε-almost period of x*) if $\rho(\pi(\tau, x), x) < \varepsilon$ (respectively, $\rho(\pi(\tau + t, x), \pi(t, x)) < \varepsilon$ for all $t \in \mathbb{T}$).

Definition 4. A point $x \in X$ is called *almost recurrent* (respectively, *Bohr almost periodic*) if for any $\varepsilon > 0$ there exists a positive number l such that any segment of length l contains an ε-shift (respectively, ε-almost period) of x.

Definition 5. A point $x \in X$ is called Lagrange stable if its trajectory $\Sigma_x := \{\pi(tx)|\ t \in \mathbb{T}\}$ is precompact.

Definition 6. If a point $x \in X$ is almost recurrent and Lagrange stable, then x is called *(Birkhoff) recurrent*.

Let $x \in X$ and denote $\mathfrak{N}_x := \{\{t_n\} \subset \mathbb{T}|$ such that $\pi(t_n, x) \to x$ as $n \to \infty\}$.

Definition 7. A point $x \in X$ is called *Levitan almost periodic* [11] (see also [1,3,10]) if there exists a dynamical system $(Y, \mathbb{T}, \sigma)$ and a Bohr almost periodic point $y \in Y$ such that $\mathfrak{N}_y \subseteq \mathfrak{N}_x$.

Definition 8. A point $x \in X$ is called *almost automorphic* if it is st. L and Levitan almost periodic.

Definition 9. A point $x \in X$ is said to be *uniformly Poisson stable* or *pseudo-periodic* in the positive (respectively, negative) direction if for arbitrary $\varepsilon > 0$ and $l > 0$ there exists an ε-almost period $\tau > l$ (respectively, $\tau < -l$) of x. The point x is said to be uniformly Poisson stable or pseudo-periodic if it is so in both directions.

Definition 10. [15, 16] A point $x \in X$ is said to be *pseudo-recurrent* if for any $\varepsilon > 0$, $p \in \Sigma_x$ and $t_0 \in \mathbb{T}$ there exists $L = L(\varepsilon, t_0) > 0$ such that

$$B(p, \varepsilon) \bigcap \pi([t_0, t_0 + L], p) \neq \emptyset,$$

where $B(p, \varepsilon) := \{x \in X : \rho(p, x) < \varepsilon\}$ and $\pi([t_0, t_0 + L], p) := \{\pi(t, p) : t \in [t_0, t_0 + L]\}$.

2.2 Shcherbakov's comparability principle of motions by their character of recurrence

In this subsection we present some notions and results stated and proved by Shcherbakov B. A. [17–21] (see also [6, Ch.I]).

Let $(X,\mathbb{T},\pi)$ and $(Y,\mathbb{T},\sigma)$ be two dynamical systems.

Definition 11. A point $x\in X$ is said to be *comparable with* $y\in Y$ *by character of recurrence* if for any $\varepsilon>0$ there exists a $\delta=\delta(\varepsilon)>0$ such that every δ-shift of y is an ε-shift for x, i.e., $\rho(\sigma(\tau,y),y)<\delta$ implies $\rho(\pi(\tau,x),x)<\varepsilon$.

Theorem 1. *[19],[21, Ch.II] Let $x\in X$ be comparable with $y\in Y$. If the point y is stationary (respectively, τ-periodic, Levitan almost periodic, almost recurrent, Poisson stable), then so is the point x.*

Definition 12. A point $x\in X$ is called *uniformly comparable with* $y\in Y$ *by character of recurrence* if for any $\varepsilon>0$ there exists a $\delta=\delta(\varepsilon)>0$ such that every δ-shift of $\sigma(t,y)$ is an ε-shift of $\pi(t,x)$ for all $t\in\mathbb{T}$, i.e., $\rho(\sigma(t+\tau,y),\sigma(t,y))<\delta$ implies $\rho(\pi(t+\tau,x),x)<\varepsilon$ for any $t\in\mathbb{T}$ (or equivalently: $\rho(\sigma(t_1,y),\sigma(t_2,y))<\delta$ implies $\rho(\pi(t_1,x),\pi(t_2,x))<\varepsilon$ for any $t_1,t_2\in\mathbb{T}$).

Denote $\mathfrak{M}_x:=\{\{t_n\}\subset\mathbb{T}:\ \{\pi(t_n,x)\}\text{ converges}\}$.

Definition 13. [2,4] A point $x\in X$ is said to be *strongly comparable with* $y\in Y$ *by character of recurrence* if $\mathfrak{M}_y\subseteq\mathfrak{M}_x$.

Theorem 2. *[19],[21, Ch.II] Let X and Y be two complete metric spaces. Let a point $x\in X$ be uniformly comparable with $y\in Y$ by character of recurrence. If y is quasi-periodic (respectively, Bohr almost periodic, almost automorphic, Birkhoff recurrent, Lagrange stable, pseudo-periodic, pseudo-recurrent), then so is x.*

Theorem 3. *[19],[21, Ch.II] Let a point $y\in Y$ be Lagrange stable, then a point $x\in X$ is uniformly comparable with $y\in Y$ if and only if $\mathfrak{M}_y\subseteq\mathfrak{M}_x$.*

Let $(Y,\mathbb{R},\sigma)$ be an autonomous two-sided dynamical system on Y and $\mathfrak{B}$ be a real or complex Banach space with the norm $|\cdot|$.

Definition 14. (Cocycle on the state space $\mathfrak{B}$ with the base $(Y,\mathbb{R},\sigma)$). The triplet $\langle\mathfrak{B},\phi,(Y,\mathbb{R},\sigma)\rangle$(or briefly ϕ) is said to be a cocycle (see, for example, [5] and [14]) on the state space $\mathfrak{B}$ with the base $(Y,\mathbb{R},\sigma)$ if the mapping $\phi:\mathbb{R}_+\times\mathfrak{B}\times Y\to\mathfrak{B}$ satisfies the following conditions:

1. $\phi(0,u,y)=u$ for all $u\in\mathfrak{B}$ and $y\in Y$;
2. $\phi(t+\tau,u,y)=\phi(t,\phi(\tau,u,y),\sigma(\tau,y))$ for any $t,\tau\in\mathbb{R}_+,u\in\mathfrak{B}$ and $y\in Y$;
3. the mapping ϕ is continuous.

Definition 15. (Skew-product dynamical system). Let $\langle \mathfrak{B}, \phi, (Y, \mathbb{R}, \sigma)\rangle$ be a cocycle on $\mathfrak{B}, X := \mathfrak{B} \times Y$ and π be a mapping from $\mathbb{R}_+ \times X$ to X defined by equality $\pi = (\phi, \sigma)$, i.e., $\pi(t, (u, y)) = (\phi(t, u, y), \sigma(t, y))$ for all $t \in \mathbb{R}_+$ and $(u, y) \in \mathfrak{B} \times Y$. The triplet $(X, \mathbb{R}_+, \pi)$ is an autonomous dynamical system and it is called [14] a skew-product dynamical system.

Definition 16. (Nonautonomous dynamical system.) Let $\mathbb{T}_1 \subseteq \mathbb{T}_2$ ($\mathbb{T}_i \in \{\mathbb{R}_+, \mathbb{R}\}$ $(i = 1, 2)$) be two subsemigroups of the group $\mathbb{T}, (X, \mathbb{T}_1, \pi)$ and $(Y, \mathbb{T}_2, \sigma)$ be two dynamical systems and $h : X \to Y$ be a homomorphism from $(X, \mathbb{T}_1, \pi)$ to $(Y, \mathbb{T}_2, \sigma)$ (i.e., $h(\pi(t, x)) = \sigma(t, h(x))$ for any $t \in \mathbb{T}_1$, $x \in X$ and h is continuous), then the triplet $\langle (X, \mathbb{T}_1, \pi),\ (Y,\ \mathbb{T}_2,\ \sigma), h\rangle$ is called (see [1] and [5]) a nonautonomous dynamical system.

Example 1. (The nonautonomous dynamical system generated by a cocycle ϕ.) Let $\langle \mathfrak{B}, \phi, (Y, \mathbb{R}, \sigma)\rangle$ be a cocycle, $(X, \mathbb{R}_+, \pi)$ be a skew-product dynamical system $(X = \mathfrak{B} \times Y, \pi = (\phi, \sigma))$ and $h = pr_2 : X \to Y$, then the triplet $\langle (X, \mathbb{R}_+, \pi), (Y, \mathbb{R}, \sigma), h\rangle$ is a nonautonomous dynamical system.

Definition 17. A continuous mapping $\gamma : Y \to X$ (respectively, $\xi : Y \to \mathfrak{B}$) is said to be an invariant section of nonautonomous dynamical system $\langle (X, \mathbb{R}_+, \pi), (Y, \mathbb{R}, \sigma), h\rangle$ (respectively, a cocycle $\langle \mathfrak{B}, \phi, (Y, \mathbb{R}, \sigma)\rangle$) if $h(\pi(t, \gamma(y))) = \sigma(t, \gamma(y))$ and $h(\gamma(y)) = y$ (respectively, $\phi(t, \xi(y), y) = \xi(\sigma(t, y))$) for any $(t, y) \in \mathbb{R}_+ \times Y$.

Remark 1. If $\gamma : Y \to X$ (respectively, $\xi : Y \to \mathfrak{B}$) is an invariant section of nonautonomous dynamical system $\langle (X, \mathbb{R}_+, \pi), (Y, \mathbb{R}, \sigma), h\rangle$ (respectively, a cocycle $\langle \mathfrak{B}, \phi, (Y, \mathbb{R}, \sigma)\rangle$), then the motion $\pi(t, \gamma(y), y)$ of the dynamical system $(X, \mathbb{R}_+, \pi)$ (respectively, $\phi(t, \xi(y), y)$ of the cocycle $\langle \mathfrak{B}, \phi, (Y, \mathbb{R}, \sigma)\rangle$) can be extended on the real axis $\mathbb{R}$ as follows: $\pi(-t, \gamma(y)) := \gamma(\sigma(-t, y))$ (respectively, $\phi(-t, \xi(y), y) := \xi(\sigma(-t, y))$ for any $(t, y) \in \mathbb{R}_+ \times Y$.

Denote by $C(\mathbb{R}, X)$ the space of all continuous functions $f : \mathbb{R} \to X$ equipped with the compact-open topology. This topology can be defined by the following distance:

$$d(f, g) := \sup_{L>0} \min\{\max_{|t|\le L} \rho(f(t), g(t)), L^{-1}\}.$$

Remark 2. The following statements are equivalent:

1. $d(f_n, f) \to 0$ as $n \to \infty$;
2. $$\lim_{n\to\infty} \max_{|t|\le L} \rho(f_n(t), f(t)) = 0$$

   for every $L > 0$.

Denote by $(C(R, X), \mathbb{R}, \sigma)$ the shift dynamical system [6, Ch.I] on the space $C(\mathbb{R}, X)$ (Bebutov's dynamical system), where $\sigma : \mathbb{R} \times C(\mathbb{R}, X) \to C(\mathbb{R}, X)$ is defined by $\sigma(h, f) := f^h$ for any $(h, f) \in \mathbb{R} \times C(\mathbb{R}, X)$ and $f^h(t) := f(t + h)$ for any $t \in \mathbb{R}$.

Lemma 1. *Let $\langle \mathfrak{B}, \phi, (Y, \mathbb{R}, \sigma)\rangle$ (shortly ϕ) be a cocycle and $\xi : Y \to \mathfrak{B}$ be a continuous invariant section of ϕ.*

Then the following statements hold:

1. *the mapping $\gamma : Y \to X := \mathfrak{B} \times Y$ defined by $\gamma(y) := (\xi(y), y)$ for any $y \in Y$ satisfies the condition $\gamma(\sigma(t, y)) = \pi(t, \gamma(y))$ for any $(t, y) \in \mathbb{R}_+ \times Y$;*
2. *for any $y \in Y$ the motion $\pi(t, x)$ (respectively, the motion $\phi(t, \xi(y), y)$ of the cocycle ϕ), where $x := (\xi(y), y)$, is extendable on the real axis;*
3. *$\mathfrak{M}_y \subseteq \mathfrak{M}_\alpha$, where $\alpha \in C(\mathbb{R}, X)$ is defined by $\alpha(t) := \gamma(\sigma(t, y))$ for any $t \in \mathbb{R}$.*

Proof. To prove the first statement it suffices to define $\pi(\cdot, x)$ for any $t \in \mathbb{R}_-$ as follows: $\pi(t, \gamma(y)) := \gamma(\sigma(t, y))$ for any $(t, y) \in \mathbb{R}_- \times Y$ (respectively, $\phi(t, \xi(y), y) := \xi(\sigma(t, y))$ for any $(t, y) \in \mathbb{R}_- \times Y$).

Let now $\{t_n\} \in \mathfrak{M}_y$, then there exists a point $q \in H(y) := \overline{\{\sigma(t, y)|\ t \in \mathbb{R}\}}$ such that $\sigma(t_n, y) \to q$ as $n \to \infty$. Let $\alpha \in C(\mathbb{R}, X)$ be defined by

$$\alpha(t) := \gamma(\sigma(t, y))$$

for any $t \in \mathbb{R}$. We will show that $\{t_n\} \in \mathfrak{M}_\alpha := \{\{\tau_k\}|$ such that α^{τ_k} converges in $C(\mathbb{R}, X)\}$. To this end denote by $\tilde{\alpha}(t) := \gamma(\sigma(t, q))$ for any $t \in \mathbb{R}$ and we will establish the relation

$$\sup_{|t| \le l} \rho(\alpha(t + t_n), \tilde{\alpha}(t)) = \sup_{|t| \le l} \rho(\gamma(\sigma(t, \sigma(t_n, y)), \gamma(\sigma(t, q)))) \to 0$$

as $n \to \infty$. If we suppose that it is not true, then there are $\varepsilon_0 > 0$, $l_0 > 0$ and $\{s_n\} \subset [-l_0, l_0]$ such that

$$\rho(\alpha(s_n + t_n), \tilde{\alpha}(s_n)) = \rho(\gamma(\sigma(s_n, \sigma(t_n, y)), \gamma(\sigma(s_n, q)))) \ge \varepsilon_0 \tag{3}$$

for any $n \in \mathbb{N}$. Since $\{s_n\} \subset [-l_0, l_0]$ then without loss of generality we can suppose that this sequence is convergent. Denote by $s_0 = \lim_{n \to \infty} s_n$. It is clear that

$$\sigma(s_n, q) \to \sigma(s_0, q),\ \sigma(t_n, y) \to q \text{ and } \sigma(s_n, \sigma(t_n, y)) \to \sigma(s_0, q) \tag{4}$$

as $n \to \infty$. Passing to the limit in (3) as $n \to \infty$ and taking into account (4) we obtain $0 \ge \varepsilon_0$. The last inequality contradicts the choice of ε_0. The obtained contradiction proves our statement. □

Corollary 1. *Let $\langle \mathfrak{B}, \phi, (Y, \mathbb{R}, \sigma)\rangle$ (shortly ϕ) be a cocycle and $\xi : Y \to \mathfrak{B}$ be a continuous invariant section of ϕ. If a point $y \in Y$ is stationary (respectively, τ-periodic, quasi-periodic with the frequency base $\{\nu_1, \ldots, \nu_m\}$, Bohr almost periodic, almost automorphic, recurrent, almost recurrent, Levitan almost periodic, pseudo-recurrent and Lagrange stable, pseudo-periodic and Lagrange stable, Poisson stable), then the motion $\pi(t, x)$ with $x = \gamma(y) = (\xi(y), y)$ (or equivalently, the function $\alpha(t) = \xi(\sigma(t, y))$ for any $t \in \mathbb{R}$) is also so.*

Proof. This statement follows directly from Theorems 1, 2, 3 and Lemma 1. □

3 Linear Systems

3.1 Linear nonautonomous dynamical systems

Let $(\mathfrak{B},|\cdot|)$ be a Banach space with the norm $|\cdot|$ and $\langle\mathfrak{B},\varphi,(Y,\mathbb{R},\sigma)\rangle$ (or shortly φ) be a linear cocycle over dynamical system $(Y,\mathbb{R},\sigma)$ with the fibre $\mathfrak{B}$, i.e., φ is a continuous mapping from $\mathbb{R}_+\times\mathfrak{B}\times Y$ into $\mathfrak{B}$ satisfying the following conditions:

1. $\varphi(0,u,y)=u$ for any $u\in\mathfrak{B}$ and $y\in Y$;
2. $\varphi(t+\tau,u,y)=\varphi(t,\varphi(\tau,u,y),\sigma(\tau,y))$ for any $t,\tau\in\mathbb{R}_+$, $u\in\mathfrak{B}$ and $y\in Y$;
3. for any $(t,y)\in\mathbb{R}_+\times Y$ the mapping $\varphi(t,\cdot,y):\mathfrak{B}\mapsto\mathfrak{B}$ is linear.

Denote by $\mathcal{L}(\mathfrak{B})$ the space of all linear bounded operators A acting on the space $\mathfrak{B}$, by $[\mathfrak{B}]$ the Banach space of all linear bounded operators $\mathcal{L}(\mathfrak{B})$ equipped with the operator norm $||A||:=\sup\limits_{|x|\le 1}|Ax|$.

Recall [12, Ch.III] that a sequence $\{A_n\}_{n\in\mathbb{N}}$ of linear bounded operators from $[\mathfrak{B}]$ strongly converges to $A\in[\mathfrak{B}]$ if $\lim\limits_{n\to\infty}A_nx=Ax$ for any $x\in\mathfrak{B}$. Denote by $[\mathfrak{B}]_s$ the space of all linear bounded operators $\mathcal{L}(\mathfrak{B})$ equipped with strong operator topology.

Theorem 4. *[12, Ch.III] (Banach-Steinhaus). If a sequence $\{A_n\}\subset[\mathfrak{B}]$ is strongly convergent, then it is bounded, i.e., there exists a constant $C>0$ such that $\|A_n\|\le C$ for any $n\in\mathbb{N}$.*

Lemma 2. *The mapping $F:[\mathfrak{B}]_s\times\mathfrak{B}\to\mathfrak{B}$ defined by $F(A,x):=Ax$ for any $(A,x)\in[\mathfrak{B}]_s\times\mathfrak{B}$ is continuous.*

Proof. Let (A_0,x_0) be an arbitrary point from $[\mathfrak{B}]_s\times\mathfrak{B}$ and $\{(A_n,x_n)\}$ be a sequence from $[\mathfrak{B}]_s\times\mathfrak{B}$ such that

$$A_n\to A_0 \text{ in } [\mathfrak{B}]_s \text{ and } x_n\to x_0 \text{ in } \mathfrak{B} \tag{5}$$

as $n\to\infty$. According to Theorem 4 there exists a constant $C>0$ such that

$$\|A_n\|\le C$$

for any $n\in\mathbb{N}$. Taking into account (6) we obtain

$$|A_nx_n-A_0x_0|=|A_n(x_n-x_0)+(A_nx_0-A_0x_0)|\le \tag{6}$$
$$\|A_n\||x_n-x_0|+|A_nx_0-A_0x_0|\le C|x_n-x_0|+|A_nx_0-A_0x_0|$$

for any $n\in\mathbb{N}$. Since $A_n\to A_0$ in $[\mathbb{B}]_s$ as $n\to\infty$, then we have

$$\lim_{n\to\infty}|A_nx_0-A_0x_0|=0. \tag{7}$$

Passing to the limit in (6) as $n\to\infty$ and taking into account (5) and (7) we obtain $\lim\limits_{n\to\infty}A_nx_n=A_0x_0$. Lemma is proved. ☐

Corollary 2. *The mapping* $\mathcal{F} : [\mathfrak{B}]_s \times [\mathfrak{B}]_s \times \mathfrak{B} \to \mathfrak{B}$ *defined by* $F(A,B,x) := A(Bx)$ *for any* $(A,B,x) \in [\mathfrak{B}]_s \times [\mathfrak{B}]_s \times \mathfrak{B}$ *is continuous.*

Proof. Let (A_0, B_0, x_0) be an arbitrary point from $[\mathfrak{B}]_s \times [\mathfrak{B}]_s \times \mathfrak{B}$ and

$$\{(A_n, B_n, x_n)\}_{n\in\mathbb{N}} \subset [\mathfrak{B}]_s \times [\mathfrak{B}]_s \times \mathfrak{B}$$

such that $(A_n, B_n, x_n) \to (A_0, B_0, x_0)$ in $[\mathfrak{B}]_s \times [\mathfrak{B}]_s \times \mathfrak{B}$ as $n \to \infty$. Denote by $y_n := B_n x_n$ and $y_0 := B_0 x_0$, then by Lemma 2 $y_n \to y_0$ in $\mathfrak{B}$ as $n \to \infty$. Let $z_n := A_n y_n$ and $z_0 := A_0 y_0$, then by Lemma 2 we have $z_0 = \lim_{n\to\infty} A_n y_n = A_0 B_0 x_0$, i.e.,

$$\lim_{n\to\infty} A_n B_n x_n = A_0 B_0 x_0.$$

Corollary is proved. □

Corollary 3. *Let* $f \in C(Y, \mathfrak{B})$ *and* $A, B \in C(Y, [\mathfrak{B}]_s)$, *then the mapping* $F \in C(Y, \mathfrak{B})$, *where* $F(y) := A(y)(B(y)f(y))$ *for any* $y \in Y$, *is continuous.*

Proof. This statement follows directly from Corollary 2. □

Remark 3. Let $\langle \mathfrak{B}, \varphi, (Y, \mathbb{R}, \sigma)\rangle$ be a cocycle over dynamical system $(Y, \mathbb{R}, \sigma)$ with the fibre $\mathfrak{B}$ and U be the mapping from $\mathbb{R}_+ \times Y$ into $\mathcal{L}(\mathfrak{B})$ defined by the equality

$$U(t,y) := \varphi(t, \cdot, y),$$

then it possesses the following properties:

a. $U(0,y) = Id_{\mathfrak{B}}$ for any $y \in Y$, where $Id_{\mathfrak{B}}$ is the unit operator acting on $\mathfrak{B}$;

b. $U(t+\tau, y) = U(t, \sigma(\tau, y))U(\tau, y)$ for any $t, \tau \in \mathbb{R}_+$;

c. for any $x \in \mathfrak{B}$ the mapping $U_x : \mathbb{R}_+ \times Y \mapsto \mathfrak{B}$ defined by $U_x(t,y) := U(t,y)x$ (for any $(t,y) \in \mathbb{R}_+ \times Y$) is continuous.

Lemma 3. *Let* $\{U(t,y) |\ t \in \mathbb{R}_+, y \in Y\}$ *be the family of operators from* $\mathcal{L}(\mathfrak{B})$ *possessing properties a.-c., then the mapping* $\varphi : \mathbb{R}_+ \times \mathfrak{B} \times Y \to \mathfrak{B}$, *defined by* $\varphi(t,x,y) := U(t,y)x$ *for any* $(t,x,y) \in \mathbb{R}_+ \times \mathfrak{B} \times Y$ *satisfies the following conditions:*

1. $\varphi(0,x,y) = x$ *for any* $(x,y) \in \mathfrak{B} \times Y$;

2. $\varphi(t+\tau, x, y) = \varphi(t, \varphi(\tau, x, y), \sigma(\tau, y))$ *for any* $t, \tau \in \mathbb{R}_+$ *and* $(x,y) \in \mathfrak{B} \times Y$;

3. the mapping $\varphi : \mathbb{R}_+ \times \mathfrak{B} \times Y \to \mathfrak{B}$ *is continuous.*

Proof. The first two conditions follow directly from the properties a. and b. respectively.

Now we will establish the continuity of the mapping φ. Let (t_0, x_0, y_0) be an arbitrary point from $\mathbb{R}_+ \times \mathfrak{B} \times Y$ and $\{(t_n, x_n, y_n)\}_{n\in\mathbb{N}}$ be a sequence from $\mathbb{R}_+ \times \mathfrak{B} \times Y$ such that $(t_n, x_n, y_n) \to (t_0, x_0, y_0)$. In virtue of the condition *c.* we have $U(t_n, y_n) \to$

$U(t_0,y_0)$ in $[\mathfrak{B}]_s$ as $n\to\infty$. By Theorem 4 the sequence of operators $\{U(t_n,y_n)\}_{n\in\mathbb{N}}$ from $\mathcal{L}(\mathfrak{B})$ is bounded, i.e., there exists a positive constant C such that

$$\|U(t_n,y_n)\|\le C \tag{8}$$

for any $n\in\mathbb{N}$. Note that

$$\begin{gathered}|\varphi(t_n,x_n,y_n)-\varphi(t_0,x_0,y_0)| = \\ |U(t_n,y_n)(x_n-x_0)+(U(t_n,y_n)-U(t_0,y_0))x_0|\le \\ \|U(t_n,y_n)\||x_n-x_0|+|(U(t_n,y_n)-U(t_0,y_0))x_0|\end{gathered} \tag{9}$$

for any $n\in\mathbb{N}$. From (8) and (9) we receive

$$|\varphi(t_n,x_n,y_n)-\varphi(t_0,x_0,y_0)|\le C|x_n-x_0|+|(U(t_n,y_n)-U(t_0,y_0))x_0| \tag{10}$$

for any $n\in\mathbb{N}$. Passing to the limit in (10) as $n\to\infty$ and taking into account the strongly continuity of the mapping $U:(t,y)\to U(t,y)$ we obtain $\lim\limits_{n\to\infty}\varphi(t_n,x_n,y_n)=\varphi(t_0,x_0,y_0)$. Lemma is proved. □

Definition 18. A family of linear bounded operators $\{U(t)\}_{t\in\mathbb{R}_+}$ is said to be a C_0-semigroup (a semigroup of strongly continuous linear bounded operators) if the following conditions are fulfilled:

1. $U(0)=Id_{\mathfrak{B}}$;
2. $U(t+\tau)=U(t)U(\tau)$ for any $t,\tau\in\mathbb{R}_+$;
3. $\lim\limits_{t\to 0^+}U(t)x=x$ for any $x\in X$.

Let $A_0:D(A_0)\to\mathfrak{B}$ be the infinitesimal generator [13, Ch.I] of the strongly continuous semigroup $\{U(t)\}_{t\in\mathbb{R}_+}$.

Definition 19. A function $F\in C(Y\times\mathfrak{B},\mathfrak{B})$ is said to be *local Lipschitzian* with respect to variable $u\in\mathfrak{B}$ uniformly with respect to $y\in Y$ if there exists a nondecreasing function $L:\mathbb{R}_+\to\mathbb{R}_+$ such that

$$|F(y,u_1)-F(y,u_2)|\le L(r)|u_1-u_2| \tag{11}$$

for any $u_1,u_2\in B[0,r]$ and $y\in Y$, where $B[0,r]:=\{u\in\mathfrak{B}:\ |u|\le r\}$.

Definition 20. The smallest constant figuring in (11) is called *Lipshchitz constant* of the function F on $Y\times B[0,r]$ (notation $Lip(r,F)$).

Let $(Y,\mathbb{R},\sigma)$ be a dynamical system on the metric space Y. Consider the differential equation

$$x'=A_0x+F(\sigma(t,y),x),\quad (y\in Y) \tag{12}$$

where $F\in C(Y\times\mathfrak{B},\mathfrak{B})$.

Definition 21. A function $u : [0, a) \mapsto \mathfrak{B}$ is said to be a weak (mild) solution of equation (12) passing through the point $x \in \mathfrak{B}$ at the initial moment $t = 0$ if $u \in C([0, T], \mathfrak{B})$ and satisfies the integral equation

$$u(t) = U(t)x + \int_0^t U(t-s)F(\sigma(s, y), u(s))ds$$

for any $t \in [0, T]$ and $0 < T < a$.

Theorem 5. *[6, Ch.VI] Let $x_0 \in \mathfrak{B}$, $r > 0$ and the conditions listed above be fulfilled. Then, there exist positive numbers $\delta = \delta(x_0, r)$ and $T = T(x_0, r)$ such that the equation (12) admits a unique solution $\varphi(t, x, y)$ ($x \in B[x_0, \delta] = \{x \in \mathfrak{B} \mid |x - x_0| \le \delta\}$) defined on the interval $[0, T]$ with the conditions: $\varphi(0, x, y) = x$, $|\varphi(t, x, y) - x_0| \le r$ for any $t \in [0, T]$ and the mapping $\varphi : [0, T] \times B[x_0, \delta] \times Y \to \mathfrak{B}$ ($(t, x, y) \mapsto \varphi(t, x, y)$) is continuous.*

Definition 22. A function $F \in C(Y \times \mathfrak{B}, \mathfrak{B})$ is said to be globally Lipschitzian with respect to variable $u \in \mathfrak{B}$ uniformly with respect to $y \in Y$ if there exists a positive constant L such that

$$|F(y, u_1) - F(y, u_2)| \le L|u_1 - u_2| \tag{13}$$

for any $u_1, u_2 \in \mathfrak{B}$ and $y \in Y$.

Definition 23. The smallest constant L with the property (13) is called Lipshchitz constant of the function F (notation $Lip(F)$).

Denote by $CL(Y \times \mathfrak{B}, \mathfrak{B})$ the Banach space of all globally Lipschitzian functions $F \in C(Y \times \mathfrak{B}, \mathfrak{B})$ and with $\sup\limits_{y \in Y} |F(y, 0)| < +\infty$ and equipped with the norm

$$||F||_{CL} := \max_{y \in Y} |F(y, 0)| + Lip(F).$$

Theorem 6. *[6, Ch.VI] Suppose that a function $F \in C(Y \times \mathfrak{B}, \mathfrak{B})$ is globally Lipschitzian. Then for any $(x, y) \in \mathfrak{B} \times Y$ there exists a unique solution $\varphi(t, x, y)$ of the equation (12) defined on the semi-axis $[0, +\infty)$ with the conditions: $\varphi(0, x, y) = x$ and the mapping $\varphi : [0, +\infty) \times \mathfrak{B} \times Y \to \mathfrak{B}$ ($(t, x, y) \mapsto \varphi(t, x, y)$) is continuous.*

Example 2. *Let $(Y, \mathbb{R}, \sigma)$ be a dynamical system on the metric space Y and $A \in C(Y, [\mathfrak{B}]_s)$. Consider the following differential equation*

$$u' = (A_0 + A(\sigma(t, y))u, \tag{14}$$

where A_0 is the infinitesimal generator of a C_0-semigroup $\{U(t)\}_{t \ge 0}$.

Denote by $C_b(Y, [\mathfrak{B}]_s)$ the family of all $A \in C(Y, [\mathfrak{B}]_s)$ with $\sup\limits_{y \in Y} \|A(y)\| < +\infty$.

Theorem 7. *Let A_0 be the infinitesimal generator of C_0-semigroup $\{U(t)\}_{t \ge 0}$ on $\mathfrak{B}$ and $A \in C_b(Y, [\mathfrak{B}]_s)$.*

Then the following statements hold:

1. *for each* $(u,y) \in \mathfrak{B} \times Y$ *the Cauchy problem*

$$x' = (A_0 + A(\sigma(t,y)))x, \quad x(0) = u$$

has a unique mild solution $\varphi(t,u,y)$*;*

2. $$\varphi(t,u,y) = U(t)u + \int_0^t U(t-\tau)A(\sigma(\tau,y))\varphi(\tau,u,y)d\tau$$

for any $(t,u,y) \in \mathbb{R}_+ \times \mathfrak{B} \times Y$*;*

3. *the triplet* $\langle \mathfrak{B}, \varphi, (Y,\mathbb{R},\sigma)\rangle$ *is a linear cocycle over* $(Y,\mathbb{R},\sigma)$ *with the fibre* $\mathfrak{B}$*.*

Proof. We can rewrite the equation (15) as follows

$$x' = A_0 x + F(\sigma(t,y),x),$$

where $F(y,x) := A(y)x$. Since $A \in C_b(Y,[\mathfrak{B}]_s)$ then there exists a constant $L > 0$ such that $\|A(y)\| \le L$ for any $y \in Y$. Thus we have

$$|F(y,x_1) - F(y,x_2)| \le L|x_1 - x_2|$$

for any $x_1, x_2 \in \mathfrak{B}$ and $y \in Y$. Now to finish the proof of Theorem it suffices to apply Theorem 6. □

3.2 Linear nonhomogeneous (affine) dynamical systems

Let $\langle \mathfrak{B}, \varphi, (Y,\mathbb{R},\sigma)\rangle$ be a linear cocycle over dynamical system $(Y,\mathbb{R},\sigma)$ with the fibre $\mathfrak{B}$, $U(t,y) := \varphi(t,\cdot,y)$ for any $(t,y) \in \mathbb{R}_+ \times Y$, $f \in C(Y,\mathfrak{B})$ and ψ be a mapping from $\mathbb{R}_+ \times \mathfrak{B} \times Y$ into $\mathfrak{B}$ defined by the equality

$$\psi(t,u,y) := U(t,y)u + \int_0^t U(t-\tau,\sigma(\tau,y))f(\sigma(\tau,y))d\tau. \tag{15}$$

From the definition of ψ the following properties follow:

1. $\psi(0,u,y) = u$ for any $(u,y) \in \mathfrak{B} \times Y$;
2. $\psi(t+\tau,u,y) = \psi(t,\psi(\tau,u,y),\sigma(\tau,y))$ for any $t,\tau \in \mathbb{R}_+$ and $(u,y) \in \mathfrak{B} \times Y$;
3. the mapping $\psi : \mathbb{R}_+ \times \mathfrak{B} \times Y \mapsto \mathfrak{B}$ is continuous;
4. $\psi(t,\lambda u + \mu v,y) = \lambda\psi(t,u,y) + \mu\psi(t,v,y)$ for any $t \in \mathbb{R}_+$, $u,v \in \mathfrak{B}$, $y \in Y$ and $\lambda,\mu \in \mathbb{R}$ (or $\mathbb{C}$) with condition $\lambda + \mu = 1$, i.e., the mapping $\psi(t,\cdot,y) : \mathfrak{B} \mapsto \mathfrak{B}$ is affine for every $(t,y) \in \mathbb{R}_+ \times Y$.

Recall that a triplet $\langle \mathfrak{B}, \psi, (Y,\mathbb{R},\sigma)\rangle$ is called an affine (nonhomogeneous) cocycle over dynamical system $(Y,\mathbb{R},\sigma)$ with the fibre $\mathfrak{B}$ if ψ is a mapping from $\mathbb{T} \times \mathfrak{B} \times Y$ into $\mathfrak{B}$ possessing the properties 1.-4.

Remark 4. If we have a linear cocycle $\langle \mathfrak{B}, \varphi, (Y, \mathbb{R}, \sigma)\rangle$ over dynamical system $(Y, \mathbb{R}, \sigma)$ with the fibre $\mathfrak{B}$ and $f \in C(Y, \mathbb{B})$, then by the equality (15) an affine cocycle $\langle \mathfrak{B}, \psi, (Y, \mathbb{R}, \sigma)\rangle$ over dynamical system $(Y, \mathbb{R}, \sigma)$ with the fibre $\mathfrak{B}$ is defined, which is called an affine (nonhomogeneous) cocycle associated with the linear cocycle φ and the function $f \in C(Y, \mathfrak{B})$.

Example 3. Let Y be a complete metric space, $(Y, \mathbb{R}, \sigma)$ be a dynamical system on Y. Consider the following linear nonhomogeneous differential equation

$$x' = (A_0 + A(\sigma(t, y)))x + f(\sigma(t, y)), \quad (y \in Y) \tag{16}$$

where $A \in C_b(Y, [\mathfrak{B}]_s)$.

Under the above assumptions equation (15) generates a linear cocycle $\langle \mathfrak{B}, \varphi, (Y, \mathbb{R}, \sigma)\rangle$ over dynamical system $(Y, \mathbb{R}, \sigma)$ with the fibre $\mathfrak{B}$. According to Remark 4 by the equality (15) a linear nonhomogeneous cocycle $\langle \mathfrak{B}, \psi, (Y, \mathbb{R}, \sigma)\rangle$ over dynamical system $(Y, \mathbb{R}, \sigma)$ with the fibre $\mathfrak{B}$ is defined. Thus every nonhomogeneous linear differential equations (16) generates a linear nonhomogeneous (affine) cocycle ψ.

3.3 Exponential dichotomy and Green's function

Definition 24. Recall (see, for example, [7, Ch.VI]) that a linear cocycle $\langle \mathfrak{B}, \varphi, (Y, \mathbb{R}, \sigma)\rangle$ is *hyperbolic* (or equivalently, satisfies the condition of *exponential dichotomy*) if there exists a continuous projection-valued function $P : Y \to [\mathfrak{B}]_s$ satisfying:

1. $P(\sigma(t, y))U(t, y) = U(t, y)P(y)$ for any $(t, y) \in \mathbb{R}_+ \times Y$:
2. for any $(t, y) \in \mathbb{R}_+ \times Y$ the operator $U_Q(t, y)$ is invertible as an operator from $ImQ(y)$ to $ImQ(\sigma(t, y))$, where $Q(y) := Id_{\mathfrak{B}} - P(y)$ and $U_Q(t, y) := U(t, y)Q(y)$;
3. there exist constants $\nu > 0$ and $\mathcal{N} > 0$ such that

$$\|U_P(t, y)\| \le \mathcal{N} e^{-\nu t} \text{ and } \|U_Q(t, y)^{-1}\| \le \mathcal{N} e^{-\nu t}$$

for any $y \in Y$ and $t \in \mathbb{R}_+$, where $U_P(t, y) := U(t, y)P(y)$ and $U(t, y) = \varphi(t, \cdot, y)$.

A *Green's function* $G(t, y)$ (see, for example, [7, Ch.VII]) for hyperbolic cocycle φ is defined by

$$G(t, y) := \begin{cases} U_P(t, y), \text{ if } (t, y) \in \mathbb{R}_+ \times Y \\ -[U_Q(-t, y)Q(\sigma(t, y))]^{-1} Q(\sigma(t, y), \text{ if } (t, y) \in \mathbb{R}_- \times Y, \end{cases}$$

where $U_Q(t, y) := U_Q(-t, \sigma(t, y))$ for any $(t, y) \in \mathbb{R}_- \times Y$.

Denote by $C_b(Y, \mathfrak{B})$ the Banach space of all continuous and bounded functions $f : Y \to \mathfrak{B}$ equipped with the sup-norm. If the metric space Y is compact, then $C_b(Y, \mathfrak{B}) = C(Y, \mathfrak{B})$.

Lemma 4. *Suppose that the linear cocycle $\langle \mathfrak{B}, \varphi, (Y, \mathbb{R}, \sigma)\rangle$ over dynamical system $(Y, \mathbb{R}, \sigma)$ with the fibre $\mathfrak{B}$ is hyperbolic and $f \in C_b(Y, \mathfrak{B})$.*

Then the following statements hold:

1. $$|G(\sigma(-\tau, y), \tau)f(\sigma(-\tau, y))| \le \mathcal{N}e^{-\nu|\tau|}\|f\|$$
   for any $(\tau, y) \in \mathbb{R} \times Y$;
2. *the integral $\int_{-\infty}^{\infty} G(\sigma(-\tau, y), \tau)f(\sigma(-\tau, y))d\tau$ converges uniformly in $y \in Y$ and*
   $$|\int_{-\infty}^{\infty} G(\sigma(-\tau, y), \tau)f(\sigma(-\tau, y))d\tau| \le 2\frac{\mathcal{N}}{\nu}\|f\|$$
   for any $y \in Y$;
3. *by the equality*
   $$\xi(y) := \int_{-\infty}^{\infty} G(\sigma(-\tau, y), \tau)f(\sigma(-\tau, y))d\tau \tag{17}$$
   a continuous mapping $\xi : Y \to \mathfrak{B}$ is well defined and $\xi \in C_b(Y, \mathfrak{B})$.

Proof. The first two statements are well known (see, for example, [7, Ch.VII]). To finish the proof of Lemma it suffices to establish that the mapping defined by (17) is continuous. To this end we note that

1. $$\xi(y) = \lim_{n\to\infty} \xi_n(y),$$
   where
   $$\xi_n(y) := \int_{-n}^{n} G(\sigma(-\tau, y), \tau)f(\sigma(-\tau, y))d\tau;$$
2. $$|\xi(y) - \xi_n(y)| \le \frac{2\mathcal{N}\|f\|}{\nu}e^{-\nu n}$$
   for any $n \in \mathbb{N}$ and, consequently, $\xi_n(y) \to \xi(y)$ as $n \to \infty$ uniformly in $y \in Y$;
3. for any $n \in \mathbb{N}$ the mapping $\xi_n : Y \to \mathfrak{B}$ is continuous;
4. the mapping $\xi : Y \to \mathfrak{B}$ is continuous and $\xi \in C_b(Y, \mathfrak{B})$.

Of the above statements, only the continuity of ξ is nontrivial. We will prove it. Since ξ is a uniform limit of the functional sequence $\{\xi_n\}$, then it is sufficient to establish the continuity of ξ_n for any $n \in \mathbb{N}$ (in fact, for sufficiently large n). Let y_0 be an arbitrary point of Y and $\{y_k\}$ be a sequence from Y such that $y_k \to y_0$ as $k \to \infty$. For any $n \in \mathbb{N}$ we have

$$|\xi_n(y_k) - \xi_n(y_0)| \le \tag{18}$$

$$\int_{-n}^{n}|G(\sigma(-\tau,y_k),\tau)f(\sigma(-\tau,y_k))-G(\sigma(-\tau,y_0),\tau)f(\sigma(-\tau,y_0))|d\tau=\int_{-n}^{n}\alpha_k(\tau)d\tau,$$

where

$$\alpha_k(\tau):=|G(\sigma(-\tau,y_k),\tau)f(\sigma(-\tau,y_k))-G(\sigma(-\tau,y_0),\tau)f(\sigma(-\tau,y_0))|.$$

The function $\alpha_k:[-n,n]\setminus\{0\}\to\mathbb{R}_+$ possesses the following properties:

1. α_k is continuous;
2. $\alpha_k(\tau)\to 0$ as $k\to\infty$ for any $\tau\in[-n,n]\setminus\{0\}$;
3. $\alpha_k(\tau)\le 2\mathcal{N}\|f\|e^{-\nu|\tau|}$ for any $\tau\in[-n,n]$ and $k\in\mathbb{N}$.

By Lebesgue dominated convergence theorem we have

$$\lim_{k\to\infty}\int_{-n}^{n}\alpha_k(\tau)d\tau=0 \tag{19}$$

for any $n\in\mathbb{N}$. From (18) and (19) we obtain $\lim_{k\to\infty}\xi_n(y_k)=\xi_n(y_0)$ for any $n\in\mathbb{N}$.

Since ξ is the uniform limit of continuous functions ξ_n, then ξ is also continuous on Y. Lemma is completely proved. □

Theorem 8. *Assume that the following conditions are fulfilled:*

1. $A\in C_b(Y,[\mathfrak{B}]_s)$;
2. *the linear cocycle* $\langle\mathfrak{B},\varphi,(Y,\mathbb{R},\sigma)\rangle$ *over dynamical system* $(Y,\mathbb{R},\sigma)$ *with the fibre* $\mathfrak{B}$ *generated by the equation*

$$x'=(A_0+A(\sigma(t,y)))x \quad (y\in Y)$$

is hyperbolic.

Then the following statements hold:

1. *by the equality*

$$\psi(t,u,y):=U(t,y)u+\int_0^t U(t-\tau,\sigma(\tau,y))f(\sigma(\tau,y))d\tau$$

an affine (linear nonhomogeneous) cocycle $\langle\mathfrak{B},\psi,(Y,\mathbb{R},\sigma)\rangle$ *over dynamical system* $(Y,\mathbb{R},\sigma)$ *with the fibre* $\mathfrak{B}$ *is defined;*

2. *if* $\xi=\mathbb{G}(f)$, *then*

$$\xi(\sigma(t,y))=U(t,y)\xi(y)+\int_0^t U(t-\tau,\sigma(\tau,y))f(\sigma(\tau,y))d\tau: \tag{20}$$

3. *if* $f \in C_b(Y, \mathfrak{B})$*, then by the equality*

$$\xi(y) = \int_{-\infty}^{\infty} G(\sigma(-\tau, y), \tau) f(\sigma(-\tau, y)) d\tau \tag{21}$$

a continuous and invariant section of the affine cocycle $\langle \mathfrak{B}, \psi, (Y, \mathbb{R}, \sigma) \rangle$ *is defined;*

4. *if a point* $y \in Y$ *of the dynamical system* $(Y, \mathbb{R}, \sigma)$ *is Poisson stable, then the solution* $\psi(t, \nu(y), y)$ *of the equation*

$$x' = (A_0 + A(\sigma(t, y)))x + f(\sigma(t, y)) \quad (y \in Y) \tag{22}$$

is compatible, i.e., $\mathfrak{N}_y \subseteq \mathfrak{N}_x$ *with* $x := (\xi(y), y)$*;*

5. *if a point* $y \in Y$ *is Lagrange stable, then the solution* $\psi(t, \xi(y), y)$ *is uniformly compatible, i.e.,* $\mathfrak{M}_y \subseteq \mathfrak{M}_x$ *and* $x = (\xi(y), y)$*.*

Proof. The first statement follows from Theorem 7 and Example 3.

The second statement follows from Proposition 7.33 [7, Ch.VII].

To prove the third statement we note that the continuity of the map $\xi : Y \to \mathfrak{B}$ defined by (21) follows from Lemma 4 (item (iii)). The invariance of the mapping ξ follows from the second statement. Indeed, in virtue of (20) we have

$$\psi(t, \xi(y), y) = U(t, y)\xi(y) + \int_0^t U(t - \tau, \sigma(\tau, y)) f(\sigma(\tau, y)) d\tau \tag{23}$$

for any $(t, y) \in \mathbb{R}_+ \times Y$. By (23) and (20) we obtain

$$\psi(t, \xi(y), y) = \xi(\sigma(t, y)) \tag{24}$$

for any $(t, y) \in \mathbb{R}_+ \times Y$.

To prove the fourth statement of Theorem it suffices to show that $\mathfrak{N}_y \subseteq \mathfrak{N}_x$, where $x := (\xi(y), y)$. Let $\{t_n\} \in \mathfrak{N}_y$, i.e., $\sigma(t_n, y) \to y$ as $n \to \infty$. Denote by $x := (\xi(y), y) \in X := \mathfrak{B} \times Y$. We note that

$$\pi(t_n, x) = (\psi(t_n, \xi(y), y), \sigma(t_n, y)) \tag{25}$$

for any $n \in \mathbb{N}$. Taking into account (24) and (25) we obtain

$$\pi(t_n, x) = (\xi(\sigma(t_n, y)), \sigma(t_n, y))) \tag{26}$$

for any $n \in \mathbb{N}$. Since $\{t_n\} \in \mathfrak{N}_y$, then passing to the limit in (26) as $n \to \infty$ we receive $\lim\limits_{n \to \infty} \pi(t_n, x) = (\xi(y), y) = x$, i.e., $\{t_n\} \in \mathfrak{N}_x$. Thus we have the inclusion $\mathfrak{N}_y \subseteq \mathfrak{N}_x$.

Let now the point y be Lagrange stable, then by Theorem 3 to prove that the solution $\psi(t, \xi(y), y)$ of the equation (22) is uniformly compatible it suffices to show that $\mathfrak{M}_y \subseteq \mathfrak{M}_x$, where $x = (\xi(y), y)$. Let $\{t_n\} \in \mathfrak{M}_y$, then there exists a point $q \in H(y)$ such that

$$\sigma(t_n, y) \to q \tag{27}$$

as $n \to \infty$. Reasoning as above we obtain

$$\pi(t_n, x) = (\psi(t_n, \xi(y), y), \sigma(t_n, y)). \tag{28}$$

Passing to the limit in (28) as $n \to \infty$ and taking into account (27) we obtain $\pi(t_n, x) \to p := (\xi(q), q) \in H(x)$ as $n \to \infty$, i.e., $\mathfrak{M}_y \subseteq \mathfrak{M}_x$. This means that the point x is uniformly comparable by character of recurrence with the point y (or equivalently, the solution $\psi(t, \xi(y), y)$ of the equation (22) is uniformly compatible). Theorem is completely proved. □

Corollary 4. *Under the conditions of Theorem 8 if a point $y \in Y$ is stationary (respectively, τ-periodic, quasi-periodic with the frequency base $\{\nu_1, \dots, \nu_m\}$, Bohr almost periodic, almost automorphic, recurrent, almost recurrent, Levitan almost periodic, Poisson stable, pseudo-recurrent and Lagrange stable, uniformly Poisson stable and Lagrange stable), then the solution $\psi(t, \xi(y), y)$ of the equation (22) is also stationary (respectively, τ-periodic, quasi-periodic with the frequency base $\{\nu_1, \dots, \nu_m\}$, Bohr almost periodic, almost automorphic, recurrent, almost recurrent, Levitan almost periodic, Poisson stable, pseudo-recurrent and Lagrange stable, uniformly Poisson stable and Lagrange stable).*

4 Semi-Linear Differential Equations

4.1 Global Lipschitzian Perturbations

Denote by d_{CL} the distance on the Banach space $(CL(Y \times \mathfrak{B}, \mathfrak{B}), \|\cdot\|_{CL})$ defined by

$$d_{CL}(F_1, F_2) := \|F_1 - F_2\|_{CL}$$

for any $F_1, F_2 \in CL(Y \times \mathfrak{B}, \mathfrak{B})$.

Theorem 9. *Suppose that the following conditions are fulfilled:*

1. *linear differential equation*

$$x' = (A_0 + A(\sigma(t, y)))x \tag{29}$$

   is hyperbolic;

2. *$F \in CL(Y \times \mathfrak{B}, \mathfrak{B})$;*

3. *$Lip(F) \le \varepsilon_0$, where $\mathcal{N}$, ν are the positive constants from Definition 24 and $0 < \varepsilon_0 < \frac{\nu}{2\mathcal{N}}$.*

Then

1. *the equation*
$$x' = (A_0 + A(\sigma(t,y)))x + F(\sigma(t,y),x) \tag{30}$$
admits a unique invariant section $\xi \in C_b(Y,\mathfrak{B})$;

2. *the mapping* $F \to \xi_F$ *from* $CL(Y \times \mathfrak{B},\mathfrak{B})$ *into* $C_b(Y,\mathfrak{B})$, *where* ξ_F *is a unique invariant section of (30), is continuous.*

Proof. Let the equation (29) be hyperbolic and $G(t,y)$ be its Green's function, then there exist positive constants $\mathcal{N}$ and ν such that

$$||G(t,y)|| \le \mathcal{N} e^{-\nu|t|}$$

for any $y \in Y$ and $t \in \mathbb{R}$. Consider the operator Φ defined on the Banach space $C_b(Y,\mathfrak{B})$ by the equality

$$(\Phi\psi)(y) = \int_{-\infty}^{+\infty} G(-\tau,\sigma(-\tau,y))F(\sigma(-\tau,y),\psi(\sigma(-\tau,y)))d\tau. \tag{31}$$

Note that the operator Φ maps $C_b(Y,\mathfrak{B})$ into itself. It is easy to see that $F(\cdot,\psi(\cdot)) \in C(Y,\mathfrak{B})$ for any $\psi \in C(Y,\mathfrak{B})$. Indeed, let y_0 be an arbitrary point from Y and $\{y_k\} \subset Y$ such that $y_k \to y_0$ as $k \to \infty$. Then we have

$$\begin{gathered} |F(y_k,\psi(y_k)) - F(y_0,\psi(y_0))| \le \\ |F(y_k,\psi(y_k)) - F(y_k,\psi(y_0))| + |F(y_k,\psi(y_0)) - F(y_0,\psi(y_0))| \le \\ Lip(F)|\psi(y_k) - \psi(y_0)| + |F(y_k,\psi(y_0)) - F(y_0,\psi(y_0))| \end{gathered} \tag{32}$$

for any $k \in \mathbb{N}$. Passing to the limit in (32) as $k \to \infty$ we receive $\lim\limits_{k\to\infty} F(y_k,\psi(y_k)) = F(y_0,\psi(y_0))$. This means that $F(\cdot,\psi(\cdot)) \in C(Y,\mathfrak{B})$. Moreover, if $\psi \in C_b(Y,\mathfrak{B})$, then $F(\cdot,\psi(\cdot)) \in C_b(Y,\mathfrak{B})$. To show this fact we notice that

$$\begin{gathered} |F(y,\psi(y))| \le |F(y,\psi(y)) - F(y,0)| + |F(y,0)| \le \\ Lip(F)|\psi(y)| + |F(y,0)| \le Lip(F)\|\psi\|_{C_b(Y,\mathfrak{B})} + A \end{gathered}$$

for any $y \in Y$ and, consequently, $\|F(\cdot,\psi(\cdot))\| \le Lip(F)\|\psi\| + A$, i.e., $F(\cdot,\psi(\cdot)) \in C_b(Y,\mathfrak{B})$.

By Lemma 4 (item (ii)) the function $\Phi\psi$ defined by (31) belongs to $C_b(Y,\mathfrak{B})$. Now we will prove that $\Phi : C_b(Y,\mathfrak{B}) \mapsto C_b(Y,\mathfrak{B})$ is a contraction. Indeed, we have

$$|(\Phi\psi_1)(y) - (\Phi\psi_1)(y)| \le \frac{2M}{\alpha}||F(\cdot,\psi_1(\cdot)) - F(\cdot,\psi_2(\cdot))||. \tag{33}$$

On the other hand

$$\begin{gathered} |F(y,\psi_1(y)) - F(y,\psi_2(y))| \le \\ Lip(F)|\psi_1(y) - \psi_2(y)| \le \varepsilon_0||\psi_1 - \psi_2|| \end{gathered}$$

for any $y \in Y$ and, consequently, we obtain

$$||F(\cdot, \psi_1(\cdot)) - F(\cdot, \psi_2(\cdot))|| \le \varepsilon_0 ||\psi_1 - \psi_2||. \tag{34}$$

From (33) and (34) it follows

$$||\Phi\psi_1 - \Phi\psi_2|| \le \frac{2\mathcal{N}}{\nu}\varepsilon_0 ||\psi_1 - \psi_2||.$$

Thus $\Phi : C_b(Y, \mathfrak{B}) \mapsto C_b(Y, \mathfrak{B})$ is a contraction, and consequently, Φ has a unique fixed point $\xi \in C_b(Y, \mathfrak{B})$: $\Phi\xi = \xi$. It is easy to see that ξ is a unique invariant section of perturbed equation (30) from $C_b(Y, \mathfrak{B})$.

Now we will prove the second statement of Theorem. Let $F_0, F_n \in CL(Y \times \mathfrak{B}, \mathfrak{B})$ with the condition $Lip(F_0), Lip(F_n) \le \varepsilon_0$ (for any $n \in \mathbb{N}$) such that $F_n \to F_0$ as $n \to \infty$. Let $\xi_{F_0} \in C_b(Y, \mathfrak{B})$ (respectively, $\xi_{F_n} \in C_b(Y, \mathfrak{B})$) be the unique invariant section of the equation

$$u' = (A_0 + A(\sigma(t, y)))u + F_0(\sigma(t, y), u)$$

(respectively, of the equation

$$u' = (A_0 + A(\sigma(t, y)))u + F_n(\sigma(t, y), u)),$$

then $\xi_{F_n} - \xi_{F_0}$ is an invariant section of the equation

$$u' = (A_0 + A(\sigma(t, y)))u + F_0(\sigma(t, y), \xi_{F_0}(\sigma(t, y))) - F_n(\sigma(t, y), \xi_{F_n}(\sigma(t, y))). \tag{35}$$

Since the equation (35) has a unique invariant section, then we obtain

$$\xi_{F_0}(y) - \xi_{F_n}(y) = \mathbb{G}[F_0(\cdot, \xi_{F_0}(\cdot)) - F_n(\cdot, \xi_{F_n}(\cdot))], \tag{36}$$

where $\mathbb{G}$ is a Green operator associate with $G(t, y)$. Note that

$$\begin{aligned} |F_0(y, \xi_{F_0}(y)) - F_n(y, \xi_{F_0}(y))| \le |F_0(y, 0) - F_n(y, 0)| + \\ |(F_0(y, \xi_{F_0}(y)) - F_n(y, \xi_{F_0}(y))) - (F_0(y, 0) - F_n(y, 0))| \le \\ \max_{y \in Y} |F_0(y, 0) - F_n(y, 0)| + Lip(F_n - F_0)||\xi_{F_0}|| \le \\ (1 + ||\xi_{F_0}||)d_{CL}(F_n, F_0) \end{aligned} \tag{37}$$

and

$$\begin{aligned} |F_0(y, \xi_{F_0}(y)) - F_n(y, \xi_{F_n}(y))| \le |F_0(y, \xi_{F_0}(y)) - F_n(y, \xi_{F_0}(y))| + \\ |F_n(y, \xi_{F_0}(y)) - F_n(y, \xi_{F_n}(y))| \le |F_0(y, \xi_{F_0}(y)) - F_n(y, \xi_{F_0}(y))| + \\ Lip(F_n)|\xi_{F_0}(y)) - \xi_{F_n}(y))| \end{aligned} \tag{38}$$

for any $y \in Y$. From (36)–(38) we obtain

$$|\xi_{F_0}(y) - \xi_{F_n}(y)| \le (1 + ||\xi_{F_0}||)||F_0 - F_n||_{CL} +$$

$$Lip(F_n)|\xi_{F_0}(y)-\xi_{F_n}(y)| \le (1+||\xi_{F_0}||)d_{CL}(F_n,F_0)+ \\ \varepsilon_0|\xi_{F_0}(y)-\xi_{F_n}(y)|$$

and, consequently,

$$|\xi_{F_0}(y)-\xi_{F_n}(y)| \le (1-\varepsilon_0)^{-1}(1+||\xi_{F_0}||)d_{CL_{r_0}}(F_n,F_0),$$

i.e.,

$$||\xi_{F_0}-\xi_{F_n}|| \le (1-\varepsilon_0)^{-1}(1+||\xi_{F_0}||)d_{CL_{r_0}}(F_n,F_0)$$

for any $n\in\mathbb{N}$. Passing to the limit in the last inequality as $n\to\infty$ we obtain $\xi_{F_n}\to\gamma_{F_0}$. Theorem is proved. □

Corollary 5. *Under the conditions of Theorem 9 if a point $y\in Y$ is stationary (respectively, τ-periodic, quasi-periodic with a frequency base $\{\nu_1,\dots,\nu_m\}$, Bohr almost periodic, almost automorphic, recurrent, almost recurrent, Levitan almost periodic, Poisson stable, pseudo-recurrent and Lagrange stable, uniformly Poisson stable and Lagrange stable), then the solution $\psi(t,\xi(y),y)$ of the equation (30) is also stationary (respectively, τ-periodic, quasi-periodic with a frequency base $\{\nu_1,\dots,\nu_m\}$, Bohr almost periodic, almost automorphic, recurrent, almost recurrent, Levitan almost periodic, Poisson stable, pseudo-recurrent and Lagrange stable, uniformly Poisson stable and Lagrange stable).*

Proof. This statement follows from Theorem 9 and Corollary 4. □

Corollary 6. *Suppose that the following conditions are fulfilled:*

1. *the linear equation (29) is hyperbolic;*
2. *$F\in CL(Y\times\mathfrak{B},\mathfrak{B})$.*

Then there exists a positive number λ_0 so that:

1. *for every $\lambda\in[-\lambda_0,\lambda_0]$ the equation*

$$u'=A(\sigma(t,y))u+f(\sigma(t,y))+\lambda F(\sigma(t,y),u) \quad (y\in Y) \tag{39}$$

admits a unique invariant section $\xi_\lambda\in C_b(Y,\mathfrak{B})$;

2.

$$||\xi_\lambda-\xi_0||\to 0$$

as $\lambda\to 0$, where ξ_0 is a unique invariant section of equation

$$u'=A(\sigma(t,y))u+f(\sigma(t,y)) \quad (y\in Y).$$

Proof. This statement follows from Theorem 9. Indeed. Denote by

$$\mathcal{F}_\lambda(y,u) := f(y) + \lambda F(y,u)$$

for any $(y,u) \in Y \times B[0,r_0]$, then

$$Lip(\mathcal{F}_\lambda) \le |\lambda| Lip(F) \le \varepsilon_0$$

for any $|\lambda| \le \lambda_0 \le \varepsilon_0 / Lip(F)$ and $||\mathcal{F}_\lambda - f||_{CL} \to 0$ as $\lambda \to 0$ since $f = \mathcal{F}_0$. □

Corollary 7. *The following statements hold.*

1. *Under the conditions of Corollary 6 there exists a positive number λ_0 such that for any $\lambda \in [-\lambda_0, \lambda_0]$ the equation (39) has a unique invariant section ξ_λ;*

2. *If a point $y \in Y$ is stationary (respectively, τ-periodic, quasi-periodic with a frequency base $\{\nu_1, \dots, \nu_m\}$, Bohr almost periodic, almost automorphic, recurrent, almost recurrent, Levitan almost periodic, Poisson stable, pseudo-recurrent and Lagrange stable, uniformly Poisson stable and Lagrange stable), then the solution $\psi(t, \xi_\lambda(y), y) = \xi_\lambda(\sigma(t,y))$ of the equation (39) is also stationary (respectively, τ-periodic, quasi-periodic with a frequency base $\{\nu_1, \dots, \nu_m\}$, Bohr almost periodic, almost automorphic, recurrent, almost recurrent, Levitan almost periodic, Poisson stable, pseudo-recurrent and Lagrange stable, uniformly Poisson stable and Lagrange stable);*

3. $$\sup_{t \in \mathbb{R}} |\psi(t, \xi_\lambda(y), y) - \psi(t, \xi_0(y), y)| \le \|\xi_\lambda - \xi_0\| \to 0$$

   as $\lambda \to 0$.

Proof. This statement follows from Corollaries 4 and 10. □

4.2 Local Lipschitzian Perturbations

Theorem 10. *Suppose that the following conditions are fulfilled:*

1. *the linear equation (29) is hyperbolic;*

2. *$F(\cdot, 0) \in C_b(Y, \mathfrak{B})$:*

3. *the function $F \in C(Y \times \mathfrak{B}, \mathfrak{B})$ is locally Lipschitzian;*

4. *$Lip(r_0, F) \le \varepsilon_0$, where $\mathcal{N}$, ν are the positive constants of hyperbolicity, $A := \sup_{y \in Y} |F(y,0)|$, $0 < \varepsilon_0 < \frac{\nu}{2\mathcal{N}}$ and $r_0 := \frac{2\mathcal{N}}{\nu} A (1 - \varepsilon_0 \frac{2\mathcal{N}}{\nu})^{-1}$.*

Then

1. *the equation (30) admits a unique invariant section $\xi \in C(Y, B[0, r_0])$;*

2. *the mapping $F \to \xi_F$ from $CL_{r_0}(Y \times B[0,r_0],\mathfrak{B})$ into $C_b(Y,\mathfrak{B})$, where ξ_F is a unique invariant section of (30), is continuous.*

Proof. Let the equation (29) be hyperbolic and $G(t,y)$ be its Green's function. Consider the operator Φ defined on the space $C_b(Y, B[0,r_0])$ by the equality

$$(\Phi\psi)(y) = \int_{-\infty}^{+\infty} G(-\tau,\sigma(-\tau,y))F(\sigma(-\tau,y),\psi(\sigma(-\tau,y)))d\tau.$$

We will show that $\Phi(C(Y,B[0,r_0]) \subseteq C(Y,B[0,r_0])$. In fact, let $\psi \in C(Y,B[0,r_0])$, then we have

$$|F(y,\psi(y))| \le A + Lip(r_0,F)r_0 \le A + \varepsilon_0 r_0$$

for any $y \in Y$ and according to choice of the number r_0 we obtain

$$||\Phi\psi|| \le \frac{2M}{\alpha}||F(\cdot,\psi(\cdot))|| \le \frac{2\mathcal{N}}{\nu}(A+\varepsilon_0 r_0) \le r_0.$$

Now we will prove that $\Phi : C(Y,B[0,r_0]) \mapsto C(Y,[0,r_0])$ is a contraction. Notice that

$$|(\Phi\psi_1)(y) - (\Phi\psi_1)(y)| \le \frac{2\mathcal{N}}{\nu}||F(\cdot,\psi_1(\cdot)) - F(\cdot,\psi_2(\cdot))|| \tag{40}$$

for any $\psi_1,\psi_2 \in C(Y,B[0,r_0])$.

On the other hand

$$|F(y,\psi_1(y)) - F(y,\psi_2(y))| \le$$
$$Lip(r_0,F)|\psi_1(y)-\psi_2(y)| \le \varepsilon_0||\psi_1-\psi_2||$$

for any $y \in Y$ and, consequently, we obtain

$$||F(\cdot,\psi_1(\cdot)) - F(\cdot,\psi_2(\cdot))|| \le \varepsilon_0||\psi_1-\psi_2||. \tag{41}$$

From (40) and (41) it follows

$$||\Phi\psi_1 - \Phi\psi_2|| \le \frac{2\mathcal{N}}{\nu}\varepsilon_0||\psi_1-\psi_2||$$

for any $\psi_1,\psi_2 \in C(Y,B[0,r_0])$. Thus $\Phi : C(Y,B[0,r_0]) \mapsto C(Y,B[0,r_0])$ is a contraction, and consequently, Φ has a unique fixed point $\xi \in C(Y,B[0,r_0])$: $\Phi\xi = \xi$. It is easy to see that ξ is a unique invariant section of the perturbed equation (30) from $C(Y,B[0,r_0])$.

Now we will prove the second statement of Theorem. Let $F_0, F_n \in CL(Y \times B[0,r_0],\mathfrak{B})$ with the condition $Lip(r_0,F_0), Lip(r_0,F_n) \le \varepsilon_0$ (for any $n \in \mathbb{N}$) such that $F_n \to F_0$ as $n \to \infty$. Let $\xi_{F_0} \in C(Y,B[0,r_0])$ (respectively, $\xi_{F_n} \in C(Y,B[0,r_0])$) be the unique invariant section of the equation

$$u' = (A_0 + A(\sigma(t,y)))u + F_0(\sigma(t,y),u)$$

(respectively, of the equation

$$u' = (A_0 + A(\sigma(t,y)))u + F_n(\sigma(t,y),u)),$$

then $\xi_{F_n} - \xi_{F_0}$ is an invariant section of the equation

$$u' = (A_0 + A(\sigma(t,y)))u + F_0(\sigma(t,y),\xi_{F_0}(\sigma(t,y))) - \\ F_n(\sigma(t,y),\xi_{F_n}(\sigma(t,y))). \tag{42}$$

Since the equation (42) has a unique invariant section, then we obtain

$$\xi_{F_0}(y) - \xi_{F_n}(y) = \mathbb{G}[F_0(\cdot,\xi_{F_0}(\cdot)) - F_n(\cdot,\xi_{F_n}(\cdot))], \tag{43}$$

where $\mathbb{G}$ is a Green operator associate with $G(t,y)$. Note that

$$|F_0(y,\xi_{F_0}(y)) - F_n(y,\xi_{F_0}(y))| \le |F_0(y,0) - F_n(y,0)| + \\ |(F_0(y,\xi_{F_0}(y)) - F_n(y,\xi_{F_0}(y))) - (F_0(y,0) - F_n(y,0))| \le \\ \max_{y\in Y} |F_0(y,0) - F_n(y,0)| + Lip(r_0, F_n - F_0)||\xi_{F_0}|| \le \\ (1 + ||\xi_{F_0}||)d_{CL_{r_0}}(F_n,F_0) \tag{44}$$

and

$$|F_0(y,\xi_{F_0}(y)) - F_n(y,\xi_{F_n}(y))| \le |F_0(y,\xi_{F_0}(y)) - F_n(y,\xi_{F_0}(y))| + \\ |F_n(y,\xi_{F_0}(y)) - F_n(y,\xi_{F_n}(y))| \le |F_0(y,\xi_{F_0}(y)) - F_n(y,\xi_{F_0}(y))| + \\ Lip(r_0,F_n)|\xi_{F_0}(y)) - \xi_{F_n}(y))| \tag{45}$$

for any $y \in Y$. From (43)–(45) we obtain

$$|\xi_{F_0}(y) - \xi_{F_n}(y)| \le (1 + ||\xi_{F_0}||)||F_0 - F_n||_{CL} + \\ Lip(F_n)|\xi_{F_0}(y) - \xi_{F_n}(y)| \le (1 + ||\xi_{F_0}||)d_{CL_{r_0}}(F_n,F_0) + \\ \varepsilon_0|\xi_{F_0}(y) - \xi_{F_n}(y)|$$

and, consequently,

$$|\xi_{F_0}(y) - \xi_{F_n}(y)| \le (1-\varepsilon_0)^{-1}(1 + ||\xi_{F_0}||)d_{CL_{r_0}}(F_n,F_0),$$

i.e.,

$$||\xi_{F_0} - \xi_{F_n}|| \le (1-\varepsilon_0)^{-1}(1 + ||\xi_{F_0}||)d_{CL_{r_0}}(F_n,F_0)$$

for any $n \in \mathbb{N}$. Passing to the limit in the last inequality as $n \to \infty$ we obtain $\xi_{F_n} \to \xi_{F_0}$. Theorem is proved. □

Corollary 8. *Under the conditions of Theorem 10 if a point $y \in Y$ is stationary (respectively, τ-periodic, quasi-periodic with a frequency base $\{\nu_1,\dots,\nu_m\}$, Bohr almost periodic, almost automorphic, recurrent, almost recurrent, Levitan almost periodic, Poisson stable, pseudo-recurrent and Lagrange stable, uniformly Poisson* *stable and Lagrange stable), then the equation (30) admits a unique invariant section* $\xi \in C(Y, B[0, r_0])$ *and the solution* $\psi(t, \xi(y), y)$ *of the equation (30) is also stationary (respectively,* τ*-periodic, quasi-periodic with a frequency base* $\{\nu_1, \dots, \nu_m\}$*, Bohr almost periodic, almost automorphic, recurrent, almost recurrent, Levitan almost periodic, Poisson stable, pseudo-recurrent and Lagrange stable, uniformly Poisson stable and Lagrange stable).*

Proof. This statement follows from Theorem 10 and Corollary 4. □

Corollary 9. *Suppose that the following conditions are fulfilled:*

1. *the linear equation (29) is hyperbolic;*
2. *there exists a number* $r_0 > 0$ *such that the function* $F \in C(Y \times \mathfrak{B}, \mathfrak{B})$ *is Lipschitzian on* $B[0, r_0]$.

Then there exists a positive number λ_0 *so that:*

1. *for every* $\lambda \in [-\lambda_0, \lambda_0]$ *equation*

$$u' = A(\sigma(t, y))u + f(\sigma(t, y)) + \lambda F(\sigma(t, y), u) \quad (y \in Y) \tag{46}$$

admits a unique invariant section $\xi_\lambda \in C(Y, B[0, r_0])$*;*

2.

$$||\xi_\lambda - \xi_0|| \to 0$$

as $\lambda \to 0$*, where* ξ_0 *is a unique invariant section of the equation*

$$u' = A(\sigma(t, y))u + f(\sigma(t, y)) \quad (y \in Y). \tag{47}$$

Proof. This statement follows from Theorem 10. Indeed, if we denote by

$$\mathcal{F}_\lambda(y, u) := f(y) + \lambda F(y, u)$$

for any $(y, u) \in Y \times B[0, r_0]$, then

$$Lip(r_0, \mathcal{F}_\lambda) \le |\lambda| Lip(r_0, F) \le \varepsilon_0$$

for any $|\lambda| \le \lambda_0 \le \varepsilon_0 / Lip(r_0, F)$ and $||\mathcal{F}_\lambda - f||_{CL_{r_0}} \to 0$ as $\lambda \to 0$ since $f = \mathcal{F}_0$. □

Corollary 10. *Under the conditions of Corollary 9 there exists a positive number* λ_0 *such that the following statements hold:*

1. *for every* $\lambda \in [-\lambda_0, \lambda_0]$ *the equation (46) has the unique invariant section* $\xi_\lambda \in C(Y, B[0, r_0])$*;*

2. *If a point $y \in Y$ is stationary (respectively, τ-periodic, quasi-periodic with a frequency base $\{\nu_1, \dots, \nu_m\}$, Bohr almost periodic, almost automorphic, recurrent, almost recurrent, Levitan almost periodic, Poisson stable, pseudo-recurrent and Lagrange stable, uniformly Poisson stable and Lagrange stable), then the solution $\psi(t, \xi_\lambda(y), y) = \xi_\lambda(\sigma(t, y))$ of the equation (46) is also stationary (respectively, τ-periodic, quasi-periodic with a frequency base $\{\nu_1, \dots, \nu_m\}$, Bohr almost periodic, almost automorphic, recurrent, almost recurrent, Levitan almost periodic, Poisson stable, pseudo-recurrent and Lagrange stable, uniformly Poisson stable and Lagrange stable);*

3.
$$\sup_{t\in\mathbb{R}} |\psi(t, \xi_\lambda(y), y) - \psi(t, \xi_0(y), y)| \leq \|\xi_\lambda - \xi_0\| \to 0$$
as $\lambda \to 0$, where ξ_0 is a unique continuous invariant section of (47).

Proof. This statement follows from Corollaries 9 and 4. □

Remark 5. Note that Corollary 9 assures existence and uniqueness of invariant section ξ_λ of the equation (46) for sufficiently small λ, but this equation can have on the space $\mathfrak{B}$ more than one invariant section. We will confirm this fact below by the corresponding example.

Example 4. Let $p \in C(Y, \mathbb{R})$ be a positive function. Consider the differential equation
$$x' = x - \lambda p(\sigma(t, y))x^3, \quad (y \in Y) \tag{48}$$
where $\lambda \in \mathbb{R}_+$. For $\lambda = 0$ it admits a unique invariant section $\xi_0(y) = 0$ for any $y \in Y$. If $\lambda > 0$, then the equation (48) admits three invariant sections: $\xi^1_\lambda(y) = 0$, $\xi^2_\lambda(y) = q_\lambda(y)$ and $\xi^3_\lambda(y) = -q_\lambda(y)$ for any $y \in Y$, where
$$q_\lambda(y) = \lambda^{-1/2}\Big(\int_{-\infty}^{0} e^{2\tau} p(\sigma(\tau, y)) d\tau\Big)^{-1/2} \quad (y \in Y).$$
Note that $||\xi^1_\lambda|| \to 0$, $||\xi^2_\lambda|| \to \infty$ and $||\xi^3_\lambda|| \to \infty$ as λ goes to 0.

Finally we note that if $\lambda < 0$, the equation (48) admits a unique invariant section $\xi_\lambda = 0$.

Below we give an example which illustrates our above results.

Example 5. Let Y be a two-dimensional torus $\mathcal{T}^2 := \mathbb{R}^2/2\pi\mathbb{Z}^2$. Let $(Y, \mathbb{R}, \sigma)$ be an irrational winding of $\mathcal{T}^2$ with the frequency $\{1, \sqrt{2}\} \in \mathbb{R}^2$, i.e., $\sigma(t, y) := (y_1 + t(mod\ 2\pi), y_2 + \sqrt{2}t(mod\ 2\pi))$ for any $(t, y) \in \mathbb{R} \times Y$.

Let $\varepsilon \in \mathbb{R}$ be a sufficiently small parameter. Consider the heat equation on the interval [0,1] with Dirichlet boundary condition:
$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial \xi^2} + \varepsilon(\cos(y_1 + t) + \sin(y_2 + \sqrt{2}t))u + \frac{1}{3}(\sin(y_2 + t) + \cos(y_2 + \sqrt{2}t)) \sin u \tag{49}$$

($y = (y_1, y_2) \in Y = \mathcal{T}^2$) on the interval [0,1] with Dirichlet boundary condition $u(t,0) = u(t,1) = 0, \quad t > 0$.

Let A_0 be a linear operator defined by $A_0\varphi(x) = \varphi''(x)$ $(0 < x < 1)$, then $A_0 : D(A_0) = H^2(0,1) \cap H_0^1(0,1) \to L^2(0,1)$ (for more details see [9, Ch.I]). Denote $H := L^2(0,1)$ and the norm on H by $|\cdot|$ and $A(y)\varphi = (\cos y_1 + \sin y_2)\varphi$ for any $y \in Y$ and $\varphi \in H$. Then the equation (49) can be written as an abstract evolution equation

$$x'(t) = (A_0 + \varepsilon A(\sigma(t,y)))x(t) + F(\sigma(t,y), x(t))$$

on the Hilbert space H, where

$$x(t) := u(t,\cdot), \quad F(y,x) := f(y,x) \ \text{ and } \ f(y,x) := \frac{1}{3}(\sin y_1 + \cos y_2)\sin x.$$

Note that $\sigma(A_0) = \{-n^2\pi^2 |\ n \in \mathbb{N}\}$ and A_0 generates a $\mathcal{C}_0$-semigroup $\{U(t)\}_{t\geq 0} = \{e^{A_0 t}\}_{t\geq 0}$ on H. It is easy to see that the semigroup $\{U(t)\}_{t\geq 0}$ is exponentially stable and consequently, for sufficiently small ε the linear equation

$$x'(t) = (A_0 + \varepsilon A(\sigma(t,y))x \quad (y \in Y)$$

is uniformly exponentially stable (see, for example, [8]). Note that $Lip(F) \leq 1$, so it is immediate to verify that conditions of Theorem 9 hold. Finally note that every point $y \in Y = \mathcal{T}^2$ is quasi-periodic in $(\mathcal{T}^2, \mathbb{R}, \sigma)$. By Theorem 9 equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial \xi^2} + \varepsilon(\cos t + \sin \sqrt{2}t)u + \frac{1}{3}\big(\sin t + \cos\sqrt{2}t\big)\sin u \quad (y_1 = 0,\ y_2 = 0)$$

for sufficiently small ε has a unique quasi-periodic solution ψ_ε with the frequency basis $\{1, \sqrt{2}\}$.

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