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An optimality criterion for disjoint bilinear programming and its application to the problem with an acute-angled polytope for a disjoint subset

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Abstract. We formulate and prove an optimality criterion for the disjoint bilinear programming problem and show how it can be efficiently used for solving the problem when one of the disjoint subsets has the structure of an acute-angled polytope. A class of integer and combinatorial problems that can be reduced to the disjoint bilinear programming problem with an acute-angled polytope is presented and it is shown how the considered optimality criterion can be applied.

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1 Introduction and problem formulation

The main results of this article are concerned with studying and solving the following well-known disjoint bilinear programming problem (see [1,2,5,9,16–18,20]):

Minimize

$$z = x^T C y + g x + e y \tag{1}$$

subject to

$$Ax \le a, \quad x \ge 0; \tag{2}$$

$$By \le b, \quad y \ge 0, \tag{3}$$

where

$$C = (c_{ij})_{n\times m}, \quad A = (a_{ij})_{q\times n}, \quad B = (b_{ij})_{l\times m},$$

$$a^T = (a_{10}, a_{20}, \dots, a_{q0}) \in R^q, \quad b^T = (b_{10}, b_{20}, \dots, b_{l0}) \in R^l,$$

$$g = (g_1, g_2, \dots g_n) \in R^n, \quad e = (e_1, e_2, \dots, e_m) \in R^m,$$

$$x^T = (x_1, x_2, \dots, x_n) \in R^n, \quad y^T = (y_1, y_2, \dots, y_m) \in R^m.$$

Throughout the article we will assume that the solution sets X and Y of the corresponding systems (2) and (3) are nonempty and bounded. Our aim is to formulate and prove an optimality criterion for problem (1)-(3) that can be efficiently used for

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studying and solving the problem in the case when the set of solutions Y has the structure of an acute-angled polytope.

Disjoint bilinear programming model (1)-(3) comprises a large class of integer and combinatorial optimization problems, including the well-known NP-complete problem of the existence of a boolean solution for a given system of linear inequalities

$$\begin{cases} \sum_{i=1}^{n} a_{ij}x_j \le a_{i0}, & i = 1,2,\dots,q; \\ x_j \ge 0, & j = 1,2,\dots,n. \end{cases} \tag{4}$$

It is easy to show that this problem can be represented as the following disjoint bilinear programming problem:
Minimize

$$z = \sum_{j=1}^{n} (x_j y_j + (1-x_j)(1-y_j)) \tag{5}$$

subject to

$$\begin{cases} \sum_{j=1}^{n} a_{ij}x_j \le a_{i0}, & i = 1,2,\dots,q, \\ 0 \le x_j \le 1, & j = 1,2,\dots,n, \end{cases} \tag{6}$$

$$0 \le y_j \le 1, \qquad j = 1,2,\dots,n, \tag{7}$$

The relationship between this bilinear programming problem and the problem of determining a boolean solution of system (4) is the following: system (4) has a boolean solution $x^* = (x_1^*, x_2^*, \dots, x_n^*)$ if and only if $x^* = (x_1^*, x_2^*, \dots, x_n^*)$ and $y^* = (y_1^*, y_2^*, \dots, y_q^*)$ with $y_j^* = 1 - x_j^*,\ j = 1,2,\dots,n$, represent an optimal solution of the disjoint bilinear programming problem (5)-(7) where $z(x^*, y^*) = 0$.

To disjoint bilinear programming problem (1)-(3) also the following classical boolean linear programming problem can be reduced:
Minimize

$$z = \sum_{j=1}^{n} c_j x_j \tag{8}$$

subject to

$$\begin{cases} \sum_{i=1}^{n} a_{ij}x_j \le a_{i0}, & i = 1,2,\dots,q; \\ x_j \in \{0,1\}, & j = 1,2,\dots,n. \end{cases} \tag{9}$$

This problem has an optimal boolean solution $x^* = (x_1^*, x_2^*, \dots, x_n^*)$ if and only if system (4) has a feasible boolean solution. If coefficients a_{ij} and c_j of problem (8),4) are integer, then $z^* = z(x^*) \in [-nH, nH]$, where $H = \max\{|c_i|, i = 1,2,\dots,n\}$. Therefore optimal solution of boolean linear programming problem (8),(4) can be

found by solving a sequence of disjoint bilinear programming problems
Minimize

$$z = \sum_{j=1}^{n}(x_j y_j + (1 - x_j)(1 - y_j)$$

subject to

$$\begin{cases} \sum_{j=1}^{n} c_j x_j \le t_k, & \\ \sum_{i=1}^{n} a_{ij} x_j \le a_{i0}, & i = 1, 2, \dots, q, \\ 0 \le x_j \le 1, & j = 1, 2, \dots, n, \\ 0 \le y_j \le 1, & j = 1, 2, \dots, n, \end{cases}$$

with integer parameters t_k on $[-nH, nH]$ by applying the bisection method with a standard integer roundoff procedure for t_k.

Boolean programming problem (8), (9) can be also formulated as the following disjoint bilinear programming problem:
Minimize

$$\overline{z} = \sum_{j=1}^{n} c_j x_j + M \sum_{j=1}^{n} (x_j y_j + (1 - x_j)(1 - y_j)) \tag{10}$$

subject to

$$\begin{cases} \sum_{j=1}^{n} a_{ij} x_j \le a_{i0}, & i = 1, 2, \dots, q, \\ 0 \le x_j \le 1, & j = 1, 2, \dots, n, \end{cases} \tag{11}$$

$$0 \le y_j \le 1, \qquad j = 1, 2, \dots, n, \tag{12}$$

where M is a suitable large value. More precisely, if the coefficients in (8), (9) are integer, then M should satisfy the condition $M \ge n \cdot H \cdot 2^{2L+1}$, where L is the length of the binary encoding of the coefficients of boolean problem (8), (4). The relationship between boolean linear programming problem (8), (9) and disjoint bilinear programming problem (10)-(12) is the following: boolean linear programming problem (8), (9) has an optimal solution $x^* = (x_1^*, x_2^*, \dots, x_n*)$ if and only if $x^* = (x_1^*, x_2^*, \dots, x_n^*)$ and $y^* = (y_1^*, y_2^*, \dots, y_n^*)$, where $y_i^* = 1 - x_i^*$, $i = 1, 2, \dots, n$, represent a solution of disjoint bilinear programming problem (10)-(12) and $\overline{z}(x^*, y^*) = \sum_{j=1}^{n} c_j x^*$. So, the boolean programming problem can be reduced in polynomial time to disjoint programming problem (1)-(3) where the matrix B is an identity one.

Another important problem which can be reduced to disjoint bilinear programming problem (1)-(3) is the following piecewise linear concave programming problem:
Minimize

$$z = \sum_{j=1}^{l} \min\{c^{jk} x + c_0^{jk}, \qquad k = 1, 2, \dots, m_j\} \tag{13}$$

subject to (2), where $x \in R^n$, $c^{jk} \in R^n$, $c_0^{jk} \in R^1$.

This problem arises as an auxiliary one when solving a class of resource allocation problems [12, 16]. In [16] it is shown that this problem can be replaced by the following disjoint bilinear programming problem:
Minimize

$$z = \sum_{j=1}^{l} \sum_{k=1}^{m_j} (c^{jk} x + c_0^{jk}) y_{jk}$$

subject to

$$Ax \leq a, \quad x \geq 0;$$

$$\begin{cases} \sum\limits_{k=1}^{m_j} y_{jk} = 1, & j = 1, 2, \ldots, l; \\ \quad y_{jk} \geq 0, & k = 1, 2, \ldots, m_j, \quad j = 1, 2, \ldots, l. \end{cases}$$

In this problem we can eliminate $y_{1,m_1}, y_{2,m_2}, \ldots, y_{l,m_l}$, taking into account that $y_{jm_j} = 1 - \sum\limits_{k=1}^{m_j-1} y_{jk}$, $j = 1, 2, \ldots l$, and we obtain the following disjoint bilinear programming problem:
Minimize

$$z = \sum_{j=1}^{l} \sum_{k=1}^{m_j-1} (c^{jk} - c^{jm_j}) x y_{jk} + \sum_{j=1}^{l} c^{jm_j} x +$$

$$+ \sum_{j=1}^{l} \sum_{k=1}^{m_j-1} (c_0^{jk} - c_0^{jm_j}) y_{jk} + \sum_{j=1}^{l} c_0^{jm_j} \tag{14}$$

subject to

$$Ax \leq a, \quad x \geq 0; \tag{15}$$

$$\begin{cases} \sum\limits_{k=1}^{m_j-1} y_{jk} \leq 1, & j = 1, 2, \ldots, l; \\ \quad y_{jk} \geq 0, & k = 1, 2, \ldots, l, \quad k = 1, 2, \ldots, m_j - 1, \end{cases} \tag{16}$$

i.e. we obtain a special case of disjoint bilinear programming problem (1)-(3) where the corresponding matrix B is step-diagonal.

Disjoint bilinear programming problem (1)-(3) has been extensively studied in [1, 2, 5, 9, 16–18, 20] and some general methods and algorithms have been developed. In this article we shall use a new optimization criterion that takes into account the structure of the disjoint subsets X and Y.

It can be observed that in the presented above examples of disjoint programming problems the matrix B is either identity one or step diagonal. This means that the set of solutions Y has the structure of an acute-angled polyhedron [3, 8]. Acute-angled polyhedra are polyhedra in which all dihedral angles are acute or right. A detailed characterization of such polyhedra has been made by Coxeter [8] and Andreev [3, 4]. Moreover, in [3, 4] it has been proven that in an acute-angled polyhedron the hyperplanes of nonadjacent facets cannot intersect. Based on this property in the present article we show that for a disjoint bilinear programming problem with the structure of an acute-angled polytope for a disjoint subset an optimality criterion can be formulated that can be efficiently used for studying and solving the problem. In fact we show that the formulated optimality criterion is valid not only for the case when Y is an acute-angle polytope but it holds also for a more general class of polyhedra which in this article are called *perfect polyhedra*; we call the corresponding disjoint bilinear programming problems with such a disjoint subset *disjoint bilinear programming problems with a perfect disjoint subset.* The aim of this paper is to show that for this class of problems an optimality criterion can be formulated that takes into account the mentioned structure of the disjoint subset Y and that can be efficiently used for solving the problem.

The article is organized as follows. In Section 2 we present the formulation of the disjoint bilinear programming problem with a prefect disjoint subset that generalizes the problem with a disjoint subset having the structure of an acute-angled polytope. In Sections 3-5 we present some basic properties of the disjoint bilinear programming problem (1)-(3) in general. In Section 7 we present new necessary conditions related to redundant inequalities for the system of linear inequalities. The main results of the article are presented in Section 8 where we formulate and prove the optimality criterion for the disjoint bilinear programming problem with a perfect disjoint subset.

2 Disjoint bilinear programming with a perfect disjoint subset

Let us consider the following disjoint bilinear programming problem:

Minimize

$$z = x^T Cy + gx + ey \tag{17}$$

subject to

$$Ax \leq a, \quad x \geq 0; \tag{18}$$

$$Dy \leq d, \tag{19}$$

This problem differs from problem (1)-(3) only by system (19) in which $D=(d_{ij})_{p\times m}$ and $d^T = (d_{10}, d_{20} \dots, d_{m0}) \in R^m$; in this problem A, C, a, g, e are the same as in problem (1)-(3). We call this problem *a disjoint bilinear programming problem with perfect disjoint subset* Y if system (19) has a full rank equal to m, where $m < p$, and this system possesses the following properties:

a) *system (19) does not contain redundant inequalities and its solution set* Y *is a bounded set with nonempty interior;*

b) *the set of solutions* Y' *of an arbitrary subsystem* $D'y \leq d'$ *of rank* m *with* m *inequalities of system (19) represents a convex cone with the origin at an extreme point* y' *of the set of solutions* Y *of system (19), i.e.* y' *is the solution of the system of equations* $D'y = d'$*;*

c) *at each extreme point* y' *of the set of solutions* Y *of system (19) exactly* m *hyperplanes of the facets of the polytope* Y *intersect*;

Note that if $Y = \{y \big| \ Dy \leq d\}$ is a nonempty set that has the structure of a nondegenerate acute-angled polytope, then it satisfies conditions a)-c) above.

3 Disjoint bilinear programming and the min-max linear problem with interdependent variables

Disjoint bilinear programming problem (1)-(3) is tightly connected with the following min-max linear programming problem with interdependent variables:

To find

$$z^* = \min_{y\in Y} \max_{u\in U(y)} (ey - a^T u) \tag{20}$$

and

$$y^* \in Y = \{y| \ By \leq b, \ y \geq 0\} \tag{21}$$

for which

$$z^* = \max_{u\in U(y^*)} (ey^* - a^T u), \tag{22}$$

where

$$U(y) = \{u \in R^q| \ -A^T u \leq Cy + g^T, \quad u \geq 0\}. \tag{23}$$

The relationship between the solutions of problem (1)-(3) and min-max problem (20)-(23) can be obtained on the basis of the following theorems.

Theorem 1. *If* (x^*, y^*) *is an optimal solution of problem (1)-(3) and* z^* *is the optimal value of the objective function in this problem, then* z^* *and* $y^* \in Y$ *represent a solution of min-max problem (20)-(23) and vice versa: if* z^* *and* $y^* \in Y$ *represent a solution of min-max problem (20)-(23), then* z^* *is the optimal value of the objective function of problem (1)-(3) and* y^* *corresponds to an optimal point in this problem.*

Proof. The proof of the theorem is obtained from the following reduction procedure of bilinear programming problem (1)-(3) to min-max problem (20)-(23). We represent disjoint bilinear programming problem (1)-(3) as the problem of determining

$$\psi_1(y^*) = \min_{y\in Y} \psi_1(y), \tag{24}$$

where

$$\begin{cases} \psi_1(y) = \min\limits_{x\in X}(x^T Cy + gx + ey), \\ \qquad\qquad X = \{x \in R^n| \ Ax \leq a, \ x \geq 0\}. \end{cases} \tag{25}$$

If we replace linear programming problem (25) with respect to x by the dual problem

$$\psi_1(y) = \max_{u\in U(y)} (ey - a^T u)$$

and after that we introduce this expression in (24), then we obtain min-max problem (20)-(23). □

Similarly we can prove the following result.

Theorem 2. *If $(x^*, y^*) \in X \times Y$ is an optimal solution of disjoint bilinear programming problem (1)-(3) and z^* is equal to the minimal value of the objective function of this problem, then z^* and x^* correspond to a solution of the following min-max linear problem:*

To find

$$z^* = \min_{x \in X} \max_{v \in V(x)} (gx - b^T v) \tag{26}$$

and

$$x^* \in X = \{x | \ Ax \le a, \ x \ge 0\} \tag{27}$$

for which

$$z^* = \max_{v \in V(x^*)} (cx^* - b^T v), \tag{28}$$

where

$$V(x) = \{v \in R^l | \ -B^T v \le C^T x + e^T, \quad v \ge 0\}. \tag{29}$$

Corollary 1. *If $(x^*, y^*) \in X \times Y$ is an optimal solution of the following disjoint bilinear programming problem:*
Minimize

$$z = x^T C y + gx + ey$$

subject to

$$Ax \le a, \quad x \ge 0, \quad Dy \le d,$$

and h^* is equal to the minimal value of the objective function of this problem, then z^* and x^* correspond to a solution of the following min-max linear problem:
To find

$$z^* = \min_{x \in X} \max_{v \in V(x)} (gx - d^T v)$$

and

$$x^* \in X = \{x | \ Ax \le a, \ x \ge 0\}$$

for which

$$z^* = \max_{v \in V(x^*)} (gx^* - d^T v),$$

where

$$V(x) = \{v \in R^l | \ -D^T v = C^T x + e^T, \quad v \ge 0\}.$$

Problems (20)-(23) and (26)-(29) can be regarded as a couple of dual min-max linear problems with interdependent variables. It is easy to see that these min-max problems always have solutions if X and Y are nonempty and bounded sets.

4 Estimation of the optimal value of the objective function

Let L be the length of the input data of disjoint bilinear programming problem (1) - (3) with integer coefficients of matrices C, A, B and vectors a, b, c', c'' [11,12], i.e.

$$L = L_1 + L_2 + (qm + q + m)(1 + \log(H + 1)),$$

where

$$L_1 = \sum_{i=0}^{q}\sum_{j=1}^{n} \log(|a_{ij}| + 1) + \log n(q + 1),$$

$$L_2 = \sum_{i=1}^{l}\sum_{j=0}^{m} \log(|b_{ij}| + 1) + \log l(m + 1),$$

$$H = \max\{|c_{ij}|, |g_i|, |e_j|,\ i = \overline{1, n},\ j = \overline{1, m}\}.$$

Then the following lemma holds.

Lemma 1. *If disjoint bilinear programming problem (1)-(3) with integer coefficients has optimal solutions, then the optimal value z' of the objective function (1) is a rational number that can be expressed by an irreducible fraction $\frac{M}{N}$ with integer M and N ($|N| \geq 1$), where $|M|$ and $|N|$ do not exceed 2^L.*

Proof. If the optimal value of the objective function of problem (1)-(3) exists then this value is attained at an extreme point (x', y') of the polyhedral set $X \times Y$ determined by (2), (3) where $x' \in X$ and $y' \in Y$ (see [1, 2, 5, 9]). Then according to Lemma 1 from [13] (see also [14]) each component x'_i of x' is a rational value and it can be expressed by a fraction of form $x' = \frac{M_i^1}{N_0^1}$ with integer M_i^1 and N_0^1, where M_i^1 is a determinant of the extended matrix of system (2), N_0^1 is a nonzero determinant of matrix A and $|M_i^1|, |N_0^1| \leq \frac{2^{L_1}}{n(q+1)}$; similarly each component y'_j of y' is a rational value and it can be expressed by a fraction of form $y' = \frac{M_j^2}{N_0^2}$ with integer M_j^2 and N_0^2, where M_j is a determinant of the extended matrix of system (3), N_0^2 is a nonzero determinant of matrix B and $|M_j^2|, |N_0^2| \leq \frac{2^{L_2}}{l(m+1)}$. Therefore

$$z' = \frac{1}{N_0^1 N_0^2}\Big(\sum_{i=1}^{q}\sum_{j=1}^{m} c_{ij} M_i^1 M_j^2 + \sum_{i=1}^{q} g_i M_i^1 N_0^2 + \sum_{j=1}^{m} e_j M_j^2 N_0^1\Big)$$

where

$$\left|N_0^1 N_0^2\right| = \left|N_0^1\right|\left|N_0^2\right| \leq \frac{2^{L_1+L_2}}{n(q+1)l(m+1)}$$

and

$$\Big|\sum_{i=1}^{q}\sum_{j=1}^{m} c_{ij}M_i^1M_j^2 + \sum_{i=1}^{q} g_iM_i^1N_0^2 + \sum_{j=1}^{m} e_jM_j^2N_0^1\Big| \leq$$

$$H\Big|\sum_{i=1}^{q}\sum_{j=1}^{m} M_i^1M_j^2 + \sum_{i=1}^{q} M_i^1N_0^2 + \sum_{j=1}^{m} M_j^2N_0^1\Big| \leq$$

$$2^{\log(H+1)}\Big(2^{L_1+L_2} + 2^{L_1+L_2} + 2^{L_1+L_2}\Big) \leq 2^L.$$

So, the optimal value z' of the objective function (1) is a rational number that can be represented by a fraction $\frac{M}{N}$ with integer M and N ($|N| \geq 1$), where $|M|$ and $|N|$ do not exceed 2^L. □

5 An optimality criterion for disjoint bilinear programming (1)-(3)

Let us assume that the optimal value of the objective function of disjoint bilinear programming problem (1)-(3) is bounded. Then problem (1)-(3) can be solved by varying the parameter $h \in [-2^L, 2^L]$ in the problem of determining the consistency (compatibility) of the system

$$\begin{cases} Ax \leq a; \\ x^TCy + gx + ey \leq h; \\ By \leq b; \\ x \geq 0, \ y \geq 0. \end{cases} \tag{30}$$

In order to study the consistency problem for system (30) we will reduce it to a consistency problem for a system of linear inequalities with a right-hand member depending on parameters using the following results.

Lemma 2. *Let solution sets X and Y of the corresponding systems (2) and (3) be nonempty. Then system (30) for a given $h \in R^1$ has no solutions if and only if the system of linear inequalities*

$$\begin{cases} -A^Tu \leq Cy + g^T; \\ a^Tu < ey - h; \\ u \geq 0 \end{cases} \tag{31}$$

is consistent with respect to u for every y satisfying (3).

Proof. System (30) has no solutions if and only if for every $y \in Y$ the system of linear inequalities

$$\begin{cases} Ax \leq a, \\ x^T(Cy + g^T) \leq h - ey, \\ x \geq 0 \end{cases} \tag{32}$$

has no solutions with respect to x. If we apply the duality principle (Theorem 2.14 from [7]) for system (32) with respect to vector of variables x then we obtain that it is inconsistent if and only if the system of linear inequalities

$$\begin{cases} A^T\lambda + (Cy + g^T)t \geq 0; \\ a^T\lambda + (h - ey)t < 0; \\ \lambda \geq 0,\ t \geq 0, \end{cases} \tag{33}$$

has solutions with respect to λ and t for every $y \in Y$. Note that for an arbitrary solution (λ^*, t^*) of system (33) the condition $t^* > 0$ holds. Indeed, if $t^* = 0$, then it means that the system

$$\begin{cases} A^T\lambda \geq 0; \\ a^T\lambda < 0,\ \lambda \geq 0, \end{cases}$$

has solutions, which, according to Theorem 2.14 from [7], involves the inconsistency of system (2) that is contrary to the initial assumption. Consequently, $t^* > 0$. Since $t > 0$ in (33) for every $y \in Y$, then, dividing each inequality of this system by t and denoting $u = (1/t)\lambda$, we obtain system (30). So, system (30) is inconsistent if and only if system (31) is consistent with respect to u for every y satisfying (3). □

Corollary 2. *Let solution sets X and Y of the corresponding systems (2) and (3) be nonempty. Then for a given h system (30) has solutions if and only if there exists $y \in Y$ for which system (31) is inconsistent with respect to u. The minimal value h^* of parameter h for which such a property holds is equal to the optimal value of the objective function in disjoint bilinear programming problem (1)-(3).*

Theorem 3. *Let solution sets X and Y of the corresponding systems (2) and (3) be nonempty and bounded. Then for a given $h \in R^1$ the system*

$$\begin{cases} Ax \leq a; \\ x^TCy + gx + ey < h; \\ By \leq b; \\ x \geq 0,\ y \geq 0. \end{cases} \tag{34}$$

is inconsistent if and only if the system of linear inequalities

$$\begin{cases} -A^Tu \leq Cy + g^T; \\ a^Tu \leq ey - h; \\ u \geq 0 \end{cases} \tag{35}$$

is consistent with respect to u for every y satisfying (3). The maximal value h^* of the parameter h for which system (35) is consistent with respect to u for every $y \in Y$ is equal to the minimal value of the objective function of disjoint bilinear programming problem (1)-(3). Moreover, for the considered systems the following properties hold:

1) If the system of linear inequalities

$$\begin{cases} -A^Tu - Cy \ \leq\ g^T; \\ a^Tu - \ ey \ < -h^*; \\ u \geq 0 \end{cases} \tag{36}$$

is inconsistent with respect to u and y, then the system

$$\begin{cases} Ax \le a; \\ C^T x = -e^T; \\ cx = h^*; \\ x \ge 0 \end{cases} \tag{37}$$

is consistent and an arbitrary solution to it x^* *together with an arbitrary* $y \in Y$ *determine a solution* (x^*, y) *of disjoint bilinear programming problem (1)-(3).*

2) If system (36) is consistent with respect to u and y, then there exists $y^* \in Y$ *for which the system*

$$\begin{cases} -A^T u \le Cy^* + g^T; \\ a^T u < ey^* - h^*; \\ u \ge 0 \end{cases} \tag{38}$$

has no solutions with respect to u. An arbitrary $y^* \in Y$ *with such a property together with a solution* x^* *of the system of linear inequalities*

$$\begin{cases} Ax \le a, \\ x^T(Cy^* + g^T) \le h^* - ey^*, \\ x \ge 0 \end{cases} \tag{39}$$

with respect to x, represent an optimal solution (x^*, y^*) *for disjoint bilinear programming problem (1)-(3).*

Proof. System (34) is inconsistent if and only if the system

$$\begin{cases} Ax \le a, \\ x^T(Cy + g^T) < h - ey, \\ x \ge 0 \end{cases} \tag{40}$$

is inconsistent with respect to x for every $y \in Y$. Taking into account that the set of solutions of system (2) is nonempty and bounded we can replace (40) by the following homogeneous system

$$\begin{cases} Ax - at \le 0, \\ x^T(Cy + g^T) - (ey - h)t < 0, \\ x \ge 0 \end{cases} \tag{41}$$

preserving the inconsistency property with respect to x and t for every $y \in Y$. Therefore system (34) is inconsistent if and only if system (41) is inconsistent for every $y \in Y$. Applying the duality principle for system (41) we obtain that it is inconsistent for every $y \in Y$ if and only if system (35) is consistent with respect to y for every y satisfying (3). Based on this property and Corollary 2 we may conclude that the maximal value h^* of parameter h in system (35) for which it has solutions with respect to u for every $y \in Y$ is equal to the minimal value of the objective function in problem (1)-(3).

Now let us prove property 1) from the theorem. Assume that system (36) is inconsistent with respect to u and y. Then the system

$$\begin{cases} -A^T u - Cy \ - g^T t \leq 0; \\ a^T u - ey + h^* t < 0; \\ u \geq 0, \qquad t \geq 0 \end{cases}$$

is inconsistent. This involves that system (37) has solutions. For an arbitrary solution $x = x^*$ of system (37) we have

$$x^{*T} Cy + gx^* + ey = y^T(C^T x^* + e^T) + gx^* = gx^* = h^*,$$

where h^* is the minimal value of the objective function in problem (1)-(3). This means that x^* together with an arbitrary $y \in Y$ determine an optimal solution (x^*, y) of problem (1)-(3). Moreover, in this case the optimal value h^* of the objective function of the problem does not depend on the constraints (3) that define Y, i.e. Y may be an arbitrary subset from R^m.

Finally let us prove property 2) of the theorem. Assume that system (36) is consistent and there exists $y^* \in Y$ for which system (38) is inconsistent. Then according to Corollary 2 we obtain that for $h = h^*$ system (30) has solutions. In this case the corresponding homogeneous system

$$\begin{cases} -A^T u - (Cy^* + g^T)t \ \leq 0; \\ au - (ey^* - h^*)t < 0; \\ u \geq 0, \qquad t \geq 0 \end{cases}$$

has no solutions and based on the duality principle we obtain that system (39) has solutions with respect to x. This means that $y^* \in Y$ together with a solution (x^*) of system (39) determine an optimal solution (x^*, y^*) of problem (1)-(3). □

Corollary 3. *The linear programming problem:*
Maximize

$$z = h \tag{42}$$

subject to

$$\begin{cases} -A^T u - Cy \leq g^T; \\ au - ey \leq -h; \\ u \geq 0, \end{cases} \tag{43}$$

has solutions if and only if the linear programming problem :
Minimize

$$z' = cx \tag{44}$$

subject to

$$\begin{cases} Ax \leq a; \\ C^T x = -e^T; \\ x \geq 0, \end{cases} \tag{45}$$

has solutions. If h^* is the optimal value of the objective function of linear programming problem (42), (43), then an optimal solution x^* of linear programming problem (44), (45) together with an arbitrary $y \in Y$ is an optimal solution of the disjoint bilinear programming problem with an arbitrary subset $Y \in R^m$.

Proof. If we dualize (42), (43), then we obtain the problem:
Minimize:

$$z' = c'x$$

subject to

$$\begin{cases} -Ax + \ at \geq 0; \\ -C^T x - e^T t \geq 0; \\ \qquad\qquad t = 1; \\ x \geq 0, \end{cases}$$

i.e. this problem is equivalent to problem (44), (45). So, problem (42), (43) has solutions if and only if problem (44), (45) has solutions. □

If in Lemma 2 we take into account that the set of solutions X of system (2) is nonempty and bounded, then we obtain the following result.

Corollary 4. *Linear programming problem (42), (43) has solutions if and only if system (45) is consistent. If system (45) is inconsistent, then the objective function (42) is unbounded on the set of feasible solutions (43).*

Remark 1. Let $U_h(y)$ be the set of solutions of system (31) with respect to u for fixed $h \in R^1$ and $y \in R^m$. Additionally, let $\overline{U}_h(y)$ be the set of solutions of system (35) with respect to u for fixed $h \in R^1$ and $y \in R^m$ and denote

$$Y_h = \{y \in R^m | U_h(y) \neq \emptyset\}; \quad \overline{Y}_h = \{y \in R^m | \overline{U}_h(y) \neq \emptyset\}.$$

Then Y_h is an open set and $\overline{U}_h(y)$ is a closed set. In terms of these sets we can formulate the results above as follows:

1. System (30) has solutions if and only if $Y \not\subset Y_h$ and the minimal value h^* of h for which this property holds is equal to the optimal value of the objective function of problem (1)-(3).

2. System (34) has no solutions if and only if $Y \subseteq \overline{Y}_h$ and the maximal value h^* of h for which this property holds is equal to the optimal value of the objective function of problem (1)-(3);

3. If system (36) is consistent, then $Y \subset \overline{Y}_h$ for $h < h^*$ and $Y \not\subset Y_h$ for $h \geq h^*$, i.e, $\overline{Y}_{h^*} \setminus Y_{h^*}$ represents the set of optimal points $y^* \in Y$ for problem (1)-(3);

4. If system (36) is inconsistent, then $Y_{h^*} = \emptyset$ and an arbitrary solution x^* of system (37) together with an arbitrary $y \in Y$ represent an optimal solution of problem (1)-(3), i.e $Y_h = \emptyset$ for $h \geq h^*$ and $Y \subset \overline{Y}_h$ for $h < h^*$.

Thus, based on Theorem 3, we can replace disjoint bilinear programming problem (1)-(3) with the problem of determining the optimal value h^* of h and vector $y^* \in Y$ for which system (35) is consistent with respect to u. System (35) can be regarded as a parametric system with right-hand members that depend on the vector of parameters $y \in Y$ and $h \in R^1$. If in (30) we regard x as a vector of parameters, then we can prove a variant of Theorem 3 in which parametrical system (35) is replaced by the parametrical system

$$\begin{cases} -B^T v \leq C^T x + e^T; \\ bv \leq gx - h; \\ v \geq 0, \end{cases} \tag{46}$$

where v is the vector of variables and x is the vector of parameters that satisfies (2). This means that for the considered parametric linear systems (35) and (46) we can formulate the following duality principle (see [15]):

Theorem 4. *The system of linear inequalities (35) is consistent with respect to u for every y satisfying (3) if and only if the system of linear inequalities (46) is consistent with respect to v for every x satisfying (2).*

All results formulated and proved in this section are valid also for the case when Y is determined by an arbitrary consistent system

$$Dy \leq d, \tag{47}$$

where $D = (d_{ij})$ is a $p \times m$-matrix and d is a column vector with p components. In the case when system (3) is replaced by system (47), the following duality principle holds.

Theorem 5. *The system of linear inequalities (35) is consistent with respect to u for every y satisfying (47) if and only if the system*

$$\begin{cases} -D^T v = C^T x + e^T; \\ dv \leq gx - h; \\ v \geq 0, \end{cases} \tag{48}$$

is consistent with respect to v for every x satisfying (2).

6 The general scheme of the approach for solving problem (1)-(3)

The approach we shall use for solving the disjoint bilinear programming problem (1)-(3) is based on Theorem 2 and is as follows:

We replace problem (1)-(3) by problem of determining the maximal value h^* of parameter h such that system (35) is consistent with respect to u for every y satisfying (3). Then we show how to determine the corresponding points x^*, y^* that satisfy the conditions of Theorem 2. To apply this approach we will develop algorithms for checking conditions 1) and 2) of the theorem, i.e. we will develop algorithms for checking the condition $Y \not\subset Y_h$ for a given $h \in [-2^L, 2^L]$. Based on such an approach and the bisection method we can determine h^* and y^*. To do this in Section 8 we state Theorem 3 for the case when Y represents a perfect polytope and formulate the condition $Y \not\subset Y_h$ in terms of the existence of a basic feasible solution for a linear system with a fixed basic component.

7 Some auxiliary results related to redundant inequalities for linear systems

In that follows we shall use some properties of redundant inequalities for linear systems. An inequality

$$\sum_{j=1}^{m} s_j y_j \le s_0 \tag{49}$$

is called *redundant* for a consistent system of linear inequalities

$$\sum_{j=1}^{m} d_{ij} y_j \le d_{i0}, \qquad i = 1, 2, \ldots, p, \tag{50}$$

if (49) holds for an arbitrary solution of system (50). We call the redundant inequality (49) *non-degenerate* if $s_j \ne 0$ at least for an index $j \in \{1, 2, \ldots, m\}$. If $s_j = 0,\ j = 1, 2, \ldots, m$, and $s_0 \ge 0$ we say that the redundant inequality (49) is *degenerate*. We call the redundant inequality (49) for consistent system (50) *strongly redundant* if there exists $\epsilon > 0$ such that the corresponding inequality

$$\sum_{j=1}^{m} s_j y_j \le s_0 - \epsilon$$

is redundant for (50); if such an ϵ does not exist, then we call inequality (49) *weakly redundant* . If an inequality

$$\sum_{j=1}^{m} d_{kj} y_j \le d_{k0} \tag{51}$$

of system (50) can be omitted without changing the set of its feasible solutions, then we say that it is redundant in (50), i.e., inequality (51) is redundant in (50) if it is redundant for the system of the rest of its inequalities.

The redundancy property of linear inequality (49) for consistent system (50) can be checked based on the following well-known result [7, 10]:

Theorem 6. *Inequality (49) is redundant for consistent system (50) if and only if the system*

$$\begin{cases} s_j = \sum\limits_{i=1}^{p} d_{ij} v_i, \qquad j = 1, 2, \ldots, m; \\ s_0 = \sum\limits_{i=1}^{p} d_{i0} v_i + v_0; \\ \qquad\qquad v_i \ge 0,\ \ i = 0, 1, 2, \ldots, p, \end{cases} \tag{52}$$

has solutions with respect to $v_0, v_1, v_2, \ldots, v_p$*. Moreover, if inequality (49) is redundant for system (50), then system (52) has a basic feasible solution* $v_0^*, v_1^*, v_2^*, \ldots, v_p^*$ *that satisfies the following conditions:*

1) the set of column vectors

$$\left\{ D_i = \begin{pmatrix} d_{i1} \\ d_{i2} \\ \vdots \\ d_{im} \end{pmatrix} : \ v_i^* > 0, \ i \in \{1, 2, \ldots, p\} \right\}$$

is linearly independent;

2) inequality (49) is redundant for the subsystem of system (49) induced by the inequalities that correspond to indices $i \in \{1, 2, \ldots, p\}$ *with* $v_i^* > 0$.

3) if $v_0^* > 0$*, then inequality (49) is strongly redundant for system (50) and if* $v_0^* = 0$*, then inequality (49) is weakly redundant for system (50).*

The proof of this theorem can be found in [7, 10].

Corollary 5. *Let redundant inequality (49) for consistent system (50) be given and consider the following linear programming problem:*
Minimize

$$z = \sum_{i=1}^{p} d_{i0} v_i \tag{53}$$

subject to

$$\begin{cases} s_j = \sum\limits_{i=1}^{p} d_{ij} v_i, \qquad j = 1, 2, \ldots, m; \\ \qquad\qquad v_i \geq 0, \quad i = 1, 2, \ldots, p. \end{cases} \tag{54}$$

Then this problem has an optimal solution $v_1^*, v_2^*, \ldots, v_p^*$ *where* $s_0 \geq \sum\limits_{i=1}^{p} d_{i0} v_i^*$*. If* $s_0 > \sum\limits_{i=1}^{p} d_{i0} v_i^*$ *then inequality (49) is strongly redundant for system (50) and if* $s_0 = \sum\limits_{i=1}^{p} d_{i0} v_i^*$ *then inequality (49) is weakly redundant for system (50).*

Theorem 6 can be extended for the case when system (50) is inconsistent.

Definition 1. Assume that system (50) is inconsistent. Inequality (49) is called redundant for inconsistent system (50) if there exists a consistent subsystem

$$\sum_{j=1}^{m} d_{i_k j} y_j \leq d_{i_k 0}, \ k = 1, 2, \ldots, p'(p' < p), \tag{55}$$

of system (50) such that inequality (49) is redundant for (55).

Theorem 7. *Inequality (49) is redundant for inconsistent system (50) if and only if system (52) has a basic feasible solution* $v_0, v_1, v_2, \ldots, v_p$ *for which the set of column vectors*

$$D^+ = \left\{ \begin{pmatrix} d_{i1} \\ d_{i2} \\ \vdots \\ d_{im} \end{pmatrix} : \ v_i > 0, \ i \in \{1, 2, \ldots, p\} \right\} \tag{56}$$

is linearly independent. Moreover, the subsystem of inconsistent system (50) induced by inequalities that correspond to indices with $v_i^0 > 0$ *is a consistent subsystem of system (50).*

Proof. $\Rightarrow$ Assume that inequality (49) is redundant for inconsistent system (50). Then there exists a consistent subsystem (55) of system (50) such that (49) is redundant for (55). Then according to Theorem 6 there exists a basic feasible solution $v_0, v_{i_1}, v_{i_2}, \ldots, v_{i_{p'}}$ for the system

$$\begin{cases} s_j = \sum_{k=1}^{p'} d_{i_k j} v_{i_k}, & j = 1, 2, \ldots, m; \\ s_0 = \sum_{k=1}^{p'} d_{i_k 0} v_{i_k} + v_0; & \\ \qquad v_0 \geq 0, \ v_{i_k} \geq 0, & k = 1, 2, \ldots, p', \end{cases}$$

such that the set of column vectors

$$\left\{ d_{i_k} = \begin{pmatrix} d_{i_k 1} \\ d_{i_k 2} \\ \vdots \\ d_{i_k m} \end{pmatrix} : \ v_{i_k} > 0, \ k \in \{1, 2, \ldots, p'\} \right\}$$

is linearly independent.

$\Leftarrow$ Let (50) be an arbitrary inconsistent system and $v_0, v_1, v_2, \ldots, v_p$ be a solution of system (52) that contains $p' \geq 1$ nonzero components $v_{i_1}, v_{i_2}, \ldots, v_{i_{p'}}$ such that the corresponding system of column vectors $\{d_{i_k} : v_{i_k} > 0, k = 1, 2, \ldots, p'\}$ is linearly independent. Then $p' \leq \min\{m, p\}$ and the corresponding system

$$\sum_{j=1}^{m} d_{i_k j} y_j \leq d_{i_k 0}, \qquad k = 1, 2, \ldots, p'$$

has solutions. Based on Theorem 6 we obtain that the inequality (49) is redundant for system (55) . This means that the inequality (49) is redundant for inconsistent system (50). □

8 The optimality criterion for disjoint bilinear programming problem with a perfect disjoint subset

In this section we present a refinement of Theorem 3 for the disjoint bilinear programming problem (17)-(19) with conditions $a) - c)$ for system (19). In fact this refinement of Theorem 3 is related to the case when $Y = \{By \le b,\ y \ge 0\}$ is replaced by $Y = \{y|Dy \le d\}$ that satisfies conditions $a) - c)$. Based on this, we show that the optimality criterion for problem (17)-(19) with conditions $a) - c)$ can be formulated in terms of the existence of a basic solution with the given basic component for a given system of linear equations.

We present the optimality criterion in new terms for problem (17)-(19) in the following extended form:
Minimize

$$z = \sum_{i=1}^{n}\sum_{j=1}^{m} c_{ij}x_iy_j + \sum_{i=1}^{n} g_ix_i + \sum_{i=1}^{n} e_jy_j \tag{57}$$

subject to

$$\begin{cases} \sum_{j=1}^{n} a_{ij}x_j \le a_{i0}, & i = 1,2,\dots,q; \\ x_j \ge 0, & j = 1,2,\dots,n; \end{cases} \tag{58}$$

$$\sum_{j=1}^{m} d_{ij}y_j \le d_{i0}, \quad i = 1,2,\dots,p \ \ (m < p), \tag{59}$$

We will assume that the set of solutions Y of system (59) in this problem satisfies the following conditions:

a) *system (59) does not contain redundant inequalities and the set of its solutions Y is a bounded set with nonempty interior;*

b) *the set of solutions of an arbitrary subsystem*

$$\sum_{j=1}^{m} d_{i_kj}y_j \le d_{i_k0}, \qquad k = 1,2,\dots,m,$$

of rank m represents a convex cone $Y^-(y^r)$ with the origin at an extreme point y^r from the set of extreme points $\{y^1, y^2, \dots, y^N\}$ of the set of solutions Y of system (59);

c) *at each extreme point $y^r \in \{y^1, y^2, \dots, y^N\}$ of the set of solutions Y of system (59) exactly m hyperplanes of the facets of of polytope Y intersect.*

It is easy to see that if $D = \begin{pmatrix} B \\ -I \end{pmatrix}$, $d = \begin{pmatrix} b \\ 0 \end{pmatrix}$, I is the identity matrix and 0 is the column vector with zero components, then problem (57)-(59) becomes problem

(1)-(3). Additionally if matrix B is an identity one or step-diagonal, then we obtain a disjoint bilinear programming problems for which conditions $a) - c)$ hold.

The main results we describe in this section are concerned with elaboration of an algorithm that determines if the property $Y \not\subset Y_h$ holds.

8.1 The properties of feasible solutions of problem (57)-(59)

Let $y^r = (y_1^r, y_2^r, \dots, y_m^r)$, $r = 1, 2, \dots N$, be the extreme points of the set of solutions Y of system (59) that satisfies conditions $a) - c)$. Then for each $y^r, r \in \{1, 2, \dots, N\}$, there exists a subsystem

$$\sum_{j=1}^{m} d_{i_k j} y_j \le d_{i_k 0}, \qquad k = 1, 2, \dots, m, \tag{60}$$

of rank m of system (59) such that $y_1^r, y_2^r, \dots, y_m^r$ is the solution of the system of linear equations

$$\sum_{j=1}^{m} d_{i_k j} y_j = d_{i_k 0}, \qquad k = 1, 2, \dots, m. \tag{61}$$

Denote

$$I(y^r) = \{i \in \{1, 2, \dots, p\} : \sum_{j=1}^{m} d_{ij} y_j^r = d_{i0}\}$$

and consider the *convex cone* $Y^-(y^r)$ *for system (59)* as the solution set of the following system

$$\sum_{j=1}^{m} d_{ij} y_j \le d_{i0}, \quad i \in I(y^r), \tag{62}$$

where $y^r = (y_1^r, y_2^r, \dots, y_m^r)$ is the *origin of the cone* $Y^-(y^r)$. We call the solution set of the system

$$\sum_{j=1}^{m} d_{ij} y_j \ge d_{i0}, \quad i \in I(y^r)$$

the symmetrical cone for $Y^-(y^r)$ and denote it by $Y^+(y^r)$.

Obviously, $Y^-(y^r)$, $Y^+(y^r)$ represent convex polyhedral sets with interior points such that $Y = \cap_{r=1}^{N} Y^-(y^r)$; $Y^+(y^r) \cap Y^-(y^r) = y^r$, $r = 1, 2, \dots, N$.

Lemma 3. *If* $Y^+(y^1)$, $Y^+(y^2)$, $\dots$, $Y^+(y^N)$ *represent the symmetrical cones for the corresponding cones* $Y^-(y^1)$, $Y^-(y^2)$, $\dots$, $Y^-(y^N)$ *of system (59) with properties* $a) - c)$, *then* $Y^+(y^r) \bigcap Y^+(y^k) = \emptyset$ *for* $r \ne k$. *Additionally if* z^1, z^2, $\dots$, z^N *represent arbitrary points of the corresponding cones* $Y^+(y^1)$, $Y^+(y^2), \dots, Y^+(y^N)$, *then the convex hull* $Conv(z^1, z^2, \dots, x^N)$ *of points* z^1, z^2, $\dots, z^N$ *contains* Y.

Proof. The property $Y^+(y^r) \cap Y^+(y^k) = \emptyset$ for $r \neq k$ can be proven by contradiction. Indeed, if we assume that $Y^+(y^r) \cap Y^+(y^k) \neq \emptyset$, then this polyhedral set contains an extreme point y^0, where $y^0 \notin Y$, because $Y^+(y^r) \cap Y^+(y^k)$ is determined by the system of inequalities consisting of inequalities that define $Y^+(y^r)$ and inequalities that define $Y^+(y^k)$. This means that $y^0 = (y_1^0, y_2^0, \dots, y_m^0)$ is the solution for the system of equations

$$\sum_{j=1}^{m} d_{i_k j} y_j = d_{i_k 0}, \qquad k \in \{1, 2, \dots, m\},$$

of rank m. Then according to properties $a) - c)$ we obtain that y^0 is a solution of system (59) which is in contradiction with the fact that $y^0 \notin Y$. So, $Y^+(y^r) \cap Y^+(y^k) = \emptyset$ for $r \neq k$.

Now let us show that if $z^1, z^2, \dots, z^N$ represent arbitrary points of the corresponding sets $Y^+(y^1), Y^+(y^2), \dots, Y^+(y^N)$, then the convex hull $Conv(z^1, z^1, z^2, \dots, z^N)$ of points $z^1, z^1, z^2, \dots, z^N$ contains Y. Indeed, if we construct the convex hull $Y^1 = Conv(z^1, y^2, \dots, y^N)$ of points $z^1, y^2, \dots, y^N$, then $y^1 \in Y^1$ and $Y \subseteq Y^1$. If after that we construct the convex hull $Y^2 = Conv(z^1, z^2, y^3 \dots, y^N)$ of points $(z^1, y^2, \dots, y^N)$, then $y^2 \in Y^2$ and $Y^1 \subseteq Y^2$ and so on. Finally at step N we construct the convex hull $Y^N = Conv(z^1, z^2, \dots, z^N)$ of points $z^1, z^2, \dots, z^N$ where $y^N \in Y^N$ and $Y^{N-1} \subseteq Y^N$, i.e. $Y \subseteq Y^1 \subseteq Y^2 \subseteq \dots \subseteq Y^N = Conv\big(z^1, z^2, \dots, z^N\big)$. □

Corollary 6. *If system (59) satisfies conditions $a) - c)$, then the system*

$$\sum_{j=1}^{m} d_{ij} y_j \geq d_{i0}, \qquad i = 1, 2, \dots, p,$$

is inconsistent and the inequalities of this system can be divided into N disjoint consistent subsystems

$$\sum_{j=1}^{m} d_{ij} y_j \geq d_{i0}, \qquad i \in I(y^r), r = 1, 2, \dots, N,$$

such that $Y^+(y^r) \cap Y^+(y^k) = \emptyset$.

8.2 Criteria to check $Y \not\subset Y_h$

We formulate some criteria for checking the condition $Y \not\subset Y_h$ that late we shall use for our basic result.

If for problem (57)-(59) we consider system (35) (see Theorem 3)

$$\begin{cases} -\displaystyle\sum_{i=1}^{q} a_{ij} u_i - \sum_{j=1}^{m} c_{ij} y_j \leq g_i, \qquad j = 1, 2, \dots, n; \\ \displaystyle\sum_{i=1}^{q} a_{i0} u_i - \sum_{j=1}^{m} e_j y_j \leq -h; \\ \qquad u_i \geq 0, \qquad i = 1, 2, \dots, q, \end{cases} \tag{63}$$

then either this system has solutions with respect to $u_1, u_2, \ldots, u_q, y_1, y_2, \ldots, y_m$ for an arbitrary $h \in R^1$ or there exists a minimal value h^* of h for which this system has a solution. According to Corollary 3 if for system (63) there exists a minimal value h^* for which it is consistent, then the optimal solution of problem (57)-(59) can be found by solving linear programming problem (44),(45). Therefore in what follows we will analyze the case when system (63) has solutions for every $h \in R^1$, i. e. the case when system (45) has no solutions.

Lemma 4. *Let disjoint bilinear programming problem (57)-(59) be such that system (59) satisfies conditions a) − c) and the set of solutions X of system (58) is nonempty and bounded. If system (45) has no solutions, then for a given h the property $Y \not\subset Y_h$ holds if and only if for system (63) there exists a non-degenerate redundant inequality*

$$\sum_{j=1}^{m} s_j y_j \leq s_0 \tag{64}$$

such that the corresponding symmetrical inequality

$$-\sum_{j=1}^{m} s_j y_j \leq -s_0 \tag{65}$$

is redundant for the inconsistent system

$$-\sum_{j=1}^{m} d_{ij} y_j \leq -d_{i0}, \qquad i = 1, 2, \ldots, p. \tag{66}$$

Proof. ⇒ Assume that system (45) has no solutions. Then according to Corollary 4 of Theorem 3 we have $Y_h \neq \emptyset$. Therefore if $Y \not\subset Y_h$, then among the extreme points $y^1, y^2, \ldots, y^N$ of Y there exists at least one that does not belong to Y_h. Denote by $y^1, y^2, \ldots, y^{N'}$ the extreme points of Y that do not belong to Y_h and by $y^{N'+1}, y^{N'+2}, \ldots, y^N$ the extreme points of Y that belong to Y_h. Each extreme point $y^l = (y_1^l, y_2^l, \ldots, y_m^l)$, $l \in \{1, 2, \ldots, N\}$, of Y represents the vertex of the cone $Y^-(y^l)$ that is determined by the solution set of system (62). At the same time each extreme point y^l of Y is the vertex of the symmetrical cone $Y^+(y^l)$ that is determined by the solution set of the subsystem of linear inequalities

$$-\sum_{j=1}^{m} d_{ij} y_j \leq -d_{i0}, \qquad i \in I(y^l) \tag{67}$$

of inconsistent system (66). According to Lemma 3 we have $Y^+(y^l) \cap Y^+(y^k) = \emptyset$ for $l \neq k$.

Let us show that among the extreme points $y^1, y^2, \ldots, y^{N'}$ there exists a point y^{j_0} for which the corresponding cone $Y^+(y^{j_0})$ has no common elements with Y_h, i.e. $Y^+(y^{j_0}) \bigcap Y_h = \emptyset$. This fact can be proved using the rule of contraries. If we assume that $Y^+(y^r) \bigcap Y_h \neq \emptyset$, $l = 1, 2, \ldots, N'$, then we can select from each set $Y^+(y^r)\bigcap Y_h$ an element z^r and construct the convex hull $Conv(z^1,\ z^2,\ \ldots,\ z^{N'}, y^{N'+1},\ y^{N'+2},\ \ldots,\ y^N)$ for the set of the points $\{z^1,\ z^2,\ \ldots, z^{N'},\ y^{N'+1},\ y^{N'+2},\ \ldots,\ y^N\}$. Taking into account that $z^r \in Y_h,\ r = 1,\ 2,\ \ldots,\ N'$ and $y^{N'+l} \in Y_h,\ r = 1, 2, \ldots, N - N'$, we have $Conv(z^1, z^2, \ldots, z^{N'}, y^{N'+1}, y^{N'+2}, \ldots, y^N) \subseteq Y_h$. However according to Lemma 3 we have $Y \subseteq Conv(z^1, z^2, \ldots, z^{N'}, y^{N'+1}, y^{N'+2}, \ldots, y^N)$, i.e. we obtain $Y \subseteq Y_h$. This is in contradiction with the condition $y^r \notin Y_h$ for $r = 1, 2, \ldots, N'$.

Thus, among $Y^+(y^1), Y^+(y^2), \ldots, Y^+(y^{N'})$ there exists a cone $Y^+(y^{j_0})$ with vertex y^{j_0} for which $Y^+(y^{j_0})\bigcap Y_h = \emptyset$. Therefore for convex sets $Y^+(y^{j_0})$ and Y_h there exists a separating hyperplane [6,19]

$$\sum_{j=1}^{m} s_j y_j = s_0$$

such that

$$\sum_{j=1}^{n} s_j y_j < s_0, \quad \forall (y_1, y_2, \ldots, y_n) \in Y_h$$

and

$$-\sum_{j=1}^{m} s_j y_j \le -s_0, \quad \forall (y_1, y_2, \ldots, y_n) \in\ Y^+(y^{j_0}).$$

So, the inequality $\sum\limits_{j=1}^{n} s_j y_j \le s_0$ is redundant for system (63) and the inequality $-\sum\limits_{j=1}^{n} s_j y_j \le -s_0$ is redundant for consistent subsystem (67) of inconsistent system (66).

$\Leftarrow$ Assume that for system (63) there exists a non-degenerate redundant inequality (64) such that the corresponding inequality (65) is redundant for inconsistent system (66). Then there exists a consistent subsystem (67) of system (66) for which the conical subset $Y^-(y^{i_0})$ has no common points with Y_h where $Y_h \neq \emptyset$, i.e. $y^{j_0} \notin Y_h$. Taking into account that $y^{j_0} \in Y$ we obtain $Y \not\subset Y_h$. □

Corollary 7. *Assume that the conditions of Lemma 4 are satisfied. Then the minimal value z^* of objective function of problem (57)-(59) is equal to the minimal value h^* of parameter h in system (63) for which there exists a non-degenerate redundant inequality*

$$\sum_{j=1}^{m} s_j^* y_j \le s_0^* \tag{68}$$

such that the corresponding symmetrical inequality

$$-\sum_{j=1}^{m} s_j^* y_j \le -s_0^* \tag{69}$$

is redundant for inconsistent system (66). An optimal point y^* for problem (57)-(59) can be found by solving the following system

$$\begin{cases} \sum_{j=1}^{m} d_{ij}y_j \le d_{i0}, \qquad i=1,2,\dots,p\ ; \\ \sum_{j=1}^{m} s_j^* y_j = s_0^*. \end{cases} \tag{70}$$

Proof. Assume that for system (63) with given h there exists a non-degenerate redundant inequality (64) such that symmetrical inequality (65) is redundant for system (66). Then according to Corollary 2 of Lemma 2 and Theorem 3 for problem (57)-(59) there exists a feasible solution such that the corresponding value of the objective function is not greater than h. Therefore the minimal value z^* of the objective function of problem (57)-(59) is equal to minimal value h^* of parameter h in system (63) for which there exists a non-degenerated redundant inequality (68) such that the corresponding symmetrical inequality (69) is redundant for inconsistent system (66). In this case $y^{j_0} = y^* \in Y \cap bd(\overline{Y}_{h^*})$ and the optimal point $y^* \in Y$ can be found by solving system (70). □

Theorem 8. *Let disjoint bilinear programming problem (57)-(59) be such that system (59) satisfies conditions a) − c) and the set of solutions X of system (58) is nonempty and bounded. If system (45) has no solutions, then for a given h the property $Y \not\subset Y_h$ holds if and only if the following system*

$$\begin{cases} \sum_{j=1}^{n} a_{ij}x_j + x_{n+i} = a_{i0}, & i=1,2,\dots,q; \\ \sum_{i=1}^{n} c_{ij}x_i + \sum_{k=1}^{p} d_{kj}v_k = -e_j, & j=1,2,\dots,m; \\ \sum_{i=1}^{n} g_i x_i \ - \sum_{k=1}^{p} d_{k0}v_k + v_{p+1} = h; & \\ \quad x_i \ge 0,\ i=0,1,2,\dots,n+q;\ v_k \ge 0,\ k=1,2,\dots,p+1 & \end{cases} \tag{71}$$

has a basic feasible solution $x_1^0, x_2^0, \dots, x_{n+q}^0, v_1^0, v_2^0, \dots, v_{p+1}^0$ for which the set of vectors

$$\left\{ D_k = \begin{pmatrix} d_{k1} \\ d_{k2} \\ \vdots \\ d_{km} \end{pmatrix} :\ v_k^0 > 0, \quad k \in \{1,2,\dots,p\} \right\} \tag{72}$$

is linearly independent. The minimal value h^* of parameter h for which $Y \not\subset Y_h$ is equal to the of the optimal value of the objective function of problem (57)-(59).

Proof. According to Lemma 4 the condition $Y \not\subset Y_h$ holds if and only if there exist $s_0, s_1, s_2, \dots, s_n$ such that (64) is non-degenerate redundant inequality for system

(63) and (65) is a redundant inequality for inconsistent system (66). If for (64) and 63) we apply the Minkowski-Farkas theorem (Theorem 6), then we obtain that (64) is redundant for (63) if and only if there exist $x_0, x_1, x_2, \ldots, x_{n+q}, t$ such that

$$\begin{cases} 0 = -\sum_{j=1}^{n} a_{ij}x_j - x_{n+i} + a_{i0}t, & i = 1, 2, \ldots, q; \\ s_j = -\sum_{i=1}^{n} c_{ij}x_i - e_j t, & j = 1, 2, \ldots, m; \\ s_0 = \sum_{i=1}^{n} g_i x_i - ht + x_0; & \\ \qquad x_j \geq 0, \ i = 0, 1, 2, \ldots, n+q; \ t \geq 0. \end{cases} \tag{73}$$

Now let us apply Theorem 7 for inequality (65) and inconsistent system (66). According to this theorem, inequality (65) is redundant for system (66) if and only if there exist $v_0, v_1, v_2, \ldots, v_m$ such that

$$\begin{cases} -s_j = -\sum_{k=1}^{p} d_{kj}v_k, & j = 1, 2, \ldots, m; \\ -s_0 = -\sum_{k=1}^{n} d_{k0}v_k + v_0; & \\ \qquad v_k \geq 0, \quad k = 0, 1, 2, \ldots, p \ , \end{cases} \tag{74}$$

where $\sum_{k=1}^{p} d_{kj}v_i \neq 0$ at least for an index $j \in \{1, 2, \ldots, m\}$ and the set of column vectors (72) is linearly independent.

So, Lemma 4 can be formulated in terms of solutions of systems (73), (74) as follows: the condition $Y \not\subset Y_h$ holds if and only if there exist s_0, s_1, $s_2, \ldots, s_m$, $x_0, x_1, x_2, \ldots, x_{n+q}$, $v_0, v_1, v_2, \ldots, v_{p+1}, t$ that satisfy (73), (74), where $\sum_{k=1}^{p} d_{kj}v_k \neq 0$ at least for an index $j \in \{1, 2, \ldots, m\}$ and the set of column vectors (72) is linearly independent. If we eliminate $s_1, s_2, \ldots, s_m$ from (73) by introducing (74) in (73) and after that denote $v_{p+1} = v_0 + x_0$, then we obtain the following system

$$\begin{cases} \sum_{j=1}^{n} a_{ij}x_j + x_{n+i} = a_{i0}t, & i = 1, 2, \ldots, q; \\ \sum_{i=1}^{n} c_{ij}x_i + \sum_{k=1}^{p} d_{kj}v_k = -e_j t, & j = 1, 2, \ldots, m; \\ \sum_{i=1}^{n} g_i x_i - \sum_{k=1}^{p} d_{k0}v_k + v_{p+1} = ht; & \\ \qquad x_j \geq 0, \ j = 1, 2, \ldots, n+q; \ v_k \geq 0, k = 1, 2, \ldots, p+1; \ t \geq 0. \end{cases} \tag{75}$$

This means that Lemma 4 in terms of solutions of system (75) can be formulated as follows: the property $Y \not\subset Y_h$ holds if and only if system (75) has a solution

$x_1,\ x_2,\ \dots, x_{n+q},\ v_1,\ v_2\ ,\dots,\ v_p,\ v_{p+1},\ t$ where $\sum_{k=1}^{p} d_{kj}v_i \neq 0$ at least for an index $j \in \{1,2,\dots,m\}$ and the set of column vectors (72) is linearly independent.

In (75) the subsystem

$$\begin{cases} \displaystyle\sum_{j=1}^{n} a_{ij}x_j + x_{n+i} = a_{i0}t, & i = 1,2,\dots,q; \\ \qquad x_j \geq 0,\ j = 1,2,\dots,n+q, t \geq 0 \end{cases} \tag{76}$$

has a nonzero solution $x_1, x_2,\ \dots,\ x_{n+q}, t$ if and only if $t > 0$, because the set of solutions of system (58) is nonempty and bounded. Therefore in (75) we can set $t = 1$ and finally obtain that $Y \not\subset Y_h$ if and only if system (71) has a solution $x_1^0,\ x_2^0,\ \dots,\ x_{n+q}^0,\ v_1^0,\ v_2^0,\ \dots,\ v_{p+1}^0$ for which the set of column vectors (72) is linearly independent and $\sum_{k=1}^{p} d_{kj}v_k^0 \neq 0$ at least for an index $j \in \{1,2,\dots,m\}$.

If h^* is the minimal value of parameter h for which system (71) has a basic feasible solution where the system of vectors (72) is linearly independent, then according to Corollary 7 of Lemma 4, h^* is equal to the optimal value of the objective function of problem (57)-(59). □

Corollary 8. *Let e disjoint bilinear programming problem (57)-(59) be such that system (59) satisfies conditions a) − c) and the set of solutions X of system (58) is nonempty and bounded. If system (45) has no solutions, then for a given h the property $Y \not\subset Y_h$ holds if and only if system (71) has a basic feasible solution $x_1^0, x_2^0, \dots, x_{n+q}^0, v_1^0, v_2^0, \dots, v_{p+1}^0$, where v_{p+1}^0 is a basic component. The minimal value h^* of parameter h with such a property is equal to the optimal value of objective function of problem (57)-(59).*

Theorem 9. *Let disjoint bilinear programming problem (57)-(59) be such that system (59) satisfies conditions a) − c) and the set of solutions X of system (2) is nonempty and bounded. Then the minimal value z^* of the objective function of problem (57)-(59) is equal to the minimal value h^* of parameter h in system (71) for which this system has a basic feasible solution $x_1^*,\ x_2^*,\ \dots,\ x_{n+q}^*, v_1^*,\ v_2^*,\ \dots, v_{p+1}^*$ where either the system of column vectors*

$$\left\{ D_k = \begin{pmatrix} d_{k1} \\ d_{k2} \\ \vdots \\ d_{km} \end{pmatrix} :\ v_k^* > 0, \quad k \in \{= 1,2,\dots,p\} \right\} \tag{77}$$

is linearly independent or this system of vectors is an empty set. If such a solution for system (71) with $h = h^*$ is known, then $x_1^*, x_2^*, \dots, x_q^*$ together with an arbitrary solution $y_1^*, y_2^*, \dots, y_m^*$ of the system

$$\begin{cases} \displaystyle\sum_{j=1}^{m} d_{kj}y_j \leq d_{k0}, & k = 1,2,\dots,p\ ; \\ \displaystyle\sum_{j=1}^{m}(\sum_{k=1}^{p} d_{kj}v_k^*)y_j = \sum_{k=1}^{p} d_{k0}v_k^* \end{cases} \tag{78}$$

represents an optimal solution $x_1^*,\ x_2^*,\ \dots,\ x_{n+q}^*,\ y_1^*,\ y_2^*,\ \dots,\ y_m^*$ *of disjoint programming problem (57)-(59). If* $\sum_{k=1}^{p} d_{kj}v_k^* = 0,\ \ j = 1,\ 2,\ \dots,\ m,$ *then* $v_1^* = 0,\ v_2^* = 0,\ \dots,\ v_{p+1}^* = 0$*; in this case* $x_1^*, x_2^*, \dots, x_q^*$ *together with an arbitrary solution* $y_1, y_2, \dots, y_m$ *of system (59) represents a solution of problem (57)-(59).*

Proof. Let h^* be the minimal value of parameter h for which system (71) has a basic feasible solution $x_1^*,\ x_2^*,\ \dots,\ x_{n+q}^*, v_1^*,\ v_2^*,\ \dots,\ v_{p+1}^*$ for which either the system of vectors (77) is linearly independent or this system of vectors is an empty set. Then according to Theorem 8 and Corollary 3 of Theorem 3 the optimal value of the objective function of problem (57)-(59) with properties $a) - c)$ is equal to h^* and $v_{p+1}^* = 0$.

Now let us prove the second part of the theorem. According to Corollary 1 of Theorem 2, for disjoint bilinear programming problem (57)-(59) we can consider the following min-max problem:

Find

$$h^* = \min_{(x_1,x_2,\dots,x_n)\in X} \ \max_{(v_1,v_2,\dots,v_p)\ \in V(x_1,x_2,\dots,x_n)} \left(\sum_{j=1}^{n} g_j x_j \ - \sum_{k=1}^{p} d_{k0} v_k\right)$$

and $\ (x_1^*,\ x_2^*,\ \dots,\ x_n^*) \in X,$

$$X = \left\{(x_1, x_2, \dots, x_n) \Big| \sum_{i=1}^{n} a_{ij} x_j \le a_{i0},\ i = 1, 2, \dots, q;\ \ x_i \ge 0,\ i = 1, 2, \dots, q\right\},$$

such that

$$h^* = \max_{(v_1,v_2\dots,v_p)\ \in V(x_1^*,x_2^*,\dots,x_n^*)} \left(\sum_{i=1}^{n} g_j x_j^* \ - \sum_{k=1}^{p} d_{k0} v_k\right),$$

where

$$V(x_1, x_2, \dots, x_n) = \left\{(v_1, v_2, \dots, v_p) \Big| - \sum_{k=1}^{p} d_{kj} v_k = \sum_{i=1}^{n} c_{ij} x_i + e_j, j = 1, 2, \dots m\right\}.$$

We can observe that $x_1^*,\ x_2^*,\ \dots,\ x_{n+q}^*, v_1^*,\ v_2^*,\ \dots,\ v_{p+1}^*$ is a solution of this min-max problem and $x^* = (x_1^*,\ x_2^*,\ \dots,\ x_{n+q}^*)$ is an optimal point for problem (57)-(59) with properties $a) - c)$. Taking into account that $\sum\limits_{k=1}^{p} d_{kj} v_k^* = -\sum\limits_{i=1}^{n} c_{ij} x_i^* - e_j,\ j = 1, 2, \dots, m,$ and $\sum\limits_{k=1}^{p} d_{k0} v_k^* = -h^* + \sum\limits_{i=1}^{p} g_i x_i^*$ we obtain that system (78) coincides with system (70), because $s_j^* = \sum\limits_{k=1}^{p} d_{kj} v_k^*$ and $s_0^* = \sum\limits_{k=1}^{p} d_{k0} v_k^*$. So, for the optimal point $x^* = (x_1^*,\ x_2^*,\ \dots,\ x_{n+q}^*)$ the corresponding optimal point $y^* = (y_1^*, y_2^*, \dots, y_m^*)$ for problem (57)-(59) can be found by solving system (78). If $\sum_{k=1}^{p} d_{kj} v_k^* = 0,\ \ j = 1,\ 2,\ \dots,\ m,$ then $v_1^* = 0,\ v_2^* = 0,\ \dots,\ v_{p+1}^* = 0$ and we obtain that $x_1^*, x_2^*, \dots, x_q^*$ together with an arbitrary solution $y_1, y_2, \dots, y_m$ of system (59) represents a solution of problem (57)-(59). □

Thus, the disjoint bilinear programming problem with the perfect disjoint subset Y (or with acute-angled polytope Y) can be solved if there exists an efficient algorithm for determining a basic feasible solution with the basic component v_{p+1} for system (71). For this problems we elaborated suitable algorithms.

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