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On the attractors of weakly hyperbolic IFS's with condensation

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Abstract. We show that for any weakly hyperbolic IFS with condensation in $\mathbb{R}^n$ whose condensation set is a union of a finite collection of convex compact sets, there exists a standard weakly hyperbolic IFS with the same attractor.

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Introduction

It is widely accepted that IFSs were conceived by John E. Hutchinson [19] in 1981. Very soon the topic was developed and popularized by Michael F. Barnsley [2] in his book "Fractals Everywhere" (1988). As it was proved in [19], every hyperbolic IFS, i.e., every finite collection of contracting mappings in a complete metric space, possesses a unique compact attractor. This set represents the fixed point of a contracting mapping that acts in the hyperspace whose "points" are non-empty compact subsets of the initial space.

The problem, whether a compact set can or cannot be represented as an IFS attractor was widely discussed in literature. M. Hata [18] has proven that if the attractor of some IFS is connected, then it is also locally connected. M. Kwiecinski [21] has purposed the construction of a locally connected continuum which cannot be the attractor of an IFS. Connected continua that cannot serve as IFS attractors have been constructed by M. J. Sanders in [27] (see also [1, 4, 6–10, 20]).

At the same time, there were many attempts to relax the condition of hyperbolicity of an IFS (see, e.g., [4, 11, 22, 24]).

M. Barnsley [2] has introduced the idea of an Iterated Function System with condensation (IFSC), which means a hyperbolic IFS, accompanied by a constant compact-valued multi-function (condensation). This idea has led to new fractals as attractors of IFS's. However, the computer simulations of such IFS's create more problems than for standard hyperbolic IFS's.

In [13] the attractor of a concrete hyperbolic IFS was presented as the attractor of a hyperbolic IFS with condensation whose condensation set is a segment. It was mentioned that the idea may be used also conversely, i.e., the construction of

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the attractor of such a hyperbolic IFS with condensation can be reduced to the construction of the attractor of an appropriate hyperbolic IFS.

For this reason the following two problems arise:

1. Under which conditions the attractor of a hyperbolic IFS with condensation can be represented as the attractor of a standard hyperbolic IFS?

2. Is it possible to reduce the construction of the attractor of a hyperbolic IFS with condensation to the construction of the attractor of an appropriate standard hyperbolic IFS?

The first question is motivated, in particular, by the fact that for hyperbolic IFS's there are more tools to study and to justify various simulations by computer (e.g., the so-called "Chaos game", see, e.g., [2–4, 23]) than for hyperbolic IFS's with condensation.

The second question is a continuation of the first one and is focused on finding a way to construct an appropriate hyperbolic IFS.

Moreover, the questions remain open for weakly hyperbolic IFS's as well.

We found a partial answer to the first question above for a certain type of compact sets in the Euclidean space $\mathbb{R}^n$ (see [17]). We proposed an answer for the case when the condensation set is a compact convex set or a finite union of such sets in $\mathbb{R}^n$. Following these results, an algorithm to construct some type of plane fractals was proposed in [15, 16] regarding the second question.

Here we purpose a generalization of these results to the case when the IFS consists of weakly (in some meaning) contracting mappings and a condensation whose image is a compact convex set or a finite union of such sets. More precisely, we purpose a construction which replaces the condensation mapping with a finite collection of contractions in such a way that the attractor of the initial weakly hyperbolic IFS with condensation coincides with the attractor of the new (standard) weakly hyperbolic IFS.

In the sequel, we will use the notion of weakly contracting mapping based on the notion of comparison function following [18, 26].

1 Preliminaries

Consider the Euclidean space $(\mathbb{R}^n, d)$.

Denote by $\mathcal{P}_{cp}(\mathbb{R}^n)$ the set of all nonempty compact subsets of $\mathbb{R}^n$, endowed with the Pompeiu-Hausdorff metric H,

$$H(A,B) = \max\left\{\max_{a\in A}\min_{b\in B} d(a,b), \max_{b\in B}\min_{a\in A} d(b,a)\right\}, \qquad A, B \subset \mathcal{P}_{cp}(\mathbb{R}^n).$$

A function $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ is called a *comparison function* (see [18, 26]) if:

1) φ is monotonically increasing, i.e., $t_1 < t_2$ implies $\varphi(t_1) \le \varphi(t_2)$;

2) the iterations $\varphi^n(t) \to 0$ as $n \to \infty$ for all $t \ge 0$.

It is known [26] that if φ is a comparison function, then $\varphi(t) < t$ for all $t > 0$.

Example 1. The function $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$, $\varphi(t) = \frac{t}{1+t}$, represents an example of comparison function.

Following [18, 26], we call a mapping $f : \mathbb{R}^n \to \mathbb{R}^n$ a *weak contraction* if there exists a comparison function φ such that

$$d(f(x), f(y)) \leq \varphi(d(x, y)), \qquad \forall\, x, y \in \mathbb{R}^n.$$

In this case one says also that f is a weak contraction with respect to the comparison function φ or that it is a φ-contraction.

Lemma 1. *For any finite collection of weak contractions $f_i : \mathbb{R}^n \to \mathbb{R}^n$ $(1 \leq i \leq m)$ there exists a common comparison function.*

Proof. Let φ_i be the comparison function with respect to which the mapping f_i is weakly contracting. Consider the function $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$, defined by $\varphi(t) = \max_i \varphi_i(t)$. Is it easily seen that φ is a comparison function for all the mappings f_i. □

A *(weakly) hyperbolic Iterated Function System (IFS or wIFS)* is a finite collection of (weak) contractions $f_i : \mathbb{R}^n \to \mathbb{R}^n$ $(1 \leq i \leq m)$, and it is denoted by $\{\mathbb{R}^n; f_1, \ldots, f_m\}$.

Associated with a (weakly) hyperbolic Iterated Function System $\mathcal{F} = \{\mathbb{R}^n; f_1, \ldots, f_m\}$ there is the mapping $F_* : \mathcal{P}_{cp}(\mathbb{R}^n) \to \mathcal{P}_{cp}(\mathbb{R}^n)$, defined by the equality $F_*(C) = \bigcup_{i=1}^{m} \bigcup_{x \in C} f_i(x)$, $C \in \mathcal{P}_{cp}(\mathbb{R}^n)$, and refereed to as the *Barnsley-Hutchinson operator* (called also the *Barnsley operator*).

Remark 1. A weakly hyperbolic Iterated Function System is a particular case of a weakly contracting relation (see [14]).

Remark 2. According to Lemma 1, any weakly hyperbolic Iterated Function System, as a weakly contracting relation, admits a comparison function (e.g., a common comparison function for all components of wIFS).

The next results follow from [14].

Theorem 1. *Let $\{X; f_1, \ldots, f_m\}$ be a weakly hyperbolic IFS with respect to a comparison function $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$. Then the corresponding Barnsley-Hutchinson operator F_* is also a φ-contraction, i.e., for any $A, B \in \mathcal{P}_{cp}(\mathbb{R}^n)$ we have*

$$H(F_*(A), F_*(B)) \leq \varphi(H(A, B)).$$

We call (see [14]) the nonempty compact set $A \subset \mathbb{R}^n$ *attractor* of the (weakly) hyperbolic IFS $\mathcal{F}$ if $F_*(A) = A$, where F_* is the corresponding Barnsley-Hutchinson operator.

The proof of the next statement follows the corresponding one regarding weakly contracting relations [14].

Theorem 2. *Any weakly hyperbolic IFS possesses an attractor, and this attractor is unique.*

2 Sum of convex compacta as attractor of hyperbolic IFS

M. Barnsley (1988) introduced the concept of Iterated Function System with condensation.

A constant compact-valued function $f_0 : \mathbb{R}^n \to \mathcal{P}_{cp}(\mathbb{R}^n)$, $f_0(x) \equiv K$ for some $K \in \mathcal{P}_{cp}(\mathbb{R}^n)$ and any $x \in \mathbb{R}^n$, is called a *condensation* with K as *condensation set.*

M. Barnsley [2] (see also [14]) has shown that any hyperbolic IFS with condensation has a unique attractor. Moreover, he stated a formula for this attractor as a generalization of Hutchinson's formula as a fixed point equation. Namely, let $\{X; f_1, \dots, f_m\}$ be a hyperbolic IFS with the attractor A, and let F_* stand for the corresponding Barnsley-Hutchinson operator. Let f_0 be a condensation with K as the corresponding value. The Barnsley formula for the attractor A_c of the hyperbolic IFS with condensation $\{X; f_0, f_1, \dots, f_m\}$ looks as

$$A_c = A \cup \big(\bigcup_{l \geq 0} F_*^l(K) \big), \tag{1}$$

where $F_*^0(K) = K$.

Remark 3. The formula (1) is also true for any weakly hyperbolic IFS $\{X; f_1, \dots, f_m\}$ which has the attractor A and is completed with a condensation f_0, with the set K as its value.

The following two theorems give us a partial answer to the questions, mentioned in the introduction.

Theorem 3. *[17] Any convex compact set in $\mathbb{R}^n$ can be represented as the attractor of a hyperbolic IFS, consisting only of contractions.*

Theorem 4. *[17] Given the finite family of convex compacta $\{K_1, \dots, K_p\}$ in $\mathbb{R}^n$, let $\mathcal{F}_i = \{\mathbb{R}^n; \xi_{i1}, \dots, \xi_{iq_i}\}$ $(1 \leq i \leq p)$ stand for the corresponding hyperbolic IFS that admits K_i as attractor.*

Then $K = \bigcup_{i=1}^p K_i$ is the attractor of the hyperbolic IFS

$$\mathcal{G} = \{\mathbb{R}^n; \psi_{11}, \dots, \psi_{1q_1}, \dots, \psi_{p1}, \dots, \psi_{pq_p}\},$$

where $\psi_{ij} = \mathrm{Pr}_i \circ \xi_{ij}$ and Pr_i is the metric projection onto K_i.

3 Attractors of weakly hyperbolic IFS with condensation

A weakly hyperbolic *Iterated Function System with condensation* (wIFSC, for short) $\{X; f_0, f_1, \dots, f_m\}$ consists of a condensation f_0 and of certain weak contractions $f_1, \dots, f_m$.

The following result generalizes a similar one for hyperbolic IFS with condensation [17].

Theorem 5. *Let $K \subset \mathbb{R}^n$ be a finite union of some convex compacta. Let $\mathcal{G}$ be a weakly hyperbolic IFS with condensation and K be its condensation set.*

Then there exists a standard weakly hyperbolic IFS which has the same attractor as the given weakly hyperbolic IFS with condensation $\mathcal{G}$.

Proof. Suppose that the compact set $K \subset \mathbb{R}^n$ is a finite union of convex compacta $K = \bigcup_{i=1}^{p} K_i$, where each K_i is the attractor of a hyperbolic IFS $\mathcal{F}_i = \{\mathrm{R}^n; \xi_{i1}, \ldots, \xi_{iq_i}\}$.

Let the given weakly hyperbolic IFS with condensation $\mathcal{G} = \{\mathbb{R}^n; f_0, f_1, \ldots, f_m\}$ consist of weak contractions $f_1, \ldots, f_m$ and the condensation f_0 with the condensation set K.

We will show that the standard weakly hyperbolic IFS

$$\mathcal{T} = \{\mathbb{R}^n; \psi_{11}, \ldots, \psi_{1q_1}, \ldots, \psi_{p1}, \ldots, \psi_{pq_p}, f_1, \ldots, f_m\}, \tag{2}$$

where $\psi_{ij} = \mathrm{Pr}_i \circ \xi_{ij}$ and Pr_i is the metric projection onto K_i, has the same attractor as the given weakly hyperbolic with condensation $\mathcal{G}$.

It is known (see, e.g., [17]) that any metric projection Pr_A on the convex compact set A is nonexpensive, i.e., for any $x, y \in \mathbb{R}^n$ one has $d(\mathrm{Pr}_A(x), \mathrm{Pr}_A(y)) \leq d(x, y)$. It implies that for any (weak) contraction φ the composition $\mathrm{Pr}_A \circ \varphi$ is also a (weak) contraction with respect to the same comparison function.

Consider the IFS, generated by the weak contractions $\{f_1, \ldots, f_r\}$. Let F_* stand for its Barnsley-Hutchinson operator and let A stand for its attractor, i.e., $F_*(A) = A$.

Denote by $\mathcal{S}$ the weak hyperbolic IFS generated by the weak contractions $\{\psi_{11}, \ldots, \psi_{1q_1}, \ldots, \psi_{p1}, \ldots, \psi_{pq_p}\}$. Let Ψ_* stand for its Barnsley-Hutchinson operator. By Theorem 4 the attractor of the weak hyperbolic IFS $\mathcal{S}$ is K, i.e., $\Psi_*(K) = K$. Moreover, for any compact set M such that $K \subset M$ we have $\Psi_*(M) = K$.

Let A_c denote the attractor of the given weakly hyperbolic IFS with condensation $\mathcal{G}$. Attractor A_c is described by (1).

It is sufficient to show for the IFS $\mathcal{T}$ from (2) with the corresponding Barnsley-Hutchinson operator $F_* \cup \Psi_*$ that

$$(F_* \cup \Psi_*)(A_c) = A_c.$$

The last is true since from (1) we have

$$(F_* \cup \Psi_*)(A_c) = F_*(A_c) \cup \Psi_*(A_c) =$$

$$(F_*(A) \cup (\bigcup_{l \geq 0} F_*^{l+1}(K))) \cup \Psi_*(A \cup K \cup (\bigcup_{l \geq 1} F_*^l(K))) =$$

$$A \cup (\bigcup_{l \geq 1} F_*^l(K)) \cup K = A \cup (\bigcup_{l \geq 0} F_*^l(K)) = A_c.$$

This completes the proof.□

Remark 4. It is worth noting that simply replacing the condensation with all contractions which generate its condensation set does not necessarily preserve the attractor.

Corollary 1. *Under the hypotheses of Theorem 5, for the given weakly hyperbolic Iterated Function System with condensation $\mathcal{G}$ there exists a weakly hyperbolic IFS $\mathcal{T}$, consisting of at most $r = m + \Sigma_{i=1}^{p} q_i$ weak contractions which has the same attractor as that of the wIFSC $\mathcal{G}$.*

4 The Pythagoras Tree

We exemplify the result, presented in the previous section, by a hyperbolic IFS with condensation whose attractor is the Pythagoras Tree.

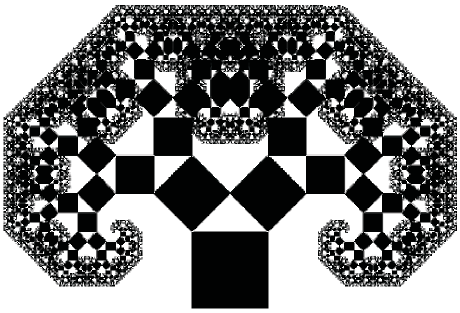

Figure 1. The classical Pythagoras Tree of A. E. Bosman

The well-known fractal called *The Pythagoras Tree* is a plane fractal which was invented and hand-drawn by the Dutch Mathematics teacher Albert E. Bosman [5] in 1942 (see Fig. 1).

This fractal, constructed from squares, can be represented as the attractor of an Iterated Function System with condensation. Given a right triangle, this IFS is determined by a constant compact-valued mapping with the "hypotenuse's square" as condensation set, together with two affine contractions which map this square onto the other two squares related to the given right triangle. The shape of the Pythagoras Tree depends on the shape of the condensation set ("steam of the tree") and of the contractions.

Fig. 2 represents two Pythagoras Trees, obtained by computer simulation as the attractor of a hyperbolic IFS with condensation (left) and a weak hyperbolic IFS with condensation (right), using the algorithm reducing the IFSC to a standard hyperbolic or weak hyperbolic IFS, consisting of six contractions or weak contractions respectively.

Fig. 3 represents the Pythagoras Tree (the black-white version), obtained by Lawrence H. Riddle [25]. As the author describes, he used a common picture of Pythagoras as the condensation set. "The trunk of the three was constructed using 10 iterations of a slight modification of the iterated function system where the

Figure 2. Pythagoras Trees as attractors of a hyperbolic IFS (left) and a weak hyperbolic IFS (right)

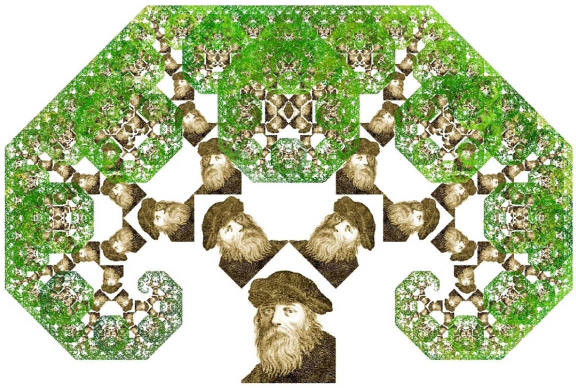

Figure 3. The Pythagoras Tree of L. H. Riddle

first function includes a horizontal reflection across a vertical line. The picture of Pythagoras was scaled and placed in just the right spot (after some experimentation) so that at each iteration the base of the new pictures will just touch at a 45° angle. The leaves of the tree consist of 500,000 points plotted using a random chaos game algorithm and colored based on Michael Barnsley's *color stealing algorithm*" (see [25]).

Remark 5. All numerical calculations and graphic objects have been done using the Computer Algebra System *Mathematica*.

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