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Completeness of the factor group of a complete topological Abelian group by a discrete subgroup

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Abstract. The factor group of a Hausdorff topological Abelian group by a discrete subgroup is complete iff the topological group is complete and the factor ring of a Hausdorff topological ring by a discrete ideal is complete iff the topological ring is complete.

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1 Introduction

The question of preserving the completeness of topological Abelian groups and the completeness of topological rings under various constructions is one of the areas of research in topological algebra (see, for example, [1]).

This article is a continuation of the research that was carried out in works [1 and 2]. It is proven that the factor group of a complete topological Abelian group by a discrete subgroup and the factor ring of a complete topological ring by a discrete ideal are complete.

2 Notations and preliminaries

To present the main results we remind the following well-known results:

Definition 2.1. As usual, a *directed set* is called a partially ordered set $(\Gamma, <)$ if for any two elements γ_1 and $\gamma_2 \in \Gamma$ there exists an element γ_3 such that $\gamma_1 < \gamma_3$ and $\gamma_2 < \gamma_3$.

Definition 2.2. If $(\Gamma, <)$ is a directed set then, as usual, a sequence $\{g_\gamma | \gamma \in \Gamma\}$ of elements of a topological Abelian group $(G(+), \tau)$ is called *a Cauchy sequence by this directed set* if for any neighborhood V of the zero of the topological group (G, τ) there exists an element $\gamma_0 \in \Gamma$ such that $g_\gamma - g_{\gamma_0} \in V$ for any element $\gamma \in \Gamma$ such that $\gamma > \gamma_0$.

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Remark 2.1. If $(\Gamma, <)$ is some directed set, then for any Cauchy sequence $\{g_\gamma | \gamma \in \Gamma\}$ of elements of a topological group (G, τ) by this directed set and any neighborhood V of the zero of the topological group (G, τ) there exists an element $\gamma_0 \in \Gamma$ such that $g_{\gamma_1} - g_{\gamma_2} \in V$ for any elements γ_1 and $\gamma_2 \in \Gamma$ such that $\gamma_1 > \gamma_0$ and $\gamma_2 > \gamma_0$.

Indeed, for the neighborhood V of the zero of the topological group (G, τ) there exists a neighborhood V_1 of the zero of the topological group (G, τ) such that $V_1 - V_1 \subseteq V$. As $\{g_\gamma | \gamma \in \Gamma\}$ is a Cauchy sequence then for the neighborhood V_1 of the zero of the topological group (G, τ) there exists an element $\gamma_0 \in \Gamma$ such that $g_\gamma - g_{\gamma_0} \in V_1$ for any element $\gamma \in \Gamma$ such that $\gamma > \gamma_0$. Then $g_{\gamma_1} - g_{\gamma_2} = (g_{\gamma_1} - g_{\gamma_0}) + (g_{\gamma_0} - g_{\gamma_2}) \in V_1 - V_1 \subseteq V$.

Definition 2.3. If $(\Gamma, <)$ is a directed set, then an element g of a topological Abelian group (G, τ) is called *the limit* of the Cauchy sequence by this directed set if for any neighborhood V of zero in the topological group (G, τ) there exists an element $\gamma_0 \in \Gamma$ such that $g - g_\gamma \in V$ for any element $\gamma \in \Gamma$ such that $\gamma > \gamma_0$.

In this case we will write $g = \lim_{\gamma \in \Gamma} g_\gamma$.

Definition 2.4. A topological group (G, τ) is called *a complete group* if for any directed set $(\Gamma, <)$, any Cauchy sequence by this directed set has a limit.

Proposition 2.1 (see [1], Proposition 3.2.22). *Let (G, τ) be a Hausdorff topological Abilian group and let N be a subgroup of the group G. If topological groups $(N, \tau|_N)$ and $(G, \tau)/N$ are complete, then the topological group (G, τ) is a complete group too.*

3 Basic results

Theorem 3.1. *If M is a subgroup of an Abelian group G and τ is a group topology on G such that M is a discrete subgroup of the topological group (G, τ), then the factor group $(G, \tau)/M$ is a complete group iff topological group (G, τ) is a complete topological group.*

Necessity. If the factor group $(G, \tau)/M$ is a complete group and because any discrete group is a complete topological group, then from Proposition 2.1 it follows that the topological group (G, τ) is a complete topological group and hence necessity is proved.

Sufficiency. Let now the topological group (G, τ) be a complete topological group. If $f : G \to G/M$ is a canonical homomorphism, then for any neighborhood V of zero of the topological group (G, τ) the set $f(V)$ is a neighborhood of zero in the topological group $(G, \tau)/M$

If $(\Gamma, <)$ is some directed set and if $\{\bar{g}_\gamma | \gamma \in \Gamma\}$ is a Cauchy sequence by this directed set in the topological group $(G, \tau)/M$, then for each element $\gamma \in \Gamma$ we choose an element $g_\gamma \in G$ such that $f(g_\gamma) = \bar{g}_\gamma$.

Because $(M, \tau|_M)$ is a discrete subgroup of the topological group (G, τ), then there exists a neighborhood V_0 of zero of the topological group (G, τ) such that $M \bigcap V_0 = \{0\}$ and there exists a neighborhood V_1 of zero in the topological group (G, τ) such that $V_1 + V_1 - V_1 \subseteq V_0$ and because $\{\bar{g} | \gamma \in \Gamma\}$ is a Cauchy sequence of the topological group $(G, \tau)/M$, then there exists an element $\gamma_0 \in \Gamma$ such that $\bar{g}_\gamma - \bar{g}_{\gamma_0} \in f(V_1)$ for any element $\gamma \in \Gamma$ such that $\gamma > \gamma_0$. Then $g_\gamma - g_{\gamma_0} \in V_1 + M$ for any element $\gamma \in \Gamma$ such that $\gamma > \gamma_0$, and hence for any element $\gamma \in \Gamma$ such that $\gamma > \gamma_0$ there exists an element $m_\gamma \in M$ such that $g_\gamma - g_{\gamma_0} \in V_1 + m_\gamma$.

For each element $\gamma \in \Gamma$ consider the element $g'_\gamma = g_\gamma - m_\gamma$ if $\gamma > \gamma_0$ and $g'_\gamma = g_\gamma$ otherwise. Then

$$g'_\gamma - g'_{\gamma_0} = (g_\gamma - m_\gamma) - g_{\gamma_0} = (g_\gamma - g_{\gamma_0}) - m_\gamma \in (V_1 + m_\gamma) - m_\gamma = V_1$$

for any element $\gamma \in \Gamma$ such that $\gamma > \gamma_0$.

Let us show that $\{g'_\gamma | \gamma \in \Gamma\}$ is a Cauchy sequence by a directed set $(\Gamma, <)$ in the topological group (G, τ).

In fact, if V is any neighborhood of zero of the topological group (G, τ), then (see Remark 2.1) for the neighborhood $f(V \bigcap V_1)$ of zero of the topological group $(G, \tau)/M$ there exists an element $\gamma_1 \in \Gamma$ such that $f(g'_{\gamma_2}) - f(g'_{\gamma_3}) \in f(V \bigcap V_1)$ for any elements γ_2 and $\gamma_3 \in \Gamma$ such that $\gamma_2 > \gamma_1$ and $\gamma_3 > \gamma_1$.

As the set $(\Gamma, <)$ is a directed set, then there exists an element γ_4 such that $\gamma_4 > \gamma_0$ and $\gamma_4 > \gamma_1$.

If now $\gamma \in \Gamma$ and $\gamma > \gamma_4$, then $\gamma > \gamma_0$ and $\gamma > \gamma_1$. Then (see definition of element γ_0) $g'_\gamma - g'_{\gamma_0} \in V_1$. In addition, (see the definition of the element γ_1) $f(g'_\gamma) - f(g'_{\gamma_4}) \in f(V \bigcap V_1)$, and hence $g'_\gamma - g'_{\gamma_4} \in (V \bigcap V_1) + M$. Then there exists an element $m'_{\gamma_4} \in M$ such that $g'_\gamma - g'_{\gamma_4} \in (V \bigcap V_1) + m'_{\gamma_4}$, and hence

$$-m'_{\gamma_4} \in \left((g'_{\gamma_4} - g'_\gamma) + (V \bigcap V_1)\right) \bigcap M =$$

$$\left((g'_{\gamma_4} - g'_{\gamma_0} + g'_{\gamma_0} - g'_\gamma) + (V \bigcap V_1)\right) \bigcap M \subseteq$$

$$(V_1 - V_1 + V_1) \bigcap M \subseteq V_0 \bigcap M = \{0\}.$$

Then $m'_{\gamma_4} = 0$, i.e. $g'_\gamma - g'_{\gamma_4} \in V \bigcap V_1$.

From the arbitrariness of the neighborhood V of the zero of the topological group (G, τ) and the arbitrariness of the element $\gamma \in \Gamma$ it follows that $\{g'_\gamma | \gamma \in \Gamma\}$ is a Cauchy sequence by a directed set $(\Gamma, <)$ in the topological group (G, τ).

Because the topological group (G, τ) is a complete group then there exists an element $g \in G$ such that $g = \lim_{\gamma \in \Gamma} g'_\gamma$. Then $f(g) = \lim_{\gamma \in \Gamma} f(g'_\gamma) = \lim_{\gamma \in \Gamma} \bar{g}_\gamma$.

The arbitrariness of the sequence $\{\bar{g}_\gamma | \gamma \in \Gamma\}$ implies the completeness of the topological group $(G, \tau)/M$.

Remark 3.1. Since the completeness of a topological ring is determined by the completeness of its additive group, then follows:

Corollary 3.1. *If M is an ideal of a ring R and τ is a ring topology on R such that M is a discrete ideal of the topological ring (R, τ), then the factor ring $(R, \tau)/M$ is a complete ring iff topological ring (R, τ) is a complete topological ring.* .

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