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Global Asymptotic Stability of Generalized Homogeneous Dynamical Systems

David Cheban

Abstract. The goal of the paper is to study the relationship between asymptotic stability and exponential stability of the solutions of generalized homogeneous nonautonomous dynamical systems. This problem is studied and solved within the framework of general non-autonomous (cocycle) dynamical system. The application of our general results for differential and difference equations is given.

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1 Introduction

This paper is dedicated to the study of the problem of asymptotic stability of a class of nonautonomous dynamical systems with some property of symmetry. Namely, we study this problem for so-called generalized homogeneous nonautonomous dynamical systems, that is, a class of nonautonomous dynamical systems invariant with respect to a group of transformations called dilations. We establish our main results in the framework of general nonautonomous (cocycle) dynamical systems.

The motive for writing of this article was the works of A. Bacciotti and L. Rosier [2], A. Polyakov [16], V. I. Zubov [24] (see also the bibliography therein) and the work of the author [5]. We prove the equivalence of uniform asymptotic stability and exponential stability for this class of nonautonomous dynamical systems. If the phase space Y of driving system $(Y,\mathbb{T},\sigma)$ for the cocycle dynamical systems $\langle\mathbb{R}^n,\varphi,(Y,\mathbb{T},\sigma)\rangle$ is compact, then we prove that the asymptotic stability and uniform asymptotic stability are equivalent. If additionally the driving system $(Y,\mathbb{S},\sigma)$ with compact phase space Y is minimal, then for asymptotic stability the uniform stability and the existence of a positive number a and an element $y_0\in Y$ such that $\lim\limits_{t\to+\infty}|\varphi(t,u,y_0)|=0$ for any $u\in B[0,a]$ are sufficient. We apply these results for differential and difference equations.

The paper is organized as follows. In the second Section, we collect some known notions and facts from dynamical systems that we use in this paper. Namely, we

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present the construction of shift dynamical systems, definitions of Poisson stable motions and some facts about compact global attractors of dynamical systems. In the third Section we establish the relation between uniformly asymptotic stability and exponential stability for general nonautonomous (cocycle) dynamical systems. The fourth Section is dedicated to the relation between asymptotic stability and exponential stability for the nonautonomous dynamical systems with the compact phase space of their driving system. In the fifth Section, we study the nonautonomous dynamical system with driving system $(Y, \mathbb{S}, \sigma)$, when Y is a compact and minimal set. Finally, in the sixth Section we apply our general results, obtained in Sections 3-5 to differential/difference equations.

2 Preliminaries

Throughout the paper, we assume that X and Y are metric spaces and for simplicity we use the same notation ρ to denote the metrics on them, which we think would not lead to confusion. Let $\mathbb{R} = (-\infty, +\infty)$, $\mathbb{Z} := \{0, \pm 1, \pm 2, \dots\}$, $\mathbb{S} = \mathbb{R}$ or $\mathbb{Z}$, $\mathbb{S}_+ := \{s \in \mathbb{S} | \ s \geq 0\}$ and $\mathbb{T} \subseteq \mathbb{S}$ be a sub-semigroup of $\mathbb{S}$ such that $\mathbb{S}_+ \subseteq \mathbb{T}$.

Let $(X, \mathbb{T}, \pi)$ be a dynamical system on X and $\mathfrak{M}$ be some family of subsets from X.

Definition 1. A dynamical system $(X, \mathbb{T}, \pi)$ is said to be $\mathfrak{M}$-dissipative if for every $\varepsilon > 0$ and $M \in \mathfrak{M}$ there exists $L(\varepsilon, M) > 0$ such that $\pi^t M \subseteq B(K, \varepsilon)$ for any $t \geq L(\varepsilon, M)$, where K is a subset from X depending only on $\mathfrak{M}$. In this case we will call K an attractor for $\mathfrak{M}$.

The most important for applications are the cases when K is a bounded or compact set and $\mathfrak{M} = \{\{x\} \mid x \in X\}$ or $\mathfrak{M} = C(X)$, or $\mathfrak{M} = \{B(x, \delta_x) \mid x \in X, \ \delta_x > 0\}$, where

1. $C(X)$ is the family of all compact subsets of X;
2. $B(x_0, \delta) := \{x \in X | \ \rho(x, x_0) < \delta\}$.

Definition 2. The system $(X, \mathbb{T}, \pi)$ is called:

– pointwise dissipative if there exists $K \subseteq X$ such that for every $x \in X$

$$\lim_{t \to +\infty} \rho(xt, K) = 0; \tag{1}$$

– compactly dissipative if the equality (1) takes place uniformly w.r.t. x on the compact subsets from X;

– locally dissipative if for any point $p \in X$ there exists $\delta_p > 0$ such that the equality (1) takes place uniformly w.r.t. $x \in B(p, \delta_p)$.

Let $(X, \mathbb{T}, \pi)$ be compactly dissipative and K be a nonempty compact set that is an attractor for compact subsets X. Then for every compact $M \subseteq X$ the equality

$$\lim_{t\to+\infty}\sup_{x\in M}\rho(xt, K) = 0$$

holds. It is possible to show [7, Ch.I] that the set J defined by the equality

$$J := \omega(K)$$

does not depend on the choice of the set K attracting all compact subsets of the space X.

Lemma 1. *[7, Ch.I] If the dynamical system $(X, \mathbb{T}, \pi)$ is pointwise dissipative, $\Omega_X \neq \emptyset$ and it is compact, then $\Omega_X \subseteq J^+(\Omega_X)$.*

Theorem 1. *[7, Ch.I] For the dynamical systems $(X, \mathbb{T}, \pi)$ with the locally compact phase space X the pointwise, compact and local dissipativity are equivalent.*

Definition 3. (Cocycle on the state space E with the base $(Y, \mathbb{S}, \sigma)$.) A triplet $\langle E, \phi, (Y, \mathbb{S}, \sigma)\rangle$ (or briefly ϕ if no confusion) is said to be a *cocycle* on state space (or fibre) E with base $(Y, \mathbb{S}, \sigma)$ (or driving system $(Y, \mathbb{S}, \sigma)$) if the mapping $\phi : \mathbb{S}_+ \times Y \times E \to E$ satisfies the following conditions:

1. $\phi(0, u, y) = u$ for all $u \in E$ and $y \in Y$;
2. $\phi(t + \tau, u, y) = \phi(t, \phi(\tau, u, y), \sigma(\tau, y))$ for all $t, \tau \in \mathbb{S}_+, u \in E$ and $y \in Y$;
3. the mapping ϕ is continuous.

Remark 1. If $\varphi(t_0, u_1, y_0) = \varphi(t_0, u_2, y_0)$ ($t_0 > 0$, $u_1, u_2 \in E$ and $y_0 \in Y$), then $\varphi(t, u_1, y_0) = \varphi(t, u_2, y_0)$ for any $t \geq t_0$.

Condition (C). (Strong uniqueness condition.) If $\varphi(t_0, u_1, y_0) = \varphi(t_0, u_2, y_0)$ ($t_0 > 0$, $u_1, u_2 \in E$ and $y_0 \in Y$), then $\varphi(t, u_1, y_0) = \varphi(t, u_2, y_0)$ for any $t \in \mathbb{T}_+$.

Everywhere below in this paper we consider the cocycles φ satisfying Condition (C).

Definition 4. (Skew-product dynamical system.) Let $\langle E, \phi, (Y, \mathbb{S}, \sigma)\rangle$ be a cocycle on E, $X := E \times Y$ and π be a mapping from $\mathbb{S}_+ \times X$ to X defined by $\pi := (\phi, \sigma)$, i.e., $\pi(t, (u, y)) = (\phi(t, u, y), \sigma(t, y))$ for all $t \in \mathbb{S}_+$ and $(u, y) \in E \times Y$. The triplet $(X, \mathbb{S}_+, \pi)$ is an autonomous dynamical system and is called *skew-product dynamical system.*

Let $x \in X$. Denote by $\Sigma_x^+ := \{\pi(t, x) :\ t \geq 0\}$ (respectively, $\Sigma_x := \{\pi(t, x_0) : t \in \mathbb{T}\}$) the positive semi-trajectory (respectively, the trajectory) of the point x and $H^+(x) := \overline{\Sigma_x^+}$ (respectively, $H(x) := \overline{\Sigma_x}$) the semi-hull of x (respectively, the hull of x), where by bar the closure of Σ_x^+ (respectively, Σ_x) in X is denoted.

Let $(X, \mathbb{S}, \pi)$ be a dynamical system. Let us recall the classes of Poisson stable motions we study in this paper, see [20, 23] for details.

Definition 5. A point $x \in X$ is called *stationary* (respectively, *τ-periodic*) if $\pi(t,x) = x$ (respectively, $\pi(t+\tau,x) = \pi(t,x)$) for all $t \in \mathbb{S}$.

Definition 6. For given $\varepsilon > 0$, a number $\tau \in \mathbb{R}$ is called an *ε-shift of x* (respectively, *ε-almost period of x*) if $\rho(\pi(\tau,x),x) < \varepsilon$ (respectively, $\rho(\pi(\tau+t,x),\pi(t,x)) < \varepsilon$ for all $t \in \mathbb{R}$).

Definition 7. A point $x \in X$ is called *almost recurrent* (respectively, *almost periodic*) if for any $\varepsilon > 0$ there exists a positive number l such that any segment of length l contains an ε-shift (respectively, ε-almost period) of x.

Definition 8. If a point $x \in X$ is almost recurrent and its trajectory Σ_x is precompact, then x is called *(Birkhoff) recurrent*.

Remark 2. It is easy to see that every almost periodic point $x \in X$ is recurrent, but the reverse statement generally speaking is not true.

Denote by $C(\mathbb{T} \times \mathbb{R}^n, \mathbb{R}^n)$ the family of all continuous functions $f : \mathbb{T} \times \mathbb{R}^n \to \mathbb{R}^n$ equipped with the compact-open topology. This topology can be generated by Bebutov distance (see, e.g., [3], [23, ChIV])

$$d(f,g) := \sup_{L>0} \min\{\max_{|t|+|x| \le L} \rho(f(t,x),g(t,x)), 1/L\}.$$

Denote by $(C(\mathbb{T} \times \mathbb{R}^n, \mathbb{R}^n), \mathbb{T}, \sigma)$ the shift dynamical system (or called Bebutov dynamical system), i.e., $\sigma(\tau,f) := f^\tau$, where $f^\tau(t,x) := f(t+\tau,x)$ for any $(t,x) \in \mathbb{T} \times \mathbb{R}^n$.

We will say that the function $f \in C(\mathbb{T} \times \mathbb{R}^n, \mathbb{R}^n)$ possesses the property (A) if the motion $\sigma(t.f)$ possesses this property in the shift dynamical system $(C(\mathbb{T} \times \mathbb{R}^n, \mathbb{R}^n), \mathbb{T}, \sigma)$. As the property (A) we will consider the Lagrange stability, periodicity in time (respectively, almost periodicity, recurrence and so on).

Note that the function $f \in C(\mathbb{T} \times \mathbb{R}^n, \mathbb{R}^n)$ is Lagrange stable if and only if the function $f_K := f_{\big|\mathbb{T}\times K}$ is bounded and uniformly continuous on $\mathbb{T} \times K$ for any compact subset K from $\mathbb{R}^n$ (see, e.g., [21], [23, ChIV]).

Definition 9. Let $(\mathbb{R}^n, \mathbb{T}, \lambda)$ be a linear dynamical system on $\mathbb{R}^n$ [7, Ch.II]. A function $F \in C(Y \times \mathbb{R}^n, \mathbb{R}^n)$ is said to be λ-homogeneous if

$$F(y, \lambda(\tau,w)) = \lambda(\tau, F(y,w)) \text{ (or equivalently } F(y, \lambda^\tau w) = \lambda^\tau F(y,w))$$

for any $(y,\tau,w) \in Y \times \mathbb{T} \times \mathbb{R}^n$.

Example 1. Let $(Y, \mathbb{T}, \sigma)$ be a dynamical system on the metric space Y and $\mathbb{T} = \mathbb{R}_+$ or $\mathbb{R}$. Consider a differential equation

$$u' = F(\sigma(t,y),u), \ \ (y \in Y) \tag{2}$$

where $F \in C(Y \times \mathbb{R}^n, \mathbb{R}^n)$ is a regular function, i.e., for any $(u,y) \in \mathbb{R}^n \times Y$ there exists a unique solution $\varphi(t,u,y)$ of equation (2) defined on $\mathbb{R}_+$ with initial data

$\varphi(0,u,y)=u$. Then (see, for example, [4], [20]-[22] the continuous mapping $\varphi : \mathbb{R}_+\times\mathbb{R}^n\times Y\to\mathbb{R}^n$ satisfying the condition $\varphi(t+\tau,u,y)=\varphi(t,\varphi(\tau,u,y),\sigma(\tau,y))$ for any $t,\tau\in\mathbb{R}_+$ and $(u,y)\in\mathbb{R}^n\times Y$ is well defined . Then the triplet $\langle\mathbb{R}^n,\varphi,(Y,\mathbb{T},\sigma)\rangle$ is a cocycle over $(Y,\mathbb{T},\sigma)$ with the fibre $\mathbb{R}^n$ (shortly φ) generated by (2).

Lemma 2. *Assume that the function $F\in C(Y\times\mathbb{R}^n,\mathbb{R}^n)$ is λ-homogeneous, then the cocycle φ generated by (2) is also λ-homogeneous.*

Proof. To prove this statement we consider the function $\psi(t):=\lambda^\tau\varphi(t,u,y)$. It is easy to check that

$$\psi^{'}(t)=\lambda^\tau\varphi^{'}(t,u,y)=\lambda^\tau F(\sigma(t,y),\varphi(t,u,y))= \\ F(\sigma(t,y),\lambda^\tau\varphi(t,u,y))=F(\sigma(t,y),\psi(t))$$

for any $t\in\mathbb{T}$. Since $\psi(0)=\lambda^\tau u$, then we obtain $\psi(t)=\varphi(t,\lambda^\tau u,y)$, i.e., $\lambda^\tau\varphi(t,u,y)=\varphi(t,\lambda^\tau u,y)$ for any $t,\tau\in\mathbb{T}$ and $(u,y)\in\mathbb{R}^n\times Y$. Lemma is proved. □

3 Uniformly Asymptotical Stability of Nonautonomous Generalized Homogeneous Dynamical Systems: General Case

Let $X:=\mathbb{R}^n$ with euclidian norm $|x|:=\sqrt{x_1^2+\ldots+x_n^2}$. Denote by

$$|x|_{r,p}:=\big(\Sigma_{i=1}^n|x_i|^{\frac{p}{r_i}}\big)^{\frac{1}{p}},$$

where $r:=(r_1,\ldots,r_n)$, $r_i>0$ for any $i=1,\ldots,n$ and $p\geq 1$.

Denote by

1. $\rho(x):=|x|_{r,p}$;
2. $S_{r,p}:=\{x\in\mathbb{R}^n|\ \rho(x)=1\}$;
3. $\mathcal{K}:=\{\alpha\in C(\mathbb{R}_+,\mathbb{R}_+)|\ \alpha(0)=0$ and α is strictly increasing$\}$ and
4. $\mathcal{K}_\infty:=\{\alpha\in\mathcal{K}|\ \alpha(t)\to+\infty$ as $t\to+\infty\}$.

There exist $a,b\in\mathcal{K}_\infty$ such that

$$a(|x|_{r,p})\leq|x|\leq b(|x|_{r,p}) \tag{3}$$

for any $x\in\mathbb{R}^n$ (see for example [10]).

A generalized weight is a vector $r=(r_1,\ldots,r_n)$ with $r_i>0$ for any $i=1,\ldots,n$. The dilation associated to the generalized weight r is the action of the multiplicative group $\mathbb{R}_+\setminus\{0\}$ on $\mathbb{R}^n$ given by:

$$\Lambda^r:\mathbb{R}_+\setminus\{0\}\to\mathbb{R}^n\ \ \big((\mu,x)\to\Lambda^r_\mu x\big),$$

where $\Lambda^r_\mu:=diag(\mu^{r_i})_{i=1}^n$.

Remark 3. The following statements hold:

1. $\Lambda_1^r = I$, where $I := diag(1, \dots, 1)$;
2. $\Lambda_{\mu_1}^r \Lambda_{\mu_2}^r = \Lambda_{\mu_1\mu_2}^r$ for any $\mu_1, \mu_2 \in \mathbb{R}_+ \setminus \{0\}$;
3. the matrix Λ_μ^r ($\mu > 0$) is invertible and $\Lambda_{\mu^{-1}}^r$ is its inverse, i.e., $\Lambda_{\mu^{-1}}^r = \left(\Lambda_\mu^r\right)^{-1}$, because $L_\mu^r \Lambda_{\mu^{-1}}^r = \Lambda_1^r = I$ for any $\mu > 0$;
4. $||\Lambda_\mu^r|| \to 0$ as $\mu \to 0$;
5. $$|\Lambda_\mu^r x| \geq \mu^\nu |x| \tag{4}$$
   for any $x \in \mathbb{R}^n$ and $\mu > 0$, where $\nu := \min\{r_i|\ i = 1, \dots, n\} > 0$;
6. $$\rho(\Lambda_\mu^r x) = \mu\rho(x) \tag{5}$$
   for any $(\mu, x) \in (0, +\infty) \times \mathbb{R}^n$, where $\rho(x) := |x|_{r,p}$;
7. $\Lambda_\mu^{(1,\dots,1)} = diag(\mu, \dots, \mu) = \mu I$ for any $\mu > 0$.

Lemma 3. *[7, Ch.II] Let $\mathfrak{D}$ be a family of functions $\eta : \mathbb{R}_+ \to \mathbb{R}_+$ satisfying the conditions:*

a. there exists $M > 0$ such that $0 < \eta(t) \leq M$ for all $t \geq 0$ and $\eta \in \mathfrak{D}$;

b. $\eta(t) \to 0$ as $t \to +\infty$ uniformly in $\eta \in \mathfrak{D}$, i.e., for any $\varepsilon > 0$ there exists $L(\varepsilon) > 0$ such that $\eta(t) < \varepsilon$ for any $t \geq L(\varepsilon)$ and $\eta \in \mathfrak{D}$.

Then we have the following statements:

1. if $\eta(t + \tau) \leq \eta(t)\eta(\tau)$ for any $t, \tau \geq 0$ and $\eta \in \mathfrak{D}$, then there exit positive numbers $\mathcal{N}$ and ν such that

$$\eta(t) \leq \mathcal{N} e^{-\nu t}$$

for any $t \geq 0$ and $\eta \in \mathfrak{D}$;

2. if $\eta(t + \tau) \leq \eta(t)\eta(\tau\eta^m(t))$ ($m > 0$) for any $t, \tau \geq 0$ and $\eta \in \mathfrak{D}$, then there exist positive numbers a and b such that

$$\eta(t) \leq M(a + bt)^{-\frac{1}{m}}$$

for any $t \geq 0$ and $\eta \in \mathfrak{D}$.

Definition 10. Following [13, 16, 18, 24] a cocycle $\langle \mathbb{R}^n, \varphi, (Y, \mathbb{T}, \sigma)\rangle$ over dynamical system $(Y, \mathbb{T}, \sigma)$ (driving system) with the fibre $\mathbb{R}^n$ is said to be r-homogeneous of degree $m \in \mathbb{R}$ if

$$\varphi(t, \Lambda_\mu^r u, y) = \mu^m \Lambda_\mu^r \varphi(t, u, y) \tag{6}$$

for any $\mu > 0$ and $(t, u, y) \in \mathbb{T}_+ \times \mathbb{R}^n \times Y$.

In this subsection we suppose that the phase space Y of the driving system $(Y,\mathbb{R},\sigma)$, generally speaking, is not compact.

Definition 11. The trivial motion $u=0$ of the cocycle φ is said to be:

1. uniformly stable if for arbitrary positive number ε there exists a positive number $\delta=\delta(\varepsilon)$ such that $|u|<\delta$ implies
$$|\varphi(t,u,y)|<\varepsilon$$
for any $(t,y)\in\mathbb{T}_+\times Y$;
2. uniformly attracting if there exists a positive number γ such that
$$\lim_{t\to+\infty}\sup_{|u|\le\gamma,y\in Y}|\varphi(t,u,y)|=0;$$
3. uniformly asymptotically stable if it is uniformly stable and uniformly attracting.

Lemma 4. *The trivial motion $u=0$ of the r-homogeneous cocycle φ of the degree zero is uniformly stable if and only if for arbitrary $\varepsilon>0$ there exists $\delta=\delta(\varepsilon)>0$ such that $\rho(u)<\delta$ implies $\rho(\varphi(t,u,y))<\varepsilon$ for any $(t,y)\in\mathbb{T}_+\times Y$.*

Proof. Let $u=0$ be uniformly stable motion of φ, $\mu>0$ and $\Delta(\mu)>0$ be a positive number figuring in the definition of the uniform stability of $u=0$. For any $\varepsilon>0$ we put $\delta(\varepsilon):=b^{-1}(\Delta(a(\varepsilon)))>0$, where a and b are some functions from $\mathcal{K}_\infty$ figuring in (3). If $\rho(u)<\delta$, then we have $|u|\le b(\rho(u))<\Delta(a(\varepsilon))$ and, consequently, $|\varphi(t,u,y)|<a(\varepsilon)$ for any $t\in\mathbb{T}_+$. Note that $\rho(\varphi(t,u,y))\le a^{-}(|\varphi(t,u,y)|)<a^{-1}(a(\varepsilon))=\varepsilon$ for any $t\ge 0$.

The inverse statement can be proved using the same arguments as above. Lemma is proved. □

Lemma 5. *If the trivial motion $u=0$ of the cocycle φ is uniformly stable, then there exists a positive number M such that*
$$|\varphi(t,u,y)|\le\tilde{M}$$
for any $|u|\le 1$ and $(t,y)\in\mathbb{T}_+\times Y$.

Proof. Since the trivial motion $u=0$ of the cocycle φ is uniformly stable, then there exists a positive number $\delta_0=\delta(1)$ such that $|u|\le\delta_0$ implies
$$|\varphi(t,u,y)|\le 1$$
for any $|u|\le\delta_0$ and $(t,y)\in\mathbb{T}_+\times Y$. Since $\|\Lambda^r_{\mu^{-1}}\|\to 0$ as $\mu\to+\infty$, then there exists a positive number μ_0 such that
$$\||\Lambda^r_{\mu^{-1}}\|\le\delta_0$$

for any $\mu \geq \mu_0$. Note that

$$|\varphi(t,u,y)| = |\varphi(t,\Lambda^r_\mu \Lambda^r_{\mu^{-1}} u,y)| = |\Lambda^r_\mu \varphi(t,\Lambda^r_{\mu^{-1}} u,y)| \leq \|\Lambda^r_\mu\| |\varphi(t,\Lambda^r_{\mu^{-1}} u,y)| \tag{7}$$

for any $\mu \geq \mu_0$ and $(t,u,y) \in \mathbb{T}_+ \times \mathbb{R}^n \times Y$. By (7) we have

$$|\Lambda^r_{\mu_0^{-1}} u| \leq \delta_0$$

for any $|u| \leq 1$ and, consequently,

$$\sup_{|u|\leq 1} |\varphi(t,\Lambda^r_{\mu_0^{-1}} u,y)| \leq 1 \tag{8}$$

for any $(t,y) \in \mathbb{T}_+ \times Y$. Finally, we note that from (7) and (8) we obtain

$$|\varphi(t,u,y)| \leq \|\Lambda^r_{\mu_0^{-1}}\| := \widetilde{M}$$

for any $|u| \leq 1$ and $(t,y) \in \mathbb{T}_+ \times Y$. Lemma is proved. □

Corollary 1. *Under the conditions of Lemma 5 for any $R > 0$ there exists a positive constant $M(R)$ such that*

$$|\varphi(t,u,y)| \leq M(R)$$

for any $u \in \mathbb{R}^n$ with $|u| \leq R$ and $(t,y) \in \mathbb{T}_+ \times Y$.

Proof. Let R be an arbitrary positive number. Since $\|\Lambda^r_{\mu^{-1}}\| \to 0$ as $\mu \to +\infty$, then there exists a positive number $\mu_0 = m_0(R)$ such that

$$\|\Lambda^r_{\mu^{-1}}\| \leq R^{-1} \tag{9}$$

for any $\mu \geq \mu_0$ and, consequently,

$$|\Lambda^r_{\mu^{-1}} u| \leq \|\Lambda^r_{\mu_0^{-1}}\| |u| \leq R^{-1} R = 1 \tag{10}$$

for any $|u| \leq R$. Note that

$$\begin{aligned} |\varphi(t,u,y)| = |\varphi(t,\Lambda^r_{\mu_0} \Lambda^r_{\mu_0^{-1}} u,y)| = |\Lambda^r_{\mu_0} \varphi(t,\lambda^r_{\mu_0^{-1}} u,y)| \leq \\ \|\Lambda^r_{\mu_0}\| |\varphi(t,\Lambda^r_{\mu_0^{-1}} u,y)| \end{aligned} \tag{11}$$

for any $(t,u,y) \in \mathbb{T}_+ \times \mathbb{R}^n \times Y$. According to (9)-(11) we obtain

$$|\varphi(t,u,y)| \leq \|\Lambda^r_{\mu_0}\| |\varphi(t,\Lambda^r_{\mu_0^{-1}} u,y)| \leq \|\Lambda^r_{\mu_0^{-1}}\| \widetilde{M} := M(R)$$

for any $|u| \leq R$ and $(t,y) \in \mathbb{T}_+ \times Y$. □

Corollary 2. *Under the conditions of Lemma 5 there exists a positive constant M such that*

$$\rho(\varphi(t,u,y)) \leq M$$

for any $u \in \mathbb{R}^n$ with $\rho(u) \leq 1$ and $(t,y) \in \mathbb{T}_+ \times Y$.

Proof. Let $u \in \mathbb{R}^n$ with $\rho(u) \le 1$ and $a, b \in \mathcal{K}_\infty$ be the function from (3), then we have

$$|u| \le b(\rho(u)) \le b(1)$$

and

$$a(\rho(\varphi(t,u,y)) \le |\varphi(t,u,y)| \le M(b(1)) \tag{12}$$

for any $(t,y) \in \mathbb{T}_+ \times Y$. From (12) we obtain

$$\rho(\varphi(t,u,y)) \le a^{-1}(M(b(1)) := M$$

for any $\rho(u) \le 1$ and $(t,y) \in \mathbb{T}_+ \times Y$. ◻

Lemma 6. *Let $\langle \mathbb{R}^n, \varphi, (Y, \mathbb{T}, \sigma)\rangle$ be a cocycle over $(Y, \mathbb{T}, \sigma)$ with the fibre $\mathbb{R}^n$. Assume that φ is an r-homogeneous of the degree zero cocycle.*

Then

1. $$\rho(\varphi(t+\tau,u,y)) = \rho(\varphi(\tau,u,y))\rho(\varphi(t,\Lambda^r_{\mu^{-1}}\varphi(\tau,u,y),\sigma(\tau,y)) \tag{13}$$
   for any $t, \tau \in \mathbb{T}_+$, where $\mu := \rho(\varphi(\tau,u,y))$;

2. $$\rho(\varphi(t,u,y)) = \rho(u)\rho(\varphi(t,\Lambda^r_{\rho(u)^{-1}}u,y))$$
   for any $u \in \mathbb{R}^n \setminus \{0\}$, $t \in \mathbb{T}_+$ and $y \in Y$.

Proof. Note that

$$\begin{gathered}\rho(\varphi(t+\tau,u,y)) = \rho(\varphi(t,\varphi(\tau,u,y),\sigma(\tau,y)) = \\ \rho(\varphi(t,\Lambda^r_\mu\Lambda^r_{\mu^{-1}}\varphi(\tau,u,y),\sigma(\tau,y)) = \rho(\Lambda^r_\mu\varphi(t,\Lambda^r_{\mu^{-1}}\varphi(\tau,u,y),\sigma(\tau,y)) = \\ \mu\rho(\varphi(t,\Lambda^r_{\mu^{-1}}\varphi(\tau,u,y),\sigma(\tau,y))\end{gathered} \tag{14}$$

for any $\mu > 0$, $t, \tau \in \mathbb{T}_+$ and $(u,y) \in \mathbb{R}^n \times Y$. In particular for $\mu = \rho(\varphi(\tau,u,y)) > 0$ we obtain from (14) the following equality

$$\rho(\varphi(t+\tau,u,y)) = \rho(\varphi(\tau,u,y))\rho(\varphi(t,\Lambda^r_{\mu^{-1}}\varphi(\tau,u,y),\sigma(\tau,y).$$

The second statement of Lemma follows from the first one if we take $\tau = 0$. ◻

Theorem 2. *Let $\langle \mathbb{R}^n, \varphi, (Y, \mathbb{T}, \sigma)\rangle$ be an r-homogeneous cocycle of the degree zero. The following statements are equivalent:*

1. *the trivial motion $u = 0$ of the cocycle φ is uniformly stable;*
2. *there exists a positive number M such that*
   $$\rho(\varphi(t,u,y)) \le M\rho(u) \tag{15}$$
   for any $(t,u,y) \in \mathbb{T}_+ \times \mathbb{R}^n \times Y$.

Proof. To prove this Theorem it is sufficient to show (i) implies (ii) because the inverse implication, taking into account Lemma 4, is evident.

Let M be the positive number from Corollary 2 and (t,u,y) be an arbitrary element from $\mathbb{T}_+ \times \mathbb{R}^n \times Y$ with $u \neq 0$, then by Lemma 6 (item (ii)) we have

$$\rho(\varphi(t,u,y)) = \rho(u)\rho(\varphi(t,\Lambda^r_{\rho(u)^{-1}}u,y)). \tag{16}$$

Since $\rho(\Lambda^r_{\rho(u)^{-1}}u) = \rho(u)^{-1}\rho(u) = 1$, then by Corollary 2 we have

$$\rho(\varphi(t,\Lambda^r_{\rho(u)^{-1}}u,y)) \leq M. \tag{17}$$

From (16) and (17) we obtain (15). Theorem is proved. □

Lemma 7. *If the trivial motion $u = 0$ of the cocycle φ is uniformly attracting, then*

$$\lim_{t\to+\infty} \sup_{|u|\leq 1, y\in Y} |\varphi(t,u,y)| = 0. \tag{18}$$

Proof. Since the trivial motion $u = 0$ of the cocycle φ is uniformly attracting, then there exists a positive number γ such that

$$\lim_{t\to+\infty} \sup_{|u|\leq \gamma, y\in Y} |\varphi(t,u,y)| = 0. \tag{19}$$

Since $\|\Lambda^r_{\mu^{-1}}\| \to 0$ as $\mu \to +\infty$, then there exists a positive number μ_0 such that

$$\||\Lambda^r_{\mu^{-1}}\| \leq \gamma \tag{20}$$

for any $\mu \geq \mu_0$ and, consequently,

$$|\Lambda^r_{\mu_0^{-1}}u| \leq \|\Lambda^r_{\mu_0^{-1}}\||u| \leq \gamma \tag{21}$$

for any $|u| \leq 1$. From (7) we have

$$|\varphi(t,u,y)| \leq \|\Lambda^r_{\mu_0}\||\varphi(t,\Lambda^r_{\mu_0^{-1}}u,y)| \tag{22}$$

and taking into account (19)-(22) we obtain (18). Lemma is proved. □

Corollary 3. *Assume that the trivial motion $u = 0$ of the cocycle φ is uniformly attracting, then*

$$\lim_{t\to+\infty} \sup_{|u|\leq R, y\in Y} |\varphi(t,u,y)| = 0 \tag{23}$$

for any $R > 0$.

Proof. Let R be an arbitrary (fixed) positive number. Since $\|\Lambda^r_{\mu^{-1}}\| \to 0$ as $\mu \to +\infty$, then there exists a positive number μ_0 such that

$$\||\Lambda^r_{\mu^{-1}}\| \leq R^{-1} \tag{24}$$

for any $\mu \geq \mu_0$ and, consequently,

$$|\Lambda^r_{\mu_0^{-1}} u| \leq \|\Lambda^r_{\mu_0^{-1}}\||u| \leq R^{-1}R = 1 \tag{25}$$

for any $|u| \leq R$. Taking into account (23)-(25) we obtain

$$|\varphi(t,u,y)| = |\varphi(t, \Lambda^r_{\mu_0}\Lambda^r_{\mu_0^{-1}}u, y)| = |\Lambda^r_{\mu_0}\varphi(t, \Lambda^r_{\mu_0^{-1}}u, y)| \leq$$
$$\|\Lambda^r_{\mu_0}\||\varphi(t, \Lambda^r_{\mu_0^{-1}}u, y)| \leq R^{-1} \sup_{|v|\leq 1, y\in Y} |\varphi(t,v,y)| \to 0$$

as $t \to +\infty$ uniformly with respect to $|u| \leq R$ and $y \in Y$. □

Corollary 4. *Under the conditions of Lemma 7 we have*

$$\lim_{t\to+\infty} \sup_{\rho(u)\leq 1, y\in Y} \rho(\varphi(t,u,y)) = 0. \tag{26}$$

Proof. Let $u \in \mathbb{R}^n$ with $\rho(u) \leq 1$, then $|u| \leq b(1)$. Since

$$a(\rho(\varphi(t,u,y)) \leq |\varphi(t,u,y)| \leq \sup_{|u|\leq b(1), y\in Y} |\varphi(t,u,y)| := \eta(t), \tag{27}$$

and by Corollary 3

$$\lim_{t\to+\infty} \eta(t) = 0. \tag{28}$$

From (27) we obtain

$$\sup_{\rho(u)\leq 1, y\in Y} \rho(\varphi(t,u,y)) \leq a^{-1}(\eta(t)) \tag{29}$$

for any $t \in \mathbb{T}_+$. Passing to the limit in (29) and taking into account (28) we obtain (26). □

Theorem 3. *Let φ be an r-homogeneous cocycle over dynamical system $(Y, \mathbb{T}, \sigma)$ with the fibre. The following statements are equivalent:*

1. *the trivial motion $u = 0$ of the cocycle φ is uniformly asymptotically stable;*
2. *there are positive numbers $\mathcal{N}$ and ν such that*

$$\rho(\varphi(t,u,y)) \leq \mathcal{N}e^{-\nu t}\rho(u) \tag{30}$$

for any $(t,u,y) \in \mathbb{T}_+ \times \mathbb{R}^n \times Y$.

Proof. It is evident that 2. implies 1.

Now we will establish that 1. implies 2. Indeed, denote by

$$m(t) := \sup_{\rho(u)\leq 1, y\in Y} \rho(\varphi(t,u,y)) \tag{31}$$

for every $t \in \mathbb{T}_+$. By (31) the mapping $m : \mathbb{T}_+ \to \mathbb{R}_+$ is well defined possessing the following properties:

a. $0 \le m(t) \le M$ for any $t \in \mathbb{T}_+$, where $M := a^{-1}(M(b(1)))$ from Corollary 2;

b. $m(t) \to 0$ as $t \to +\infty$;

c. $m(t+\tau) \le m(t)m(\tau)$ for any $t, \tau \in \mathbb{T}_+$.

The statement a. (respectively, statement b.) follows from Corollary 2 (respectively, Corollary 4). To prove the statement c. we note that

$$\begin{aligned} m(t+\tau) = \sup_{\rho(u)\le 1, y\in Y} \rho(\varphi(t+\tau,u,y) = \\ \sup_{\rho(u)\le 1, y\in Y} \rho(\varphi(t,\varphi(\tau,u,y),\sigma(\tau,y))) = \\ \sup_{\rho(u)\le 1, y\in Y} \rho(\varphi(t,\Lambda^r_\mu \Lambda^r_{\mu^{-1}}\varphi(\tau,u,y),\sigma(\tau,y))) = \\ \sup_{\rho(u)\le 1, y\in Y} \rho(\Lambda^r_\mu \varphi(t,\Lambda^r_{\mu^{-1}}\varphi(\tau,u,y),\sigma(\tau,y))), \end{aligned} \tag{32}$$

where

$$\mu := \rho(\varphi(\tau,u,y)). \tag{33}$$

By the equality (5) we have

$$\begin{aligned} \sup_{\rho(u)\le 1, y\in Y} \rho(\Lambda^r_\mu \varphi(t,\Lambda^r_{\mu^{-1}}\varphi(\tau,u,y),\sigma(\tau,y))) = \\ \mu\rho(\varphi(t,\Lambda^r_{\mu^{-1}}\varphi(\tau,u,y),\sigma(\tau,y))). \end{aligned} \tag{34}$$

Note that

$$\rho(\Lambda^r_{\mu^{-1}}\varphi(\tau,u,y)) = \mu^{-1}\rho(\varphi(\tau,u,y) = 1 \tag{35}$$

and, consequently,

$$\rho(\varphi(t,\Lambda^r_{\mu^{-1}}\varphi(\tau,u,y),\sigma(\tau,y))) \le \sup_{\rho(v)\le 1, q\in Y} \rho(\varphi(t,v,q)) = m(t). \tag{36}$$

From (32)-(36) we obtain

$$m(t+\tau) \le m(\tau)m(t)$$

for any $t, \tau \in \mathbb{T}_+$.

According to Lemma 6 (item (ii)) we have

$$\rho(\varphi(t,u,y)) = \rho(u)\rho(\varphi(t,\Lambda^r_{\rho(u)^{-1}}u,y)) \le m(t)\rho(u)$$

for any $u \in \mathbb{R}^n \setminus \{0\}$ and $(t,y) \in \mathbb{T}_+ \times Y$ because $\rho(\Lambda^r_{\rho(u)^{-1}}u) = 1$ and, consequently,

$$\rho(\varphi(t,\Lambda^r_{\rho(u)^{-1}}u,y)) \le \sup_{\rho(v)\le 1, y\in Y} \rho(\varphi(t,v,y)) = m(t). \tag{37}$$

By Lemma 3 there are positive numbers $\mathcal{N}$ and ν such that $m(t) \le \mathcal{N}e^{-\nu t}$ for any $t \in \mathbb{T}_+$, and taking into account (37) we obtain (30). Theorem is proved. $\square$

4 Asymptotic Stability of Nonautonomous Generalized Homogeneous Dynamical Systems: The Case of the Compact Phase Space of Driving System

Let $\langle \mathbb{R}^n, \varphi, (Y, \mathbb{T}, \sigma)\rangle$ be a cocycle over $(Y, \mathbb{T}, \sigma)$ with the fibre $\mathbb{R}^n$ and Y be a compact metric space. Assume that the cocycle φ admits the trivial motion 0, i.e., $\varphi(t, 0, y) = 0$ for any $(t, y) \in \mathbb{T}_+ \times Y$.

Remark 4. Let $\langle \mathbb{R}^n, \varphi, (Y, \mathbb{T}, \sigma)\rangle$ be r homogeneous of order m, then φ admits the trivial motion.

Denote by

$$W^s_y(0) := \{u \in \mathbb{R}^n | \lim_{t\to+\infty} |\varphi(t, u, y)| = 0\}.$$

Definition 12. A trivial motion 0 of the cocycle φ is said to be:

1. uniformly stable if for arbitrary $\varepsilon > 0$ there exists $\delta = \delta(\varepsilon) > 0$ such that $|u| < \delta$ implies $|\varphi(t, u, y)| < \varepsilon$ for any $t \in \mathbb{T}_+$ and $y \in Y$;
2. attracting if there exists $\gamma > 0$ such that $\lim\limits_{t\to+\infty} |\varphi(t, u, y)| = 0$ for any $|u| < \gamma$ and $y \in Y$;
3. asymptotically stable if it is uniformly stable and attracting;
4. globally asymptotically stable if it is asymptotically stable and $W^s_y(0) = \mathbb{R}^n$ for any $y \in Y$.

Let $\langle \mathbb{R}^n, \varphi, (Y, \mathbb{T}, \sigma)\rangle$ be a cocycle over $(Y, \mathbb{T}, \sigma)$ with the fiber $\mathbb{R}^n$ and $\varphi(t, 0, y) = 0$ for any $(t, y) \in \mathbb{T}_+ \times Y$.

Lemma 8. *Let $\langle \mathbb{R}^n, \varphi, (Y, \mathbb{T}, \sigma)\rangle$ be an $r \in (0, +\infty)^n$ homogeneous (of the degree zero) cocycle over $(Y, \mathbb{T}, \sigma)$ with the fiber $\mathbb{R}^n$. Assume that $W^s_y(0)$ is neighborhood of 0, then $W^s_y(0) = \mathbb{R}^n$.*

Proof. Let $u \in \mathbb{R}^n$ be an arbitrary point. Under the condition of Lemma there exists a positive number δ_y such that $B(0, \delta_y) \subseteq W^s_y(0)$, where $B(0, \delta) := \{u \in \mathbb{R}^n | \, |u| < \delta\}$. Since the cocycle φ is r homogeneous of the degree zero, then there exists a positive number $\mu_0 < 1$ such that

$$\Lambda^r_\mu u \in B(0, \delta_y) \tag{38}$$

for any $0 < \mu < \mu_0$. Note that

$$\varphi(t, u, y) = \varphi(t, \Lambda^r_{\mu^{-1}} \Lambda^r_\mu u, y)) = \Lambda^r_{\mu^{-1}} \varphi(t, \Lambda^r_\mu u, y). \tag{39}$$

From (38)-(39) we obtain $u \in W^s_y(0)$, that is, $\mathbb{R}^n = W^s_y(0)$. Lemma is proved. □

Theorem 4. *Let $\langle \mathbb{R}^n, \varphi, (Y, \mathbb{T}, \sigma)\rangle$ be a cocycle over $(Y, \mathbb{T}, \sigma)$ with the fibre $\mathbb{R}^n$ and $r \in (0, +\infty)^n$. Assume that the cocycle φ is r homogeneous of the degree zero and Y is compact. Then the following conditions are equivalent:*

1. *the trivial motion $u = 0$ of the cocycle φ is attracting;*
2. *the skew-product dynamical system $(X, \mathbb{T}_+, \sigma)$ generated by φ is pointwise dissipative.*

Proof. To prove this statement it is sufficient to show that (i) implies (ii). Let $x = (u, y) \in X = E \times Y$ be an arbitrary point. By Lemma 8 we have $W^s_y(0) = \mathbb{R}^n$ and, consequently $u \in W^s_y(0)$, i.e.,

$$\lim_{t \to +\infty} |\varphi(t, u, y)| = 0.$$

Since the space Y is compact, then the motion $\pi(t, x)$ ($x = (u, y)$ and $\pi(t, x) = (\varphi(t, u, y), \sigma(t, y))$) is positively Lagrange stable and $\emptyset \neq \omega_x \subseteq \Theta := \{0\} \times Y$. Thus $\Omega_X \subseteq \Theta$ and, consequently, the dynamical system $(X, \mathbb{T}, \sigma)$ is pointwise dissipative. Theorem is proved. □

Theorem 5. *Let φ be an r homogeneous cocycle over $(Y, \mathbb{T}, \sigma)$ of the degree zero and Y be a compact metric space. Then the following statements are equivalent:*

1. *the trivial motion of φ is attracting;*
2. *the skew-product dynamical system $(X, \mathbb{T}_+, \pi)$ generated by cocycle φ ($X := \mathbb{R}^n \times Y$ and $\pi = (\varphi, \sigma)$) and its Levinson center $J \subseteq \Theta := \{0\} \times Y$.*

Proof. To prove this statement it is sufficient to show that 1. implies 2. Indeed, by Lemma 8 we have $W^s_y(0) = \mathbb{R}^n \times Y$ for any $y \in Y$. Since the space Y is compact, then the skew-product dynamical system $(X, \mathbb{T}_+, \pi)$ ($X = \mathbb{R}^n \times Y$ and $\pi = (\varphi, \sigma)$) is pointwise dissipative. Since the phase space $X = \mathbb{R}^n \times Y$ is locally compact, then by Theorem 1 the dynamical system $(X, \mathbb{T}_+, \pi)$ is compactly dissipative. Denote by J its Levinson center. Since J is a compact subset of X, then there exists a positive number γ_0 such that $J \subseteq B[0, \gamma_0] \times Y$, where $B[0, \gamma_0] := \{u |\ |u| \leq \gamma_0\}$. Now we will show that $J \subseteq \Theta$. If we suppose that it is not true, then there exists a point $x_0 = (u_0, y_0) \in J \setminus \Theta$. This means that $u_0 \neq 0$ and through the point x_0 passes a full trajectory $\{\pi(t, x_0) = (\varphi(t, u_0, y_0), \sigma(t, y_0)|\ t \in \mathbb{S}\}$ which belongs to J. Since the cocycle φ is r-homogeneous of the degree zero, then

$$\varphi(t, \Lambda^r_\mu u_0, y_0) = \Lambda^r_\mu \varphi(t, u_0, y_0) \tag{40}$$

for any $t \in \mathbb{S}$. From (40) it follows that the full trajectory $\{(\varphi(t, \Lambda^r_\mu u_0, y_0), \sigma(t, y_0)|\ t \in \mathbb{S}\}$ is precompact and, consequently,

$$(\Lambda^r_\mu u_0, y_0) \in J$$

for any $\varepsilon \in (0, +\infty)$. Note that

$$|\Lambda^r_\mu u_0| \geq \mu^\nu |u_0| \tag{41}$$

for any $\mu > 0$, where $\nu = \min\{r_1, \ldots, r_n\} > 0$. Passing to the limit in (41) as $\mu \to +\infty$ we conclude that the set J is not compact. This contradicts the fact that the Levinson center is the maximal compact invariant set of $(X, \mathbb{T}_+, \pi)$. The obtained contradiction proves our statement. Theorem is completely proved. □

Theorem 6. *Let φ be an r homogeneous cocycle over $(Y, \mathbb{T}, \sigma)$ of the degree zero and Y be a compact metric space. Then the trivial motion $u = 0$ of the cocycle φ is asymptotically stable if and only if it is uniformly asymptotically stable.*

Proof. To prove this statement it is sufficient to show that the asymptotic stability of the trivial motion $u = 0$ of φ implies its uniformly asymptotic stability. Assume that the trivial motion $u = 0$ of the cocycle φ is asymptotically stable. Then by Theorem 5 the skew-product dynamical system $(X, \mathbb{T}_+, \pi)$ generated by cocycle φ $(X := \mathbb{R}^n \times Y,\ \pi = (\varphi, \sigma))$ and its Levinson center $J \subseteq \Theta := \{0\} \times Y$. Let γ be an arbitrary positive number, then

$$\lim_{t \to +\infty} \sup_{|u| \le \gamma, y \in Y} |\varphi(t, u, y)| = 0.$$

Suppose that it is not true, then there exist positive numbers ε_0, γ_0 and sequences $\{u_k\}$ (with $|u_k| \le \gamma_0$ for any $k \in \mathbb{N}$), $\{y_k\} \subset Y$ and $t_k \ge k$ such that

$$|\varphi(t_k, u_k, y_k)| \ge \varepsilon_0 \tag{42}$$

for any $k \in \mathbb{N}$. Since the set $K_0 := B[0, \gamma_0] \times Y$ is compact and the skew-product dynamical system $(X, \mathbb{T}_+, \pi)$ is compactly dissipative, then without loss of generality we may assume that the sequences $\{u_k\}$, $\{y_k\}$, $\{\sigma(t_k, y_k)\}$ and $\{\varphi(t_k, u_k, y_k)\}$ are convergent. Denote by $\bar{y} = \lim\limits_{k \to \infty} \sigma(t_k, y_k)$ and

$$\bar{u} = \lim_{k \to \infty} \varphi(t_k, u_k, y_k). \tag{43}$$

It is clear $\pi(t_k, (u_k, y_k)) = (\varphi(t_k, u_k, y_k), \sigma(t_k, y_k)) \in \Sigma^+_{K_0} := \bigcup \{\pi(t, K_0) |\ t \ge 0\}$ and $(\bar{u}, \bar{y}) \in \omega(K_0) \subseteq J \subseteq \Theta := \{0\} \times Y$. This means, in particular, that

$$|\bar{u}| = 0. \tag{44}$$

On the other hand passing to the limit in (42) as $k \to \infty$ and taking into account (43) we obtain

$$|\bar{u}| \ge \varepsilon_0 > 0$$

which contradicts (44). The obtained contradiction proves our statement. Theorem is completely proved. □

Theorem 7. *Let φ be an r homogeneous cocycle over $(Y, \mathbb{T}, \sigma)$ of the degree zero and Y be a compact metric space.*

Then the trivial motion $u = 0$ of the cocycle φ is asymptotically stable if and only if it is attracting.

Proof. To prove this statement it is sufficient to show that under the conditions of Theorem if the trivial motion $u = 0$ of the cocycle φ is attracting, then it is asymptotically stable. If we suppose that it is not true, then there are $\varepsilon_0 > 0$, $\delta_k \to 0$ $(\delta_k > 0)$ and $t_k \to +\infty$ as $k \to \infty$, $u_k \in \mathbb{R}^n$ and $y_k \in Y$ such that

$$|u_k| \le \delta_k \quad \text{and} \quad |\varphi(t_k, u_k, y_k)| \ge \varepsilon_0. \tag{45}$$

Reasoning as in the proof of Theorem 6 we can suppose that the sequence $\{\varphi(t_k, u_k, y_k)\}$ converges. Denote its limit by $\bar{u} = \lim_{k\to\infty} \varphi(t_k, u_k, y_k)$. Passing to the limit in (45) as $k \to \infty$ we obtain $\bar{u} \neq 0$. On the other hand $(\bar{u}, \bar{y}) \in J \subseteq \Theta = \{0\}\times Y$ (see the proof of Theorem 6) and, consequently, $\bar{u} = 0$. The obtained contradiction completes the proof of Theorem. □

Corollary 5. *Let $r \in (0, +\infty)^n$ and φ be an r homogeneous cocycle over $(Y, \mathbb{T}, \sigma)$ with the fibre $\mathbb{R}^n$. If the space is compact, then the following statements are equivalent:*

1. *the trivial motion $u = 0$ of the cocycle φ is asymptotically stable;*
2. *the skew-product dynamical system $(X, \mathbb{T}_+, \pi)$ generated by φ is pointwise dissipative.*

Proof. This statement follows from Theorems 4 and 7. □

Lemma 9. *Let $\langle \mathbb{R}^n, \varphi, (Y, \mathbb{T}, \sigma)\rangle$ be a cocycle over $(Y, \mathbb{T}, \sigma)$ with the fibre $\mathbb{R}^n$, then the following statements hold:*

1. *the trivial motion $u = 0$ of the cocycle φ is positively uniformly stable if and only if for any $\varepsilon > 0$ there exists $\delta(\varepsilon) > 0$ such that $\rho(u) < \delta$ implies $\rho(\varphi(t, u, y)) < \varepsilon$ for any $(t, u) \in \mathbb{T}_+ \times Y$;*
2. $\lim_{t\to+\infty} |\varphi(t, u, y)| = 0$ *if and only if* $\lim_{t\to+\infty} \rho(\varphi(t, u, y)) = 0$.

Proof. Assume that the trivial motion of the cocycle φ is positively uniformly stable, then for arbitrary $\varepsilon > 0$ there exists $\delta(\varepsilon) > 0$ such that $\rho(u) < \delta$ implies $\rho(\varphi(t, u, y)) < \varepsilon$ for any $(t, u) \in \mathbb{T}_+ \times Y$. If we suppose that it is not true, then there exist $\varepsilon_0 > 0$, $\delta_k \to 0$ $(\delta_k > 0)$, $\rho(u_k) < \delta_k$ $(u_k \in \mathbb{R}^n)$, $(t_k, y_k) \in \mathbb{T}_+ \times Y$ such that

$$\rho(\varphi(t_k, u_k, y_k)) \geq \varepsilon_0 \tag{46}$$

for any $k \in \mathbb{N}$. Let $a, b \in \mathcal{K}_\infty$ be the functions figuring in (3), then from (3) and (46) we obtain

$$0 < a(\varepsilon_0) \leq a(\rho(\varphi(t_k, u_k, y_k))) \leq |\varphi(t_k, u_k, y_k)|. \tag{47}$$

On the other hand by positively uniform stability of trivial motion for φ we can choose a positive number $\delta(\varepsilon_0)$ such that

$$|\varphi(t, u, y)| < a(\varepsilon_0)$$

for any $|u| < \delta(\varepsilon_0)$ and $(t, y) \in \mathbb{T}_+ \times Y$. Note that $|u_k| \leq b(\rho(u_k)) < b(\delta_k) \to 0$ as $k \to \infty$ and, consequently, there exists a number $k_0 \in \mathbb{N}$ such that $|u_k| < \delta(\varepsilon_0)$ for any $k \geq k_0$. Thus we have

$$|\varphi(t, u_k, y)| < a(\varepsilon_0) \tag{48}$$

for any $k \geq k_0$ and $(t, y) \in \mathbb{T}_+ \times Y$. In particular, from (48) we receive

$$|\varphi(t_k, u_k, y_k)| < a(\varepsilon_0) \tag{49}$$

for any $k \geq k_0$. The inequalities (47) and (49) are contradictory. The obtained contradiction proves our statement. The converse statement can be proved using absolutely the same arguments as above.

Let $(u, y) \in \mathbb{R}^n \times Y$ be so that

$$\lim_{t \to +\infty} |\varphi(t, u, y)| = 0. \tag{50}$$

Since $a(\rho(\varphi(t, u, y))) \leq |\varphi(t, u, y)|$, then

$$\rho(\varphi(t, u, y)) \leq a^{-1}(|\varphi(t, u, y)|) \tag{51}$$

for any $(t, u, y) \in \mathbb{T}_+ \times \mathbb{R}^n \times Y$. Passing to the limit in (51) as $t \to +\infty$ and taking into account (50) we obtain $\lim_{t \to +\infty} \rho(\varphi(t, u, y)) = 0$. Then we have $|\varphi(t, u, y)| \leq b(\rho(\varphi(t, u, y)))$ and, consequently, $\lim_{t \to +\infty} |\varphi(t, u, y)| = 0$. Lemma is completely proved. □

Theorem 8. *Assume that the following conditions are fulfilled:*

1. *the cocycle $\langle \mathbb{R}^n, \varphi, (Y, \mathbb{T}, \sigma) \rangle$ is r-homogeneous of the degree zero;*
2. *the space Y is compact.*

Then the following statements are equivalent:

a. *the trivial motion of the cocycle φ is asymptotically stable;*

b. *there exit positive numbers $\mathcal{N}$ and ν such that*

$$\rho(\varphi(t, u, y)) \leq \mathcal{N} e^{-\nu t} \rho(u)$$

for any $u \in \mathbb{R}^n$, $y \in Y$ and $t \geq 0$.

Proof. To prove the theorem it is sufficient to establish the implication $a. \Rightarrow b.$, since the converse statement is obvious.

Since the cocycle φ is r homogeneous of the degree zero and the trivial motion $u = 0$ is attracting, then from Lemmas 9 and 6 we have $W^s_y(0) = \mathbb{R}^n$ for any $y \in Y$. Consider the skew-product dynamical system $(X, \mathbb{T}_+, \pi)$ generated by the cocycle φ ($X := \mathbb{R}^n \times Y$ and $\pi := (\varphi, \sigma)$). Taking into account that Y is a compact space and $W^s_y(0) = \mathbb{R}^n$ (for any $y \in Y$) according to Theorem 5 we conclude that the dynamical system $(X, \mathbb{T}_+, \pi)$ is compactly dissipative and its Levinson center $J \subseteq \Theta := \{0\} \times Y$. This means that for any compact subset $K \subset X = \mathbb{R}^n \times Y$ the following statements hold:

1.
$$M(K) := \sup_{(t,u,y)\in \mathbb{T}_+\times K} |\varphi(t,u,y)| < +\infty;$$

2.
$$m_K(t) := \sup_{(u,y)\in K} |\varphi(t,u,y)| \to 0$$

as $t \to +\infty$.

Note that $S_{r,p}\times Y$ is a compact subset of $X = \mathbb{R}^n \times Y$, because $S_{r,p}$ is a compact subset of $\mathbb{R}^n$. Denote by

$$m(t) := \sup_{(u,y)\in S_{r,p}\times Y} \rho(\varphi(t,u,y)) \tag{52}$$

and

$$M := \sup_{(t,u,y)\in \mathbb{T}_+\times S_{r,p}\times Y} \rho(\varphi(t,u,y)). \tag{53}$$

Let a,b be the functions from $\mathcal{K}_\infty$ figuring in (3), then we obtain

$$\rho(\varphi(t,u,y)) \le a^{-1}(|\varphi(t,u,y)|) \le a^{-1}(M(S_{r,p})) \tag{54}$$

and

$$\rho(\varphi(t,u,y)) \le a^{-1}(|\varphi(t,u,y)|) \le a^{-1}(m_{S_{r,p}}(t)) \tag{55}$$

for any $t\in\mathbb{T}_+$, $u\in S_{r,p}$ and $y\in Y$. From (52)-(55) we have the following statements:

1. $0 < m(t) \le M$ for any $t\in\mathbb{T}_+$;
2. $m(t)\to 0$ as $t\to+\infty$.

From Lemma 6 (item (ii)) we obtain

$$\rho(\varphi(t,u,y)) \le m(t)\rho(u)$$

for any $t\in\mathbb{T}_+$ and $u\neq 0$, where

$$m(t) := \sup\{\rho(\varphi(t,u,y))|\ (u,y)\in S_{r,p}\times Y\}.$$

Indeed, $\Lambda^r_{\rho(u)^{-1}}u \in S_{r,p}$ for any $u\neq 0$ and, consequently,

$$\rho(\varphi(t,\Lambda^r_{\rho(u)^{-1}}u,y)) \le \sup_{(v,y)\in S_{r,p}\times Y} \rho(\varphi(t,v,y)) = m(t) \tag{56}$$

for any $u\neq 0$ and $(t,y)\in\mathbb{T}_+\times Y$. In particular from (56) we obtain

$$\rho(\varphi(t,\Lambda^r_{\rho(\varphi(\tau,u,y))^{-1}}\varphi(\tau,u,y),\sigma(\tau,y))) \le \sup_{(\tilde{u},\tilde{y})\in S_{r,p}\times Y} \rho(\varphi(t,\tilde{u},\tilde{y}) = m(t)$$

for any $t,\tau\in\mathbb{T}_+$ and $(u,y)\in S_{r,p}\times Y$.

Finally, by the equality (13) we have

$$m(t+\tau) = \sup_{(u,y)\in S_{r,p}\times Y} \rho(\varphi(t+\tau,u,y)) =$$
$$\sup_{(u,y)\in S_{r,p}\times Y} \rho(\varphi(\tau,u,y))\rho(\varphi(t,\Lambda^r_{\mu^{-1}}\varphi(\tau,u,y),\sigma(\tau,y))) \le$$
$$\sup_{(u,y)\in S_{r,p}\times Y} \rho(\varphi(\tau,u,y)) \times \sup_{(u,y)\in S_{r,p}\times Y} \rho(\varphi(t,\Lambda^r_{\mu^{-1}}\varphi(\tau,u,y),\sigma(\tau,y))) \le m(\tau)m(t)$$

because

$$\Lambda^r_{\mu^{-1}}\varphi(\tau,u,y) \in S_{r,p}$$

if $\mu = \rho(\varphi(\tau,u,y))$ and

$$\sup_{(u,y)\in S_{r,p}\times Y} \rho(\varphi(t,\Lambda^r_{\mu^{-1}}\varphi(\tau,u,y),\sigma(\tau,y))) \le \sup_{(\tilde{u},\tilde{y})\in S_{r,p}\times Y} \rho(\varphi(t,\tilde{u},\tilde{y})) = m(t).$$

By Lemma 3 there exist positive numbers $\mathcal{N}$ and ν such that $m(t) \le \mathcal{N}e^{-\nu t}$ for any $t \in \mathbb{T}_+$. ☐

5 Asymptotic Stability of Nonautonomous Generalized Homogeneous Dynamical Systems: The Case of the Compact and Minimal Phase Space of Driving System

In this Section we suppose that the complete metric space Y is compact and the dynamical system $(Y,\mathbb{T},\sigma)$ is minimal, i.e., every trajectory $\Sigma_y := \{\sigma(t,y) :\ t \in \mathbb{T}\}$ is dense in Y (this means that $H(y) = Y$ for all $y \in Y$, where $H(y) := \overline{\Sigma}_y$).

Theorem 9. *[6, Ch.II, pp.94-95] Let $\langle \mathbb{R}^n, \varphi, (Y,\mathbb{S},\sigma)\rangle$ be a cocycle over two-sided dynamical system $(Y,\mathbb{S},\sigma)$ with the fibre $\mathbb{R}^n$. Assume that the following conditions are fulfilled:*

1. *the trivial motion $u = 0$ of the cocycle φ is uniformly stable;*
2. *there exist positive number δ_0 and point $y_0 \in Y$ such that $B(0,\delta_0) \subset W^s_{y_0}$, where $B(0,r) := \{u \in \mathbb{R}^n |\ |u| < r\}$.*

Then the trivial motion $u = 0$ of the cocycle φ is asymptotically stable, i.e., there exists a positive number β such that $B(0,\beta) \subset W^s_y(0)$ for any $y \in Y$.

Theorem 10. *Let $\langle \mathbb{R}^n, \varphi, (Y,\mathbb{S},\sigma)\rangle$ be a cocycle over two-sided dynamical system $(Y,\mathbb{S},\sigma)$ with the fibre $\mathbb{R}^n$. Assume that the following conditions are fulfilled:*

1. *the cocycle $\langle \mathbb{R}^n, \varphi, (Y,\mathbb{T},\sigma)\rangle$ is r-homogeneous of the degree zero;*
2. *the trivial motion $u = 0$ of the cocycle φ is stable;*
3. *there exit a point $y_0 \in Y$ and positive number δ_{y_0} such that $B(0,\delta_{y_0}) \subset W^s_{y_0}(0)$.*

Then the trivial motion $u = 0$ of the cocycle φ is globally uniformly asymptotically stable, i.e., $W^s_y(0) = \mathbb{R}^n$ for any $y \in Y$.

Proof. By Theorem 9 there exists a positive number δ_0 such that $B(0, \delta_0) \subset W^s_y(0)$ for any $y \in Y$. According to Lemma 8 we have $W^s_y(0) = \mathbb{R}^n$ for any $y \in Y$. Theorem is proved. □

Theorem 11. *Let $\langle \mathbb{R}^n, \varphi, (Y, \mathbb{S}, \sigma)\rangle$ be an r-homogeneous cocycle of the degree zero over two-sided dynamical system $(Y, \mathbb{S}, \sigma)$.*

Then the following statements are equivalent:

1. *the trivial motion $u = 0$ of the cocycle φ is uniformly stable and there exists a point $y_0 \in Y$ and positive number δ_{y_0} such that $B(0, \delta_{y_0}) \subset W^s_{y_0}(0)$;*
2. *there exist positive numbers $\mathcal{N}$ and ν such that $\rho(\varphi(t, u, y)) \leq \mathcal{N} e^{-\nu t} \rho(u)$ for any $u \in \mathbb{R}^n, y \in Y$ and $t \geq 0$.*

Proof. According to Theorem 10 under the conditions of Theorem 11 the trivial motion $u = 0$ of the cocycle φ is (globally) uniformly asymptotically stable. To finish tha proof of Theorem it is sufficient to Apply Theorem 8. □

6 Applications

6.1 Ordinary Differential Equations

Let $\mathbb{R}^n$ be n-dimensional real or complex Euclidean space. Let us consider a differential equation

$$u' = f(t, u), \tag{57}$$

where $f \in C(\mathbb{R} \times \mathbb{R}^n, \mathbb{R}^n)$. Along with the equation (57) we consider its *H-class* [4, 15, 21, 22], i.e., the family of the equations

$$v' = g(t, v), \tag{58}$$

where $g \in H(f) := \overline{\{f^\tau |\ \tau \in \mathbb{R}\}}$, $f^\tau(t, u) = f(t + \tau, u)$ for any $(t, u) \in \mathbb{R} \times \mathbb{R}^n$ and by bar we denote the closure in $C(\mathbb{R} \times \mathbb{R}^n, \mathbb{R}^n)$. We will suppose also that the function f is *regular* [20, ChIV], i.e., for every equation (58) the conditions of existence, uniqueness (on the maximal interval of definition of the solutions) and extendability on $\mathbb{R}_+$ are fulfilled. Denote by $\varphi(t, v, g)$ the solution of equation (58), passing through the point $v \in \mathbb{R}^n$ at the initial moment $t = 0$. Then from the general properties of solutions of *ordinary differential equations (ODEs)* it follows that the mapping $\varphi : \mathbb{R}_+ \times \mathbb{R}^n \times H(f) \to \mathbb{R}^n$ is well defined and it satisfies the following conditions (see for example [4, ChIV] and [20, ChIV]):

1) $\varphi(0, v, g) = v$ for any $v \in \mathbb{R}^n$ and $g \in H(f)$;

2) $\varphi(t, \varphi(\tau, v, g), g^\tau) = \varphi(t + \tau, v, g)$ for every $v \in \mathbb{R}^n$, $g \in H(f)$ and $t, \tau \in \mathbb{R}_+$;

3) the mapping $\varphi : \mathbb{R}_+ \times \mathbb{R}^n \times H(f) \to \mathbb{R}^n$ is continuous.

Denote by $Y := H(f)$ and $(Y, \mathbb{R}, \sigma)$ the dynamical system of translations on Y, induced by the dynamical system of translations $(C(\mathbb{R} \times \mathbb{R}^n, \mathbb{R}^n), \mathbb{R}, \sigma)$. The triplet $\langle \mathbb{R}^n, \varphi, (Y, \mathbb{R}, \sigma)\rangle$ is a cocycle over $(Y, \mathbb{R}_+, \sigma)$ with the fibre $\mathbb{R}^n$. Thus the equation (57) generates a cocycle $\langle \mathbb{R}^n, \varphi, (Y, \mathbb{R}, \sigma)\rangle$ satisfying Condition (C).

Note that under the conditions listed above the equation (57) (respectively, H-class (58)) can be written in the form (2). Indeed, let $Y := H(f)$ and $(Y, \mathbb{R}, \sigma)$ be the dynamical system of translations on Y. Denote by F the mapping from $Y \times \mathbb{R}^n$ into $\mathbb{R}^n$ defined by the equality

$$F(g, u) := g(0, u). \tag{59}$$

It is not difficult to check that the mapping $F : H(f) \times \mathbb{R}^n \to \mathbb{R}^n$ is continuous. Finally, note that we can rewrite the equation (58) as follows

$$u' = F(\sigma(t, g), u) \quad (g \in H(f)). \tag{60}$$

Definition 13. A function $f \in C(\mathbb{R} \times \mathbb{R}^n, \mathbb{R}^n)$ is said to be r homogeneous ($r \in (0, +\infty)^n$) of degree $m \in \mathbb{R}$ if $f(t, \Lambda^r_\varepsilon u) = \lambda^m \Lambda^r_\varepsilon f(t, u)$ for any $(\varepsilon, t, u) \in (0, +\infty) \times \mathbb{R} \times \mathbb{R}^n$.

Remark 5. If the function $f \in C(\mathbb{R} \times \mathbb{R}^n, \mathbb{R}^n)$ is r homogeneous of a degree $m \geq 0$, then $f(t, 0) = 0$ for any $t \in \mathbb{R}$.

Lemma 10. *If the function $f \in C(\mathbb{R} \times \mathbb{R}^n, \mathbb{R}^n)$ is r homogeneous of a degree m, then the mapping $F : Y \times \mathbb{R}^n \to \mathbb{R}^n$ ($Y = H(f)$) defined by the equality (59) is r homogeneous of a degree m with respect to $u \in \mathbb{R}^n$ uniformly in $y \in Y$.*

Proof. Let $g \in H(f)$, then there exists a sequence $\{t_k\} \subset \mathbb{R}$ such that

$$g(t, u) = \lim_{k \to \infty} f(t + t_k, u)$$

uniformly with respect to (t, u) on every compact subset from $\mathbb{R} \times \mathbb{R}^n$. Notice that

$$F(g, \Lambda^r_\varepsilon u) = \lim_{k \to \infty} f(t + t_k, \Lambda^r_\varepsilon u) = \lambda^m \Lambda^r_\varepsilon \lim_{k \to \infty} f(t + t_k, u) = \lambda^m \Lambda^r_\varepsilon F(g, u)$$

for any $(\varepsilon, g, u) \in (0, +\infty) \times H(f) \times \mathbb{R}^n$. Lemma is proved. □

Corollary 6. *Assume that the function $f \in C(\mathbb{R} \times \mathbb{R}^n, \mathbb{R}^n)$ is r homogeneous of the degree zero, then the cocycle $\langle \mathbb{R}^n, \varphi, (H(f), \mathbb{R}, \sigma)\rangle$ generated by the equation (57) is r homogeneous of the degree zero.*

Proof. This statement follows from Lemmas 2 and 10. □

Let $f(t, 0) \equiv 0$ and the function $f \in C(\mathbb{R} \times \mathbb{R}^n, \mathbb{R}^n)$ be regular.

Definition 14. The trivial solution of the equation (57) is said to be:

1. *uniformly stable* if for any positive number ε there exists a number $\delta = \delta(\varepsilon)$ ($\delta \in (0, \varepsilon)$) such that $|x| < \delta$ implies $|\varphi(t, x, f^\tau)| < \varepsilon$ for any $t \in \mathbb{R}_+$ and $\tau \in \mathbb{R}$;

2. *attracting (respectively, uniformly attracting)* if there exists a positive number a such that
$$\lim_{t\to+\infty}|\varphi(t,x,f^{\tau})|=0$$
for any $|x|\le a$ and $\tau\in\mathbb{R}$ (respectively, uniformly with respect to $|x|\le a$ and $t\in\mathbb{R}$);

3. *asymptotically stable (respectively, uniformly asymptotically stable*, if it is uniformly stable and attracting (respectively, uniformly attracting).

Remark 6. If the function $f\in C(\mathbb{R}\times\mathbb{R}^n,\mathbb{R}^n)$ is regular and $f(t,0)=0$ for any $t\in\mathbb{R}$, then it is easy to show that the trivial solution of equation (57) is uniformly attracting if and only if there exists a positive number a such that
$$\lim_{t\to+\infty}\sup_{|x|\le a,\ g\in H(f)}|\varphi(t,u,g)|=0. \tag{61}$$

Remark 7. 1. Note that from the results given in the works [1, 19] it follows the equivalence of standard definition (see, for example,[12, Ch.V]) of the uniform stability (respectively, global uniform asymptotically stability) and of the one given above for the equation (57) with regular right hand side.

2. From the results of G. Sell [19,20] it follows that for the differential equations (57) with the regular and Lagrange stable right hand side f the following statements are equivalent:

1. the trivial solution of equation (57) is uniformly asymptotically stable;
2. the trivial motion of the cocycle $\langle\mathbb{R}^n,\varphi,(H(f),\mathbb{R},\sigma)\rangle$ generated by (57) is uniformly asymptotically stable.

Theorem 12. *Let $f\in C(\mathbb{R}\times\mathbb{R}^n,\mathbb{R}^n)$. Assume that the following conditions are fulfilled:*

1. *the function f is regular and $f(t,0)=0$ for any $t\in\mathbb{R}$;*
2. *the function f is r homogeneous of the degree zero.*

Then the following statements are equivalent:

1. *the trivial solution of the equation (57) is uniformly asymptotically stable;*
2. *there exit positive numbers $\mathcal{N}$ and ν such that*
$$\rho(\varphi(t,u,g))\le\mathcal{N}e^{-\nu t}\rho(u)$$
for any $u\in\mathbb{R}^n$, $g\in H(f)$ and $t\ge 0$, where $\rho(u)=|u|_{r,p}$.

Proof. Let $Y := H(f)$ and $(Y, \mathbb{R}, \sigma)$ be the shift dynamical system on $Y = H(f)$. Denote by $\langle \mathbb{R}^n, \varphi, (H(f), \mathbb{R}, \sigma)\rangle$ (shortly φ) the cocycle generated by the differential equation (57). Since the function f is r homogeneous of the degree zero, then by Corollary 6 the cocycle φ generated by the equation (57) is r homogeneous of the degree zero. To finish the proof of Theorem 12 it is sufficient to take into account Remarks 6–7 and apply Theorem 3. □

Theorem 13. *Let $f \in C(\mathbb{R} \times \mathbb{R}^n, \mathbb{R}^n)$ be a regular function. Assume that the following conditions are fulfilled:*

1. *$f(t, 0) = 0$ for any $t \in \mathbb{R}$;*
2. *the function f is r homogeneous of the degree zero and Lagrange stable.*

Then the following statements are equivalent:

1. *the trivial solution of the equation (57) is asymptotically stable;*
2. *there exit positive numbers $\mathcal{N}$ and ν such that*
$$\rho(\varphi(t, u, g)) \leq \mathcal{N} e^{-\nu t} \rho(u)$$
for any $u \in \mathbb{R}^n$, $g \in H(f)$ and $t \geq 0$.

Proof. Let $Y := H(f)$ and $(Y, \mathbb{R}, \sigma)$ be the shift dynamical system on $Y = H(f)$. Note that the space Y is compact because the function f is Lagrange stable. Since the function f is r homogeneous of the degree zero, then by Corollary 6 the cocycle $\langle \mathbb{R}^n, \varphi, (H(f), \mathbb{R}, \sigma)\rangle$ generated by the equation (57) is r homogeneous of the degree zero. To finish the proof of Theorem 13 it suffices to take into account Remark 7 and apply Theorem 8. □

Remark 8. 1. If the function f is τ-periodic, then the equivalence of the conditions (i) and (ii) was established in the work [17].

2. If the function f is homogeneous of the degree zero (in the classical sense, i.e., $f(t, \varepsilon x) = \varepsilon f(t, x)$ for any $\varepsilon > 0$ and $(t, x) \in \mathbb{R} \times \mathbb{R}^n$), then the equivalence of the uniform asymptotically stability and exponential stability was established in the work [12, Ch.VII]. If the function f is r homogeneous of the degree zero the equivalence of the uniform asymptotic stability and exponential stability was established in the work [9]

Recall that the function $f \in C(\mathbb{T} \times \mathbb{R}^n, \mathbb{R}^n)$ is said to be recurrent in time if the motion $\sigma(t, f)$ generated by f in the shift dynamical system $(C(\mathbb{T} \times \mathbb{R}^n, \mathbb{R}^n), \mathbb{T}, \sigma)$ is recurrent.

Remark 9. Note that the function f is recurrent in time if and only if its hull $H(f)$ is a compact and minimal set of the shift dynamical system $(C(\mathbb{T} \times \mathbb{R}^n, \mathbb{R}^n), \mathbb{T}, \sigma)$ (see for example [8, Ch.I]).

Theorem 14. *Let $f \in C(\mathbb{R} \times \mathbb{R}^n, \mathbb{R}^n)$ be a regular function. Assume that the following conditions are fulfilled:*

1. *the function f is recurrent in time and $f(t,0)=0$ for any $t\in\mathbb{R}$;*
2. *the function f is r homogeneous of the degree zero.*

Then the following statements are equivalent:

1. *the trivial solution of equation (57) is uniformly stable and there exists a positive number a such that*
$$\lim_{t\to+\infty}|\varphi(t,u,f)|=0 \tag{62}$$
for any $u\in B[0,a]:=\{u\in\mathbb{R}^n|\ |u|\le a\}$;
2. *there exit positive numbers $\mathcal{N}$ and ν such that*
$$\rho(\varphi(t,u,g))\le\mathcal{N}e^{-\nu t}\rho(u)$$
for any $u\in\mathbb{R}^n$, $g\in H(f)$ and $t\ge 0$.

Proof. Let $Y:=H(f)$ and $(Y,\mathbb{R},\sigma)$ be the shift dynamical system on $Y=H(f)$. Note that the space Y is a compact and minimal set because the function f is recurrent in time (see Remark 9). Since the function f is r homogeneous of the degree zero, then by Corollary 6 the cocycle $\langle\mathbb{R}^n,\varphi,(H(f),\mathbb{R},\sigma)\rangle$ generated by the equation (57) is r homogeneous of the degree zero. To finish the proof of Theorem 14 it is sufficient to take into account Remark 7 and to apply Theorem 11. □

Here is an example illustrating the theorems proved in this subsection.

Example 2. Denote by $C(\mathbb{R},\mathbb{R})$ the space of all continuous functions $\psi:\mathbb{R}\to\mathbb{R}$ equipped with the compact-open topology and $(C(\mathbb{R},\mathbb{R}),\mathbb{R},\sigma)$ the shift dynamical system on $C(\mathbb{R},\mathbb{R})$. Consider the system of differential equations
$$\begin{cases}\dot{x}_1=-x_1+p(t)\sqrt{|x_2|}\\ \dot{x}_2=-x_2\end{cases}, \tag{63}$$
where $p\in C(\mathbb{R},\mathbb{R})$.

Note that the function $F\in C(\mathbb{R}\times\mathbb{R}^2,\mathbb{R}^2)$, where $F(t,x):=(-x_1+p(t)\sqrt{|x_2|},-x_2)$ and $x:=(x_1,x_2)$, is $r=(1,2)$ homogeneous. This means that $F(t,\Lambda_\mu x))=\Lambda_\mu F(t,x)$ for any $(t,\mu,x)\in\mathbb{R}\times(0,+\infty)\times\mathbb{R}^n$, where $\Lambda_\mu x=(\mu x_1,\mu^2x_2)$.

Recall that the function p is called Lagrange stable if the set $H(p):=\overline{\{p^h|\ h\in\mathbb{R}\}}$ ($p^h(t):=p(t+h)$ for any $t\in\mathbb{R}$) is a compact subset of $C(\mathbb{R},\mathbb{R})$.

Along this the system (63) we consider its H-class, i.e., the family of systems of differential equations
$$\begin{cases}\dot{x}_1=-x_1+q(t)\sqrt{|x_2|}\\ \dot{x}_2=-x_2\end{cases}\qquad (q\in H(p)). \tag{64}$$

Denote by $Y:=H(p)$, $(Y,\mathbb{R},\sigma)$ the shift dynamical system on $Y=H(p)$ and $\varphi(t,u,q)$ the unique solution of the system (64) passing through the point $u\in\mathbb{R}^2$ at the initial moment $t=0$. Then $\langle\mathbb{R}^2,\varphi,(Y,\mathbb{R},\sigma)$ is a cocycle over $(Y,\mathbb{R},\sigma)$ with the fibre $\mathbb{R}^2$.

Lemma 11. *Assume that the function p is Lagrange stable, then the skew-product dynamical system $(X,\mathbb{R}_+,\pi)$ generated by the cocycle φ ($X := \mathbb{R}^2 \times Y$ and $\pi = (\varphi,\sigma)$) is pontwise dissipative.*

Proof. Consider a function $V : \mathbb{R}^2 \times Y \to \mathbb{R}_+$ defined by the equality

$$V(u_1,u_2,q) := u_1^2 + u_2^2$$

for any $(u_1,u_2,q) \in \mathbb{R}^2 \times H(p)$. Note that

$$\frac{dV}{dt}\Big|_{t=0} := \lim_{t\to 0^+} \frac{V(\pi(t,x)) - V(x)}{t} = -2(u_1^2+u_2^2) + 2q(0)u_1\sqrt{|u_2|}. \tag{65}$$

Since the function p is bounded, then there exists a positive number R_0 such that

$$-2(u_1^2+u_2^2) + 2q(0)u_1\sqrt{|u_2|} \le -{u_1}^2 - {u_2}^2 \tag{66}$$

for any $|u| := (u_1^2+u_2^2)^{1/2} \ge R_0$. From (65) and (66) we obtain

$$\frac{dV}{dt}\Big|_{t=0} \le -u_1^2 - u_2^2$$

for any $|u| := (u_1^2+u_2^2)^{1/2} \ge R_0$. According to Theorem 5.3 from [7, Ch.V] the skew-product dynamical system $(X,\mathbb{R}_+,\pi)$ generated by the cocycle φ is pointwise dissipative. Lemma is proved. □

Corollary 7. *The trivial motion $u = 0$ of the cocycle φ generated by the system (63) is attracting.*

Proof. This statement follows from Lemma 11 and Theorem 4. □

Corollary 8. *If the function p is Lagrange stable, then there are positive numbers $\mathcal{N}$ and ν such that*

$$\rho(\varphi(t,u,q)) \le \mathcal{N}e^{-\nu t}\rho(u)$$

for any $(t,u,q) \in \mathbb{R}_+ \times \mathbb{R}^2 \times H(p)$, where $\rho(u) := (u_1^4 + u_2^2)^{1/4}$.

Proof. This statement follows from Corollary 7 and Theorems 13 and 7. □

6.2 Difference Equations

6.2.1 Discrete Nonautonomous Dynamical Systems

Definition 15. Let $\mathbb{T} \subseteq \mathbb{Z}$ and $(\mathbb{R}^n,\mathbb{T},\lambda)$ be a discrete linear dynamical system on $\mathbb{R}^n$. A function $F \in C(Y \times \mathbb{R}^n,\mathbb{R}^n)$ is said to be λ-homogeneous if

$$F(y,\lambda(\tau,w)) = \lambda(\tau,F(y,w)) \text{ (or equivalently } F(y,\lambda^\tau w) = \lambda^\tau F(y,w))$$

for any $(y,\tau,w) \in Y \times \mathbb{T} \times \mathbb{R}^n$.

Consider the difference equation

$$u(t+1) = F(\sigma(t,y), u(t)),\ (y \in Y) \tag{67}$$

where $F \in C(Y \times \mathbb{R}^n, \mathbb{R}^n)$. We will suppose also that the function F is *regular*, i.e., for every equation (67) the conditions of existence and uniqueness (on the maximal interval of definition of solutions) are fulfilled. Denote by $\varphi(t,u,y)$ the unique solution of the equation (67) with the initial data $\varphi(0,u,y) = u$, then the continuous mapping $\varphi : \mathbb{Z}_+ \times \mathbb{R}^n \times Y \to \mathbb{R}^n$ satisfying the condition $\varphi(t+\tau,u,y) = \varphi(t,\varphi(\tau,u,y),\sigma(\tau,y))$ for any $t,\tau \in \mathbb{Z}_+$ and $(u,y) \in \mathbb{R}^n \times Y$ is well defined. Then the triplet $\langle \mathbb{R}^n, \varphi, (Y,\mathbb{Z},\sigma)\rangle$ is a cocycle generated by (67) and satisfying Condition (C).

Lemma 12. *Assume that the function $F \in C(Y \times \mathbb{R}^n, \mathbb{R}^n)$ is λ-homogeneous, then the cocycle $\langle \mathbb{R}^n, \varphi, (Y,\mathbb{Z},\sigma)\rangle$ generated by the equation (67) is λ-homogeneous.*

Proof. To prove this statement we consider the function $\psi(t) := \lambda^\tau \varphi(t,u,y)$. It is easy to check that

$$\psi(t+1) = \lambda^\tau \varphi(t+1,u,y) = \lambda^\tau F(\sigma(t,y), \varphi(t,u,y)) = \\ F(\sigma(t,y), \lambda^\tau \varphi(t,u,y)) = F(\sigma(t,y), \psi(t))$$

for any $t \in \mathbb{Z}_+$. Since $\psi(0) = \lambda^\tau u$, then we obtain $\psi(t) = \varphi(t, \lambda^\tau u, y)$, i.e., $\lambda^\tau \varphi(t,u,y) = \varphi(t,\lambda^\tau u, y)$ for any $t,\tau \in \mathbb{Z}_+$ and $(u,y) \in \mathbb{R}^n \times Y$. Lemma is proved. □

6.2.2 Homogeneous Difference Equations

Let us consider a difference equation

$$u(t+1) = f(t, u(t)), \tag{68}$$

where $f \in C(\mathbb{Z} \times \mathbb{R}^n, \mathbb{R}^n)$. Along with equation (68) we consider its *H-class* [4, 15, 21, 22], i.e., the family of equations

$$v(t+1) = g(t, v(t)), \tag{69}$$

where $g \in H(f) := \overline{\{f^\tau |\ \tau \in \mathbb{Z}\}}$, $f^\tau(t,u) = f(t+\tau, u)$ for any $(t,u) \in \mathbb{Z} \times \mathbb{R}^n$ and by bar we denote the closure in $C(\mathbb{Z} \times \mathbb{R}^n, \mathbb{R}^n)$. Assume that the function $f \in C(\mathbb{Z} \times \mathbb{R}^n, \mathbb{R}^n)$ is regular, that is, for any $u \in \mathbb{R}^n$ and $g \in H(f)$ the equation (69) has a unique (on the maximal domain of definition) solution $\varphi(t,v,g)$ passing through the point $v \in \mathbb{R}^n$ at the initial moment $t = 0$. Then from the general properties of solutions of *difference equations (DEs)* it follows that the mapping $\varphi : \mathbb{Z}_+ \times \mathbb{R}^n \times H(f) \to \mathbb{R}^n$ is well defined and it satisfies the following conditions (see for example [4, ChIV] and [20, ChIV]):

1) $\varphi(0,v,g) = v$ for any $v \in \mathbb{R}^n$ and $g \in H(f)$;

2) $\varphi(t,\varphi(\tau,v,g),g^{\tau}) = \varphi(t+\tau,v,g)$ for every $v\in\mathbb{R}^n$, $g\in H(f)$ and $t,\tau\in\mathbb{Z}_+$;

3) the mapping $\varphi:\mathbb{Z}_+\times\mathbb{R}^n\times H(f)\to\mathbb{R}^n$ is continuous.

Denote by $Y := H(f)$ and $(Y,\mathbb{Z},\sigma)$ the dynamical system of translations on Y induced by the dynamical system of translations $(C(\mathbb{Z}\times\mathbb{R}^n,\mathbb{R}^n),\mathbb{Z},\sigma)$. The triplet $\langle\mathbb{R}^n,\varphi,(Y,\mathbb{Z},\sigma)\rangle$ is a cocycle over $(Y,\mathbb{Z},\sigma)$ with the fibre $\mathbb{R}^n$. Thus equation (68) generates a cocycle $\langle\mathbb{R}^n,\varphi,(Y,\mathbb{Z},\sigma)\rangle$. Note that under the conditions listed above the equation (68) (respectively, H-class (69)) can be written in the form

$$u(t+1) = F(\sigma(t,y),u(t)) \quad (y\in Y = H(f)). \tag{70}$$

Indeed, let $Y := H(f)$ and $(Y,\mathbb{Z},\sigma)$ be the dynamical system of translations on Y. Denote by F the mapping from $Y\times\mathbb{R}^n$ into $\mathbb{R}^n$ defined by the equality

$$F(g,u) := g(0,u). \tag{71}$$

It is easy to check that the mapping $F: H(f)\times\mathbb{R}^n\to\mathbb{R}^n$ is continuous.

Definition 16. A function $f\in C(\mathbb{Z}\times\mathbb{R}^n,\mathbb{R}^n)$ is said to be r homogeneous ($r\in(0,+\infty)^n$) of the degree zero if $f(t,\Lambda_\mu^r u) = \Lambda_\mu^r f(t,u)$ for any $(\mu,t,u)\in(0,+\infty)\times\mathbb{Z}\times\mathbb{R}^n$.

Remark 10. If the function $f\in C(\mathbb{Z}\times\mathbb{R}^n,\mathbb{R}^n)$ is r homogeneous of the degree zero, then $f(t,0)=0$ for any $t\in\mathbb{Z}$.

Lemma 13. *If the function $f\in C(\mathbb{Z}\times\mathbb{R}^n,\mathbb{R}^n)$ is r homogeneous of the degree zero, then the mapping $F: Y\times\mathbb{R}^n\to\mathbb{R}^n$ ($Y = H(f)$) defined by the equality (70) is r homogeneous of the degree zero with respect to $u\in\mathbb{R}^n$ uniformly in $y\in Y$.*

Proof. This statement can be proved using the same arguments as in the proof of Lemma 10. □

Corollary 9. *Assume that the function $f\in C(\mathbb{Z}\times\mathbb{R}^n,\mathbb{R}^n)$ is r homogeneous of the degree zero, then the cocycle $\langle\mathbb{R}^n,\varphi,(H(f),\mathbb{Z},\sigma)\rangle$ generated by the equation (68) is r homogeneous of the degree zero.*

Proof. This statement follows from Lemmas 12 and 13. □

6.2.3 Asymptotic Stability of Nonautonomous Difference Equations

Let $f\in C(\mathbb{Z}\times\mathbb{R}^n,\mathbb{R}^n)$ and $f(t,0)\equiv 0$ for any $t\in\mathbb{Z}$.

Definition 17. The trivial solution of equation (68) is said to be:

1. *uniformly stable* if for any positive number ε there exists a number $\delta=\delta(\varepsilon)$ ($\delta\in(0,\varepsilon)$) such that $|x|<\delta$ implies $|\varphi(t,x,f_\tau)|<\varepsilon$ for any $(t,\tau)\in\mathbb{Z}_+\times\mathbb{Z}$;

2. *attracting* (respectively, *uniformly attracting*) if there exists a positive number a such that
$$\lim_{t\to+\infty}|\varphi(t,x,f_\tau)|=0 \tag{72}$$
for any $|x|\le a$ and $\tau\in\mathbb{Z}$;

3. *asymptotically stable* if it is uniformly stable and attracting (respectively, the equality (72) holds uniformly with respect to $|u|\le a$ and $\tau\in\mathbb{Z}$).

Remark 11. If the function $f\in C(\mathbb{Z}\times\mathbb{R}^n,\mathbb{R}^n)$ is regular and $f(t,0)=0$ for any $t\in\mathbb{Z}$, then it is easy to show that the trivial solution of the equation (68) is uniformly attracting if and only if there exists a positive number a such that
$$\lim_{t\to+\infty}\sup_{|x|\le a,\ g\in H(f)}|\varphi(t,u,g)|=0. \tag{73}$$

Remark 12. 1. By slight modifications of the reasoning from the works [1,19] we can establish the equivalence of the standard definition (see for example [11, Ch.V] and [14, Ch.IV]) of uniform stability (respectively, global uniform asymptotic stability) and of the one given above for the difference equation (68).

2. Using the same ideas as in the works of G. Sell [19, 20] we can prove that for the difference equations (68) with the Lagrange stable right hand side f the following statements are equivalent:

1. the trivial solution of the equation (68) is uniformly asymptotically stable;
2. the trivial motion of the cocycle $\langle\mathbb{R}^n,\varphi,(H(f),\mathbb{Z},\sigma)\rangle$ generated by (68) is uniformly asymptotically stable.

Theorem 15. *Let $f\in C(\mathbb{Z}\times\mathbb{R}^n,\mathbb{R}^n)$. Assume that the following conditions are fulfilled:*

1. *the function f is regular and $f(t,0)=0$ for any $t\in\mathbb{Z}$;*
2. *the function f is r homogeneous of the degree zero.*

Then the following statements are equivalent:

1. *the trivial solution of the equation (68) is uniformly asymptotically stable;*
2. *there exit positive numbers $\mathcal{N}$ and ν such that*
$$\rho(\varphi(t,u,g))\le\mathcal{N}e^{-\nu t}\rho(u)$$
for any $u\in\mathbb{R}^n$, $g\in H(f)$ and $t\in\mathbb{Z}_+$.

Proof. Let $Y:=H(f)$ and $(Y,\mathbb{Z},\sigma)$ be the shift dynamical system on $Y=H(f)$. Denote by $\langle\mathbb{R}^n,\varphi,(H(f),\mathbb{Z},\sigma)\rangle$ the cocycle generated by the difference equation (68). Since the function f is r homogeneous of the degree zero, then by Corollary 6 the cocycle φ generated by the equation (68) is r homogeneous of the degree zero. To finish the proof of Theorem 15 it suffices to take into account Remarks 11 – 12 and apply Theorem 3. $\square$

Theorem 16. *Let $f \in C(\mathbb{Z}\times\mathbb{R}^n,\mathbb{R}^n)$. Assume that the following conditions are fulfilled:*

1. *$f(t,0)=0$ for any $t\in\mathbb{Z}$;*
2. *the function f is r homogeneous of the degree zero and Lagrange stable.*

Then the following statements are equivalent:

1. *the trivial solution of the equation (68) is uniformly asymptotically stable;*
2. *the trivial solution of the equation (68) is globally uniformly asymptotically stable;*
3. *there exit positive numbers $\mathcal{N}$ and ν such that*
$$\rho(\varphi(t,u,g)) \le \mathcal{N}e^{-\nu t}\rho(u) \tag{74}$$
for any $u\in\mathbb{R}^n$, $g\in H(f)$ and $t\ge 0$, where $\rho(u)=|u|_{r,p}$.

Proof. Let $Y := H(f)$ and $(Y,\mathbb{Z},\sigma)$ be the shift dynamical system on $Y = H(f)$. Since the function f is Lagrange stable, then the set Y is compact. Denote by $\langle\mathbb{R}^n,\varphi,(H(f),\mathbb{Z},\sigma)\rangle$ the cocycle generated by the difference equation (68). Since the function f is r homogeneous of the degree zero, then by Corollary 9 the cocycle φ generated by equation (68) is r homogeneous of the degree zero. To finish the proof of Theorem 16 it suffices to take into account Remark 12 and apply Theorem 8. □

Theorem 17. *Let $f \in C(\mathbb{Z}\times\mathbb{R}^n,\mathbb{R}^n)$ be a regular function. Assume that the following conditions are fulfilled:*

1. *the function f is recurrent in time and $f(t,0)=0$ for any $t\in\mathbb{Z}$;*
2. *the function f is r homogeneous of the degree zero.*

Then the following statements are equivalent:

1. *the trivial solution of the equation (68) is uniformly stable and there exists a positive number a such that*
$$\lim_{t\to+\infty}|\varphi(t,u,f)|=0 \tag{75}$$
for any $u\in B[0,a]$;
2. *there exit positive numbers $\mathcal{N}$ and ν such that*
$$\rho(\varphi(t,u,g)) \le \mathcal{N}e^{-\nu t}\rho(u)$$
for any $u\in\mathbb{R}^n$, $g\in H(f)$ and $t\ge 0$.

Proof. Let $Y := H(f)$ and $(Y,\mathbb{Z},\sigma)$ be the shift dynamical system on $Y = H(f)$. Note that the space Y is a compact and minimal set because the function f is recurrent in time (see Remark 9). Since the function f is r homogeneous of the degree zero, then by Corollary 6 the cocycle $\langle\mathbb{R}^n,\varphi,(H(f),\mathbb{Z},\sigma)\rangle$ generated by the equation (68) is r homogeneous of the degree zero. To finish the proof of Theorem 17 it suffices to take into account Remark 12 and apply Theorem 8. □

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8 Conflict of Interests

The author declare that he does not have conflict of interests.

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State University of Moldova
Faculty of Mathematics and Computer Science
Laboratory of "Fundamental and Applied Mathematics"
A. Mateevich Street 60
MD–2009 Chişinău, Moldova
E-mail: *david.ceban@usm.md, davidcheban@yahoo.com*

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