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B-spline collocation method for solving Fredholm integral equations with discontinuous right-hand side

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Abstract. In this paper, we propose a method for approximating the solution of the linear Fredholm integral equation of the second kind which is defined on a closed contour Γ in the complex plane. The right-hand side of the equation is a piecewise continuous function that is numerically defined on a finite set of points on Γ. To approximate the solution, we use a linear combination of B-spline functions and Heaviside step functions defined on Γ. We discuss both theoretical and practical aspects of the pointwise convergence of the method, including its performance in the vicinity of the points where discontinuities occur.

Mathematics subject classification: 65R20, 65D07.
Keywords and phrases: Fredholm integral equation, piecewise continuous function, closed contour, complex plane, numerical approximation, B-spline, step function, convergence.

1 Introduction and problem formulation

Let a closed and piecewise smooth contour Γ be the boundary of the simply connected domain $\Omega^+ \subset \mathbb{C}$, and let the point $z = 0 \in \Omega^+$. Consider the Riemann function $z = \psi(w)$ that performs the conformal map of the domain D^- from the outside of the circle $\Gamma_0 := \{w \in \mathbb{C} : \ |w| = 1\}$ onto the domain Ω^- from the outside of the contour Γ, such that $\psi(\infty) = \infty, \ \psi'(\infty) > 0$. The function $\psi(w)$ transforms the circle Γ_0 onto the contour Γ. Next, we consider that the points of the contour Γ are defined by means of the function $\psi(w)$.

Let $f : \Gamma \to \mathbb{C}$ be a continuous or piecewise continuous function on Γ, and in this context, we will use the notation $f \in PC(\Gamma)$. If the function $f \in PC(\Gamma)$ is discontinuous on Γ, we consider that it has finite jump discontinuities, being left-continuous at the discontinuity points.

Let's consider the linear Fredholm integral equation of the second kind

$$\varphi(t) - \lambda \int_\Gamma K(t,s)\,\varphi(s)\,ds = f(t), \ \ t \in \Gamma, \tag{1}$$

which is defined on the contour Γ described above. The kernel function is continuous in both variables, $K \in C(\Gamma \times \Gamma)$. The right-hand side function $f \in PC(\Gamma)$, and the constant $\lambda \in \mathbb{C}$ satisfies the sufficient condition for equation (1) to have a unique solution $\varphi \in PC(\Gamma)$.

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Considering that the right-hand side f is numerically defined on the set of points $\{t_j\}$ on the contour Γ, we aim to develop an efficient method for computing a sequence of approximations φ_n to the solution φ which converges pointwise to φ on Γ as $n \to \infty$.

Global or piecewise polynomial approximation is generally ineffective for approximating piecewise continuous functions, except when studying convergence in the norm of Lebesgue spaces $L_p,\ 1 < p < \infty$. In such cases, it is shown that the sequence of interpolation polynomials converges to the solution φ, with the exception of a countable set of points [1].

It is known that if algebraic polynomials or spline functions of order $m \geq 2$ are used to approximate the piecewise continuous function φ, then in the vicinity of the discontinuity points, the approximation error does not tend to zero, no matter how much we increase the amount of informations required for constructing the approximation.

For applications, it is of interest to define analytically a sequence of approximation functions φ_n that converge pointwise to the piecewise continuous function φ, including in the vicinity of the discontinuity points.

Linear spline functions can be employed as an approximation technique, but in this case, the convergence rate of the approximation process can be exceedingly slow [2]. Some numerical results show that the oscillatory effect disappears and pointwise convergence of the approximations is attained, even in the vicinity of discontinuity points, when the approximation φ_n is constructed as a linear combination of B-spline functions of order $m \geq 2$ [2,3]. However, in the vicinity of discontinuity points, the convergence rate of the approximations is exceedingly slow. Furthermore, it should be noted that continuous curves in the complex plane frequently lead to a heavily distorted approximation of discontinuous curves.

The proposed approximation method entails constructing a sequence of piecewise continuous approximations for the function φ, with the objective of incorporating the convergence properties of B-spline functions. Specifically, we define the sequence of approximations φ_n as a linear combination of B-spline functions and Heaviside step functions. Previous studies have examined these approximations on intervals of the real axis [4]. In this paper, we investigate the case where the approximations are defined on the contour Γ in the complex plane.

Let $\{t_j\}_{j=1}^{n_B}$ be the set of distinct points on the contour Γ where the values of the function $f \in PC\left(\Gamma\right)$ are defined. We consider that the points t_j are generated based on the relation

$$t_j = \psi\left(w_j\right),\ w_j = e^{i\theta_j},\ \theta_j = 2\pi\left(j-1\right)/n_B,\ j = 1, ..., n_B.$$

We denote by $\Gamma_j := arc\left[t_j, t_{j+1}\right]$ the set of points of the contour Γ, located between the points t_j and t_{j+1} (see Figure 1).

We admit that the values $f\left(t_r^d\right)$ of the function f are known at the discontinuity points $t_r^d,\ r = 1, ..., n_{pd}$, on the contour Γ. For the function f, defined numerically, in [5] and [6] several algorithms have been proposed for establishing the locations of the discontinuity points on Γ.

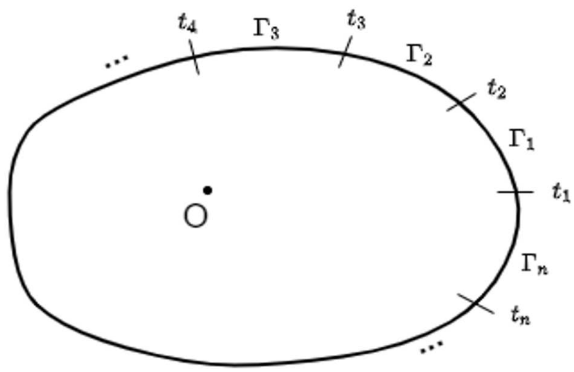

Figure 1: The contour and notations used

2 The computational scheme for approximating the solution of the integral equation

The algorithm we propose for approximating the solution φ of equation (1) is based on the concept of B-spline functions of order $m \geq 2$ which are defined at the points t_j of the contour Γ. These B-spline functions are defined using the recursive formula

$$B_{m,j}(t) := \frac{m}{m-1}\left(\frac{t-t_j^B}{t_{j+m}^B-t_j^B}B_{m-1,j}(t)+\frac{t_{j+m}^B-t}{t_{j+m}^B-t_j^B}B_{m-1,j+1}(t)\right),\ j=1,...,n_B, \tag{2}$$

where $B_{1,j}(t) = \begin{cases} \frac{1}{t_{j+1}^B-t_j^B} & \text{if } t \in arc\left[t_j^B, t_{j+1}^B\right) \\ 0 & \text{otherwise} \end{cases}$. The set of nodes $\left\{t_j^B\right\}_{j=1}^{n_B+m}$ satisfies the condition $t_j^B = t_j,\ j=1,...,n_B,\ t_{n_B+1}^B = t_1^B, t_{n_B+2}^B = t_2^B, ..., t_{n_B+m}^B = t_m^B$ (see [3]). For a fixed $m \geq 2$, the B-spline functions (2) have an explicit representation [3].

We define the Heaviside step function H on the contour Γ, constructed using the discontinuity points $t_r^d,\ r=1,...,n_{pd}$:

$$H\left(t-t_r^d\right) := \begin{cases} 0 & \text{if } t \in \Gamma_1 \cup ... \cup \Gamma_{s-1} \cup arc\left[t_s^B, t_r^d\right) \\ 1 & \text{if } t \in arc\left[t_r^d, t_{s+1}^B\right) \cup \Gamma_{s+1} \cup ... \cup \Gamma_{n_B} \end{cases},$$

where $\Gamma_s = arc\left[t_s^B, t_{s+1}^B\right],\ t_r^d \in \Gamma_s$.

Taking into account that the solution φ of equation (1) is a function with jump discontinuities on the contour Γ, and the linear combination of B-spline functions generates a continuous curve, we will seek the approximation of the solution φ of equation (1) in the form

$$\varphi_{n_B}^H(t) := \sum_{k=1}^{n_B}\alpha_k B_{m,k}(t) + \sum_{r=1}^{n_{pd}}\beta_r H\left(t-t_r^d\right), \tag{3}$$

where the coefficients $\alpha_k \in \mathbb{C},\ k = 1, ..., n_B$, and $\beta_r \in \mathbb{C},\ r = 1, ..., n_{pd}$, are determined by imposing the interpolation conditions

$$\varphi_{n_B}^H\left(t_j^C\right) - \lambda \int_\Gamma K\left(t_j^C, s\right)\varphi_{n_B}^H(s)\, ds = f\left(t_j^C\right),\ \ j = 1, ..., n. \tag{4}$$

In relation (4), where $n := n_B + n_{pd}$, the following elements of the B-spline knot set are selected as interpolation points $t_j^C,\ j = 1, ..., n$:

1. the first n_B interpolation points $t_j^C,\ j = 1, ..., n_B$, are the nodes $t_j^B = t_j,\ j = 1, ..., n_B$;
2. the remaining n_{pd} interpolation points $t_j^C,\ j = n_B + 1, ..., n$, are the discontinuity points $t_r^d,\ r = 1, ..., n_{pd}$, of the function f.

If among the interpolation points $t_j^C,\ j = 1, ..., n_B$, there are discontinuity points $t_j^d = \psi\left(e^{i\theta_j^d}\right)$ of the function f on Γ, then instead of them we consider the points $\tilde{t}_j^d = \psi\left(e^{i\left(\theta_j^d - \varepsilon_2\right)}\right)$, where $\varepsilon_2 > 0$ is a small value, for example, $\varepsilon_2 = 0.01$. Since the function is left continuous, for a sufficiently small ε_2, it can be considered that the value of the function f at point $\tilde{t}_j^d$ coincides with its value at point t_j^d.

Taking into account the representation

$$\int_\Gamma K(t, s)\,\varphi_{n_B}^H(s)\, ds = \int_\Gamma K(t, s)\left(\sum_{k=1}^{n_B}\alpha_k B_{m,k}(s) + \sum_{r=1}^{n_{pd}}\beta_r H\left(s - t_r^d\right)\right) ds =$$

$$= \sum_{k=1}^{n_B}\alpha_k I_k^{1,m}(t) + \sum_{r=1}^{n_{pd}}\beta_r I_r^2(t),$$

where $I_k^{1,m}(t) := \int_\Gamma K(t, s)\, B_{m,k}(s)\, ds,\ \ I_r^2(t) := \int_\Gamma K(t, s)\, H\left(s - t_r^d\right) ds,$ we can write the interpolation conditions (4) in the form

$$\sum_{k=1}^{n_B}\left(B_{m,k}\left(t_j^C\right) - \lambda I_k^{1,m}\left(t_j^C\right)\right)\alpha_k + \sum_{r=1}^{n_{pd}}\left(H\left(t_j^C - t_r^d\right) - \lambda I_r^2\left(t_j^C\right)\right)\beta_r =$$

$$= f\left(t_j^C\right),\ \ j = 1, ..., n. \tag{5}$$

The relation (5) can be written in matrix form as $B\bar{x} = \bar{f}$, where

$$B = \{m_{j,k}\}_{j,k=1}^n,\ \ m_{j,k} := B_{m,k}\left(t_j^C\right) - \lambda I_k^{1,m}\left(t_j^C\right),\ \ j = 1, ..., n,\ \ k = 1, ..., n_B,$$

$$m_{j,k} := H\left(t_j^C - t_r^d\right) - \lambda I_r^2\left(t_j^C\right),\ \ j = 1, ..., n,\ \ k = n_B + 1, ..., n,$$

$$\bar{x} = \left(\alpha_1, ..., \alpha_{n_B}, \beta_1, ..., \beta_{n_{pd}}\right)^T,\ \ \bar{f} = \left(f\left(t_1^C\right), ..., f\left(t_n^C\right)\right)^T.$$

If we consider $t_j^C = t_{j+1}^B$, $j = 1, ..., n_B$, for $m = 2$, and $t_j^C = t_{j+2}^B$, $j = 1, ..., n_B$, for $m = 3$ and $m = 4$, then the elements of the matrix B can be calculated as follows:

$$B = B_1 - \lambda B_2. \tag{6}$$

The properties of matrix

$$B_1 = \begin{pmatrix} B_{m,1}\left(t_1^C\right) & \cdots & B_{m,n_B}\left(t_1^C\right) & H\left(t_1^C - t_1^d\right) & \cdots & H\left(t_1^C - t_{n_{pd}}^d\right) \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ B_{m,1}\left(t_n^C\right) & \cdots & B_{m,n_B}\left(t_n^C\right) & H\left(t_n^C - t_1^d\right) & \cdots & H\left(t_n^C - t_{n_{pd}}^d\right) \end{pmatrix}$$

have been examined in [3], and the elements of matrix

$$B_2 = \begin{pmatrix} I_1^{1,m}\left(t_1^C\right) & \cdots & I_{n_B}^{1,m}\left(t_1^C\right) & I_1^2\left(t_1^C\right) & \cdots & I_{n_{pd}}^2\left(t_1^C\right) \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ I_1^{1,m}\left(t_n^C\right) & \cdots & I_{n_B}^{1,m}\left(t_n^C\right) & I_1^2\left(t_n^C\right) & \cdots & I_{n_{pd}}^2\left(t_n^C\right) \end{pmatrix}$$

can be determined as follows:

For the elements $I_k^{1,m}\left(t_j^C\right)$, $j = 1, ..., n$, $k = 1, ..., n_B$, the following relations hold:

$$I_k^{1,m}\left(t_j^C\right) = \int_\Gamma K\left(t_j^C, s\right) B_{m,k}(s)\, ds = \int_{arc\left[t_k^B, t_{k+m}^B\right]} K\left(t_j^C, s\right) B_{m,k}(s)\, ds =$$

$$= \sum_{r=1}^{m} \int_{arc\left[t_{k+r-1}^B, t_{k+r}^B\right]} K\left(t_j^C, s\right) p_k^{(r)}(s)\, ds = \sum_{r=1}^{m} \int_{\theta_{k+r-1}^B}^{\theta_{k+r}^B} g_{j,k}^r(\theta)\, d\theta,$$

where $g_{j,k}^r := K\left(t_j^C, \psi\left(e^{i\theta}\right)\right) p_k^{(r)}\left(\psi\left(e^{i\theta}\right)\right) \psi'\left(e^{i\theta}\right) ie^{i\theta}$, and $p_k^{(r)}(s)$ represents the components of the B-spline function $B_{m,k}(s)$ of the corresponding order m. For example, for $m = 4$, we have:

$$p_k^{(1)}(s) = \frac{4\left(s - t_k^B\right)^3}{\left(t_{k+4}^B - t_k^B\right)\left(t_{k+3}^B - t_k^B\right)\left(t_{k+2}^B - t_k^B\right)\left(t_{k+1}^B - t_k^B\right)},$$

$$p_k^{(2)}(s) = 4\left(I_1 + I_2\right),$$

where

$$I_1 := \frac{s - t_k^B}{t_{k+4}^B - t_k^B}\left(I_1^1 + I_1^2\right),$$

$$I_1^1 = \frac{\left(s - t_k^B\right)\left(t_{k+2}^B - s\right)}{\left(t_{k+3}^B - t_k^B\right)\left(t_{k+2}^B - t_k^B\right)\left(t_{k+2}^B - t_{k+1}^B\right)},$$

$$I_1^2 = \frac{\left(s - t_{k+1}^B\right)\left(t_{k+3}^B - s\right)}{\left(t_{k+3}^B - t_k^B\right)\left(t_{k+3}^B - t_{k+1}^B\right)\left(t_{k+2}^B - t_{k+1}^B\right)},$$

$$I_2 := \frac{\left(t_{k+4}^B - s\right)\left(s - t_{k+1}^B\right)^2}{\left(t_{k+4}^B - t_k^B\right)\left(t_{k+4}^B - t_{k+1}^B\right)\left(t_{k+3}^B - t_{k+1}^B\right)\left(t_{k+2}^B - t_{k+1}^B\right)},$$

$$p_k^{(3)}(s) = 4\left(I_3 + I_4\right),$$

and

$$I_3 := \frac{\left(t_{k+3}^B - s\right)^2\left(s - t_k^B\right)}{\left(t_{k+4}^B - t_k^B\right)\left(t_{k+3}^B - t_k^B\right)\left(t_{k+3}^B - t_{k+1}^B\right)\left(t_{k+3}^B - t_{k+2}^B\right)},$$

$$I_4 := \frac{t_{k+4}^B - s}{t_{k+4}^B - t_k^B}\left(I_4^1 + I_4^2\right),$$

$$I_4^1 = \frac{\left(s - t_{k+1}^B\right)\left(t_{k+3}^B - s\right)}{\left(t_{k+4}^B - t_{k+1}^B\right)\left(t_{k+3}^B - t_{k+1}^B\right)\left(t_{k+3}^B - t_{k+2}^B\right)}$$

$$I_4^2 = \frac{\left(s - t_{k+2}^B\right)\left(t_{k+4}^B - s\right)}{\left(t_{k+4}^B - t_{k+1}^B\right)\left(t_{k+4}^B - t_{k+2}^B\right)\left(t_{k+3}^B - t_{k+2}^B\right)},$$

$$p_k^{(4)}(s) = \frac{4\left(t_{k+4}^B - s\right)^3}{\left(t_{k+4}^B - t_k^B\right)\left(t_{k+4}^B - t_{k+1}^B\right)\left(t_{k+4}^B - t_{k+2}^B\right)\left(t_{k+4}^B - t_{k+3}^B\right)}.$$

The integrals $\int_{\theta_{k+r-1}^B}^{\theta_{k+r}^B} g_{j,k}^r(\theta)\,d\theta$ are approximated using the generalized trapezoidal rule, which is also applicable to functions with complex values [7]:

$$I := \int_{\theta_{in}}^{\theta_f} g(\theta)\,d\theta \approx I_N := h\left(0.5\left(g(\theta_{in}) + g(\theta_f)\right) + \sum_{j=1}^{N-1} g(\theta_{in} + jh)\right), \qquad (7)$$

$$h := \left(\theta_f - \theta_{in}\right)/N.$$

As $N \to \infty$, it has been shown in [7] that $I_N \to I$ at the rate of a geometric progression.

For the elements $I_r^2\left(t_j^C\right),\ j = 1, ..., n,\ r = 1, ..., n_{pd}$, the relations

$$I_r^2\left(t_j^C\right) = \int_\Gamma K\left(t_j^C, s\right) H\left(s - t_r^d\right) ds = \int_{arc\left[t_r^d, \psi(1)\right]} K\left(t_j^C, s\right) ds = \int_{\theta_r^d}^{2\pi} q_j(\theta)\,d\theta,$$

hold true, where $q_j(\theta) := K\left(t_j^C, \psi\left(e^{i\theta}\right)\right)\psi'\left(e^{i\theta}\right) ie^{i\theta}$. Similarly, the integrals $\int_{\theta_r^d}^{2\pi} q_j(\theta)\,d\theta$ will be approximated using the generalized trapezoidal rule (7).

It should be noted that the functions $g_{j,k}^r(\theta)$ and $q_j(\theta)$ do not depend on the function $f(t)$. Therefore, they can be evaluated at any point $\theta \in [0, 2\pi]$, allowing for the approximation of the integrals $I_k^{1,m}\left(t_j^C\right)$ and $I_r^2\left(t_j^C\right)$, respectively.

3 About the convergence of the method and a numerical example

After determining the solution $\alpha_k,\ k=1,...,n_B,\ \beta_r,\ r=1,...,n_{pd}$, of the system (5), we construct the approximation (3) of the function $\varphi(t)$ and calculate its values at the points $t\in\Gamma$. The convergence of the approximation sequence $\varphi_{n_B}^H$, defined by (3), to the function $\varphi\in PC(\Gamma)$ as $n_B\to\infty$ has been established in [3].

We exhibit the convergence of the proposed method through a numerical example. Consider the Riemann function $z=\psi(w)$ that performs the conformal transformation of the set $\{w\in\mathbb{C}:\ |w|>1\}$ on the domain Ω^- from the outside of the contour Γ as $\psi(w)=w+1/\left(3w^3\right)$. Thus, $\psi(w)$ transforms the unit circle Γ_0 onto the astroid Γ (see Figure 2).

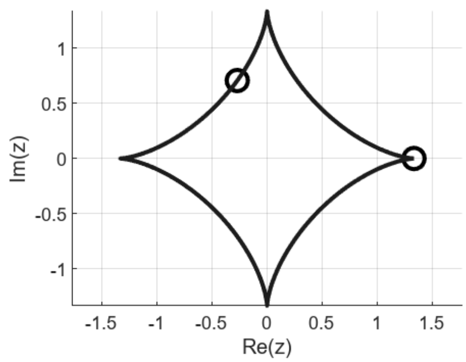

Figure 2: The contour and discontinuity points

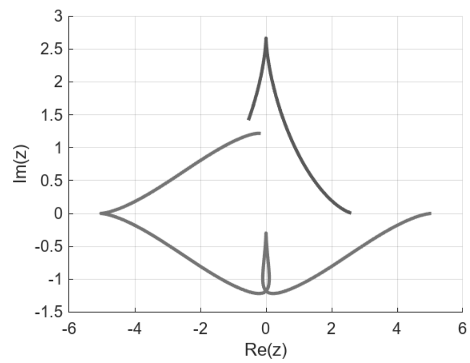

Figure 3: Graph of the solution

For testing purposes, we consider in the integral equation (1) the kernel function $K(t,s)=t^2+s^2$, the constant $\lambda=0.5$, and the right-hand side $f(t)$ given analytically on Γ:

$$f(t)=\begin{cases} 2t-\lambda u \ \text{ if } \ \theta\in\left(0,\theta_1^d\right] \\ t^3+2t-\lambda u \ \text{ if } \ \theta\in\left(\theta_1^d,\theta_2^d\right] \\ t^3+2t-\lambda u \ \text{ if } \ \theta=0 \end{cases}.$$

We have $\theta_1^d=0.7\pi,\ \ \theta_2^d=2\pi$ and $u:=(0.78148-0.081271i)\,t^2+0.91818+0.025237i$. The function f has $n_{pd}=2$ jump discontinuity points on Γ, $t_j^d=\psi\left(e^{i\theta_j^d}\right),\ j=1,2$ (see Figure 2 and Figure 3).

Likewise, the exact solution $\varphi\in PC(\Gamma)$ for the given test problem is known to be

$$\varphi(t)=\begin{cases} 2t \ \text{ if } \ \theta\in\left(0,\theta_1^d\right] \\ t^3+2t \ \text{ if } \ \theta\in\left(\theta_1^d,\theta_2^d\right] \\ t^3+2t \ \text{ if } \ \theta=0 \end{cases}.$$

It has two discontinuity points, the same as the right-hand side f.

The approximation algorithm for the solution of equation (1) takes as initial data the values f_j of the function f at the points

$$t_j = \psi\left(e^{i\theta_j}\right) \in \Gamma,\ \theta_j = 2\pi\left(j-1\right)/n_B,\ n_B \in \mathbb{N},\ k = 1, ..., n_B.$$

The coefficients of the approximation for the solution of equation (1) are determined as a linear combination according to (3), where B-spline functions of order $m = 4$ are considered. The number of points where the value of the function f is given on Γ is $n_B = 320$. Consequently, the solution to the system of equations $B\bar{x} = \bar{f}$ is determined, where $\bar{x} = \left(\alpha_1, ..., \alpha_{n_B}, \beta_1, ..., \beta_{n_{pd}}\right)^T$, $\bar{f} = \left(f\left(t_1^c\right), ..., f\left(t_n^c\right)\right)^T$, $n = n_B + n_{pd}$, and the matrix B has the form specified in (6).

The integrals $I_k^{1,m}\left(t_j^C\right)$ and $I_r^2\left(t_j^C\right)$, which define the components of the matrix B, are approximated using the generalized trapezoidal rule (7), with the parameter $N = 200$.

For values $n_B = 160$ and $n_B = 320$ in Figure 4 and Figure 5 the error obtained at the approximation of the solution φ by $\varphi_{n_B}^H$ is presented. It can be seen that the maximum error decreases significantly for $n_B = 320$.

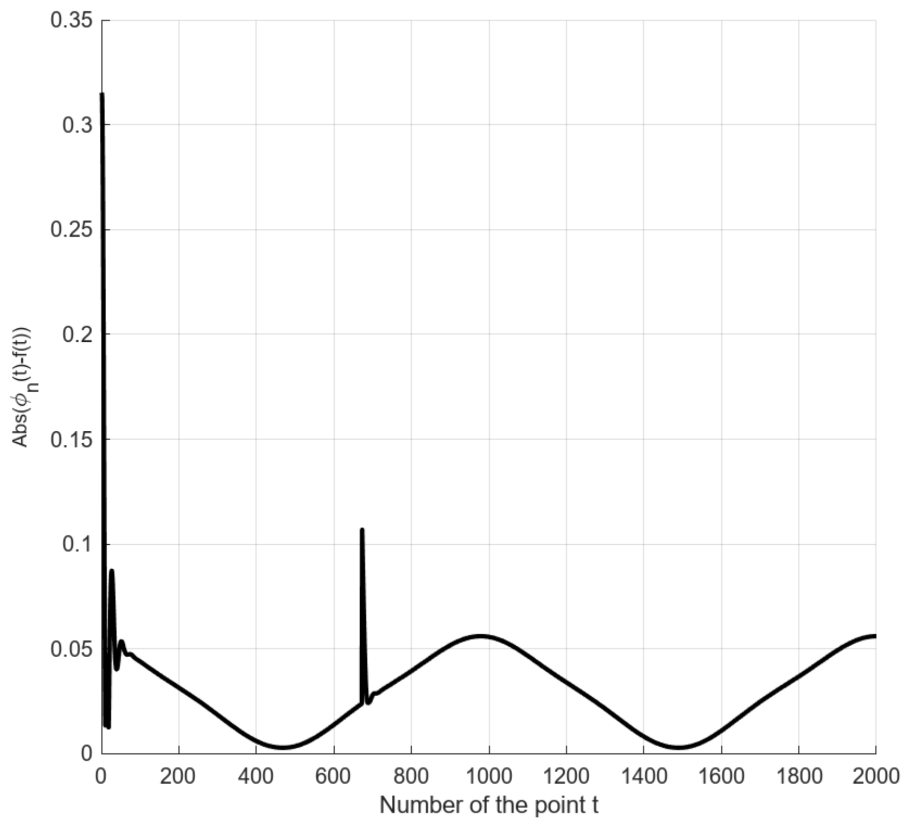

Figure 4: The approximation error for nB=160

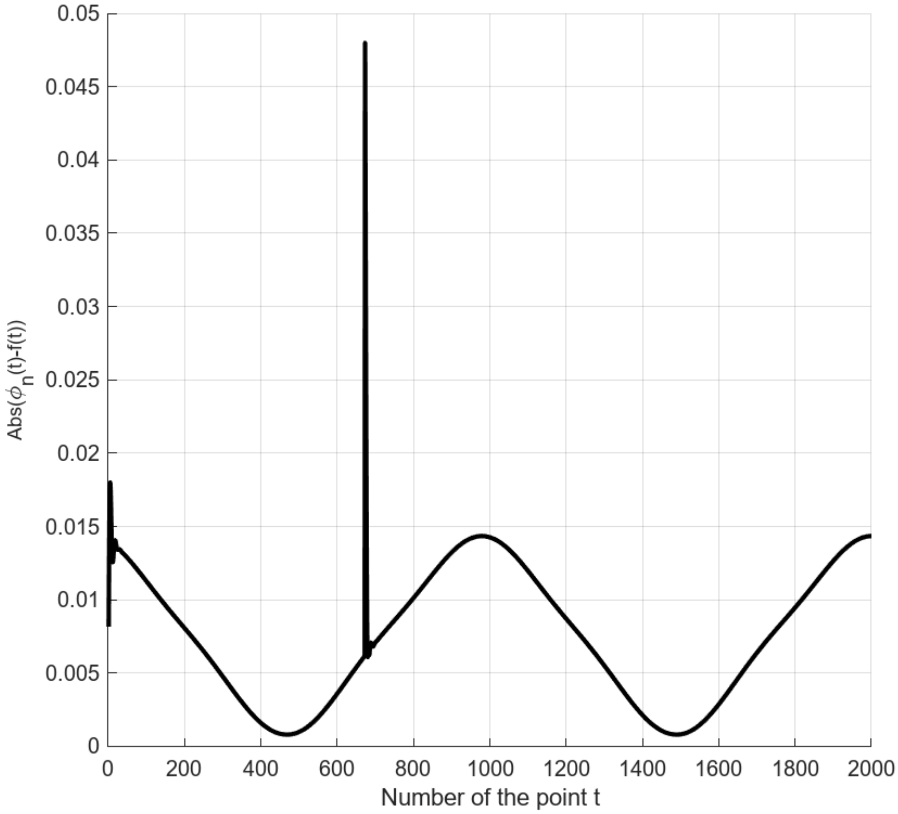

Figure 5: The approximation error for nB=320

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