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On the Existence of Stationary Nash Equilibria for Mean Payoff Games on Graphs

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Abstract. In this paper we extend the classical concept of positional strategies for a mean payoff game to a general mixed stationary strategy approach, and prove the existence of mixed stationary Nash equilibria for an arbitrary m-player mean payoff game on graphs. Traditionally, a positional strategy represents a pure stationary strategy in a classical mean payoff game, where a Nash equilibrium in pure stationary strategies in general may not exist. Based on a constructive proof of the existence of specific equilibria for an m-player mean payoff game we propose a new approach for determining the optimal mixed stationary strategies. Additionally we characterize and extend the general problem of the existence of pure stationary Nash equilibria for some special classes of mean payoff games.

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1 Introduction

In [3,5,11] the following game of two players on a graph has been considered: Let $G = (X, E)$ be a finite directed graph in which every vertex $x \in X$ has at least one outgoing directed edge $e = (x, y) \in E$. On the edge set E a function $c : E \to R$ is given which assigns a value $c(e)$ to each edge $e \in E$. Furthermore, the vertex set X is divided into two disjoint subsets X_1 and X_2 $(X = X_1 \cup X_2,\ X_1 \cap X_2 = \emptyset)$ which are regarded as position sets of the two players. The game starts in a given position $x_0 \in X$. If $x_0 \in X_1$ then the move is done by the first player, otherwise it is done by second one. Move means the passage from position x_0 to a neighbor position x_1 through the directed edge $e_0 = (x_0, x_1) \in E$. After that if $x_1 \in X_1$ then the move is done by the first player, otherwise it is done by the second one and so on indefinitely. The first player has the aim to maximize $\lim_{t\to\infty}\inf \frac{1}{t}\sum_{\tau=0}^{t-1} c(e_\tau)$ while the second player has the aim to minimize $\lim_{t\to\infty}\sup \frac{1}{t}\sum_{\tau=0}^{t-1} c(e_\tau)$. In [3] it has been proven that for this game there exists a value $v(x_0)$ such that the first player has a strategy (of moves) that insures $\lim_{t\to\infty}\inf \frac{1}{t}\sum_{\tau=0}^{t-1} c(e_\tau) \geq v(x_0)$ and the second player has a strategy that

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insures $\lim_{t\to\infty}\sup \frac{1}{t}\sum_{\tau=0}^{t-1} c(e_\tau) \leq v(x_0)$. Furthermore it has been shown that players in this games can achieve the values $v(x_0)$ applying the strategies of moves which do not depend on t but depend only on the vertex (position) from which the player is able to move. Therefore, in [3,11] such strategies are called positional strategies and the game sometimes is called positional game; in [5, 10] these strategies are called *stationary strategies*. More precisely the stationary strategies can be specified as *pure stationary strategies* because each move through a directed edge at a vertex of the game is chosen from the set of feasible strategies of moves by the corresponding player with the probability equal to 1 and in each position such a strategy does not change in time.

A generalization of a zero-sum mean payoff game to a non-zero-sum m-player positional game, where $m \geq 2$, is now the following: Consider a finite directed graph $G = (X, E)$ in which every vertex has at least one outgoing directed edge. Assume that the vertex set X is divided into m disjoint subsets $X_1, X_2, \dots, X_m$ ($X = X_1 \cup X_2 \cup \dots \cup X_m$; $X_i \cap Xj = \emptyset, i \neq j$) which we regard as position sets of the m players. Additionally, we assume that on the edge set m functions $c^i : F \to R$, $i = 1, 2, \dots, m$, are defined that assign to each directed edge $e = (x, y) \in E$ the values $c^1_e, c^2_e, \dots, c^m_e$ that are regarded as the rewards for the corresponding players $1, 2, \dots, m$.

On G we consider the following m-person dynamic game: The game starts at a given position $x_0 \in X$ at the moment of time $t = 0$ where the player $i \in \{1, 2, \dots, m\}$ who is the owner of the starting position x_0 makes a move from x_0 to a neighbor position $x_1 \in X$ through the directed edge $e_0 = (x_0, x_1) \in E$. After that players $1, 2, \dots, m$ receive the corresponding rewards $c^1_{e_0}, c^2_{e_0}, \dots, c^m_{e_0}$. Then at the moment of time $t = 1$ the player $k \in \{1, 2, \dots, m\}$ who is owner of position x_1 makes a move from x_1 to a position $x_2 \in V$ through the directed edge $e_1 = (x_1, x_2) \in E$, players $1, 2, \dots, m$ receive the corresponding rewards $c^1_{e_1}, c^2_{e_1}, \dots, c^m_{e_1}$, and so on, indefinitely. Such a play of the game on G produces the sequence of positions $x_0, x_1, x_2, \dots, x_t, \dots$ where each x_t is the position at the moment of time t.

An m-player mean payoff game on G is the game with payoffs

$$\omega^i_{x_o} = \lim_{t\to\infty}\inf \frac{1}{t}\sum_{\tau=0}^{t-1} c^i_{e_\tau}, \quad i = 1, 2, \dots, m.$$

The positional game on graph G formulated above in the cas $m = 2$ and $c^1_e = -c^2_e = c_e$, $\forall e \in E$, is transformed into a a two-player zero-sum mean payoff game on graph G for which Nash equilibria in pure stationary strategies exist. In general, a non-zero-sum mean payoff game on a graph may have no Nash equilibrium in pure stationary strategies. This fact has been shown in [5], where an example of two-player non-zero-sum mean payoff game that has no Nash equilibria in pure strategies is constructed. A pure stationary Nash equilibrium may exist only for some special cases of non-zero mean payoff games (see [1, 5, 10]).

In this contribution we consider the non-zero-sum positional games in mixed stationary strategies. We define a mixed stationary strategy of moves in a position $x \in X_i$ for the player $i \in \{1, 2, \ldots, m\}$, as a probability distribution over the set of feasible moves from x. We show that an arbitrary m-player mean payoff game on a graph possesses a Nash equilibrium in mixed stationary strategies. Based on a constructive proof of this result we propose an approach for determining the optimal mixed stationary strategies of the players.

The paper is organized as follows: In Section 2 an average stochastic positional game that generalizes non-zero-sum mean payoff games is formulated. Then in Sections 3 the known results of the existence of stationary Nash equilibria for an average stochastic positional game and an approach for determining the optimal strategies of players in such a game are presented. In Sections 4, 5, based on results from the Sections 3 the existence of Nash equilibria in mixed stationary strategies for non-zero-sum mean payoff games is proven and an approach for determining the optimal strategies of the players is proposed.

2 A Generalization of Mean Payoff Game on Graphs to Average Stochastic Positional Games

The problem of determining Nash equilibria in mixed stationary strategies for mean payoff games on graphs leads to a special class of stochastic games from [7–9] called *average stochastic positional games*. In [8] it is shown that such class of games possesses Nash equilibria in mixed stationary strategies. Therefore in the paper we shall use the average stochastic positional games for studying the existence of mixed stationary Nash equilibria in non-zero-sum mean payoff games. An m-player average stochastic positional game consists of the following elements:

– a state space X (which we assume to be finite);

– a partition $X = X_1 \cup X_2 \cup \cdots \cup X_m$ where X_i represents the position set of player $i \in \{1, 2, \ldots, m\}$;

– a finite set $A(x)$ of actions in each state $x \in X$;

– a step reward $f^i(x, a)$ with respect to each player $i \in \{1, 2, \ldots, m\}$ in each state $x \in X$ and for an arbitrary action $a \in A(x)$;

– a transition probability function $p : X \times \prod_{x \in X} A(x) \times X \to [0, 1]$ that gives the probability transitions $p^a_{x,y}$ from an arbitrary $x \in X$ to an arbitrary $y \in X$ for a fixed action $a \in A(x)$, where $\sum_{y \in X} p^a_{x,y} = 1, \forall x \in X, a \in A(x)$;

– a starting state $x_0 \in X$.

The game starts at the moment of time $t = 0$ in the state x_0 where the player $i \in \{1, 2, \ldots, m\}$ who is the owner of the state position x_0 $(x_0 \in X_i)$ chooses an action $a_0 \in A(x_0)$ and determines the rewards $f^1(x_0, a_0)$, $f^2(x_0, a_0)$, $\ldots$, $f^m(x_0, a_0)$ for the corresponding players $1, 2, \ldots, m$. After that the game passes to a state $y = x_1 \in X$ according to a certain probability distribution $\{p^{a_0}_{x_0,y}\}$. At the moment of time $t = 1$ the player $k \in \{1, 2, \dots, m\}$ who is the owner of the state position x_1 ($x_1 \in X_k$) chooses an action $a_1 \in A(x_1)$ and players $1, 2, \dots, m$ receive the corresponding rewards $f^1(x_1, a_1), f^2(x_1, a_1), \dots, f^m(x_1, a_1)$. Then the game passes to a state $y = x_2 \in X$ according to a probability distribution $\{p^{a_1}_{x_1,y}\}$ and so on indefinitely. Such a play of the game produces a sequence of states and actions $x_0, a_0, x_1, a_1, \dots, x_t, a_t, \dots$ that defines a stream of stage rewards $f^1(x_t, a_t), f^2(x_t, a_t), \dots, f^m(x_t, a_t)$, $t = 0, 1, 2, \dots$.

The average stochastic positional game is the game with payoffs of the players

$$\omega^i_{x_0} = \lim_{t\to\infty} \inf \ \mathsf{E}\Big(\frac{1}{t}\sum_{\tau=0}^{t-1} f^i(x_\tau, a_\tau)\Big), \quad i = 1, 2, \dots, m$$

where E is the expectation operator with respect to the probability measure in the Markov process induced by actions chosen by players in their position sets and given starting state x_0.

In the following we will consider the stochastic positional game when the players use pure and mixed stationary strategies of choosing the actions in the states.

3 Existence and Determining Mixed Stationary Nash Equilibria for Average Stochastic Positional Games

In this section we present the main results concerned with the existence of stationary Nash equilibria for stochastic positional games with average payoffs. Note that in general for an average stochastic game a stationary Nash equilibrium may not exist (see [4]).

3.1 Stochastic Positional Games in Pure and Mixed Stationary Strategies

A strategy of player $i \in \{1, 2, \dots, m\}$ in a stochastic positional game is a mapping s^i that gives for every state $x_t \in X_i$ a probability distribution over the set of actions $A(x_t)$. If these probabilities take only values 0 and 1, then s^i is called *a pure strategy*, otherwise s^i is called *a mixed strategy*. If these probabilities depend only on the state $x_t = x \in X_i$ (i.e. s^i does not depend on t), then s^i is called *a stationary strategy*, otherwise s^i is called a *non-stationary strategy*.

Thus, we can identify the set of mixed stationary strategies $\mathbf{S}^i$ of player i with the set of solutions of the system

$$\begin{cases} \sum\limits_{a\in A(x)} s^i_{x,a} = 1, & \forall x \in X_i; \\ \qquad\quad s^i_{x,a} \ge 0, & \forall x \in X_i, \ \ \forall a \in A(x). \end{cases} \tag{1}$$

Each basic solution s^i of this system corresponds to a pure stationary strategy of player $i \in \{1, 2, \dots, m\}$. So, the set of pure stationary strategies S^i of player i corresponds to the set of basic solutions of system (1).

Let $\mathbf{s}= (s^1, s^2, \dots, s^m) \in \mathbf{S} = \mathbf{S}^1 \times \mathbf{S}^2 \times \cdots \times \mathbf{S}^m$ be a profile of stationary strategies (pure or mixed strategies) of the players. Then the elements of probability transition matrix $P^{\mathbf{s}} = (p^{\mathbf{s}}_{x,y})$ in the Markov process induced by $\mathbf{s}$ can be calculated as follows:

$$p^{\mathbf{s}}_{x,y} = \sum_{a \in A(x)} s^i_{x,a} p^a_{x,y} \quad \text{for} \quad x \in X_i, \quad i = 1, 2, \dots, m. \tag{2}$$

If we denote by $Q^{\mathbf{s}} = (q^{\mathbf{s}}_{x,y})$ the limiting probability matrix of matrix $P^{\mathbf{s}}$ then the average payoffs per transition $\omega^1_{x_0}(\mathbf{s}),\ \omega^2_{x_0}(\mathbf{s}), \dots, \omega^m_{x_0}(\mathbf{s})$ for the players induced by profile $\mathbf{s}$ are determined as follows

$$\omega^i_{x_0}(\mathbf{s}) = \sum_{k=1}^{m} \sum_{y \in X_k} q^{\mathbf{s}}_{x_0,y} f^i(y, s^k), \quad i = 1, 2, \dots, m, \tag{3}$$

where

$$f^i(y, s^k) = \sum_{a \in A(y)} s^k_{y,a} f^i(y, a), \quad \text{for} \quad y \in X_k,\ k \in \{1, 2, \dots, m\} \tag{4}$$

expresses the average reward (step reward) of player i in the state $y \in X_k$ when player k uses the strategy s^k.

The functions $\omega^1_{x_0}(\mathbf{s}),\ \omega^2_{x_0}(\mathbf{s}),\ \dots,\ \omega^m_{x_0}(\mathbf{s})$ on $\mathbf{S} = \mathbf{S}^1 \times \mathbf{S}^2 \times \cdots \times \mathbf{S}^m$, defined according to (10), (11), determine a game in normal form that we denote by $\langle \{\mathbf{S}^i\}_{i=\overline{1,m}},\ \{\omega^i_{x_0}(\mathbf{s})\}_{i=\overline{1,m}} \rangle$. This game corresponds to the *average stochastic positional game in mixed stationary strategies* that in extended form is determined by the tuple $(\{X_i\}_{i=\overline{1,m}},\ \{A(x)\}_{x \in X},\ \{f^i(x,a)\}_{i=\overline{1,m}},\ p,\ x_0)$. The functions $\omega^1_{x_0}(\mathbf{s}),\ \omega^2_{x_0}(\mathbf{s}), \dots, \omega^m_{x_0}(\mathbf{s})$ on $S = S^1 \times S^2 \times \cdots \times S^m$, determine the game $\langle \{S^i\}_{i=\overline{1,m}}, \{\omega^i_{x_0}(\mathbf{s})\}_{i=\overline{1,m}} \rangle$ that corresponds to the *average stochastic positional game in pure strategies*. In the extended form this game is also determined by the tuple $(\{X_i\}_{i=\overline{1,m}},\ \{A(x)\}_{x \in X},\ \{f^i(x,a)\}_{i=\overline{1,m}},\ p,\ x_0)$.

A stochastic positional game can be considered also for the case when the starting state is chosen randomly according to a given distribution $\{\theta_x\}$ on X. So, for a given stochastic positional game we may assume that the play starts in the state $x \in X$ with probability $\theta_x > 0$ where $\sum_{x \in X} \theta_x = 1$. If the players use mixed stationary strategies then the payoff functions

$$\psi^i_\theta(\mathbf{s}) = \sum_{x \in X} \theta_x \omega^i_x(\mathbf{s}), \quad i = 1, 2, \dots, m$$

on $\mathbf{S}$ define a game in normal form $\langle \{\mathbf{S}^i\}_{i=\overline{1,m}},\ \{\psi^i_\theta(\mathbf{s})\}_{i=\overline{1,m}} \rangle$ that in extended form is determined by $(\{X_i\}_{i=\overline{1,m}}, \{A(x)\}_{x \in X}, \{f^i(x,a)\}_{i=\overline{1,m}}, p, \{\theta_x\}_{x \in X})$. In the case $\theta_x = 0, \forall x \in X \setminus \{x_0\},\ \theta_{x_o} = 1$ the considered game becomes a stochastic positional game with a fixed starting state x_0.

3.2 Stationary Nash Equilibria for an Average Stochastic Positional Game and Determining the Optimal Strategies of the Players

We present a Nash equilibria existence result and an approach for determining the optimal mixed stationary strategies of the players for the average stochastic positional game when the starting state of the game is chosen randomly according to a given distribution $\{\theta_x\}$ on the set of states X. In this case for the game in normal form $\langle \{\mathbf{S}^i\}_{i=\overline{1,m}},\ \{\psi^i_\theta(\mathbf{s})\}_{i=\overline{1,m}}\ \rangle$, the set of strategies $\mathbf{S}^i$ and the payoff functions $\psi^i_\theta(\mathbf{s}),\ \ i=1,2,\dots,m$, can be specified as follows:

Let $\mathbf{S}^i,\ i\in\{1,2,\dots m\}$ be the set of solutions of the system

$$\begin{cases} \sum\limits_{a\in A(x)} s^i_{x,a}=1, & \forall x\in X_i;\\ s^i_{x,a}\geq 0, & \forall x\in X_i,\ \ \forall a\in A(x). \end{cases} \tag{5}$$

On $\mathbf{S}=\mathbf{S}^1\times\mathbf{S}^2\times\cdots\times\mathbf{S}^m$ we define m payoff functions

$$\psi^i_\theta(s^1,s^2,\dots,s^m)=\sum_{k=1}^m\sum_{x\in X_k}\sum_{a\in A(x)} s^k_{x,a}f^i(x,a)q_x, \qquad i=1,2,\dots,m, \tag{6}$$

where q_x for $x\in X$ are determined uniquely from the following system of linear equations

$$\begin{cases} q_y-\sum\limits_{k=1}^m\sum\limits_{x\in X_k}\sum\limits_{a\in A(x)} s^k_{x,a}\ p^a_{x,y}\ q_x=0, & \forall y\in X;\\ q_y+w_y-\sum\limits_{k=1}^m\sum\limits_{x\in X_k}\sum\limits_{a\in A(x)} s^k_{x,a}\ p^a_{x,y}\ w_x=\theta_y, & \forall y\in X \end{cases} \tag{7}$$

for an arbitrary fixed profile $\mathbf{s}=(s^1,s^2,\dots,s^m)\in\mathbf{S}$.

The functions $\psi^i_\theta(s^1,s^2,\dots,s^m),\ i=1,2,\dots,m,$ represent the payoff functions for the average stochastic game in normal form $\langle \{\mathbf{S}^i\}_{i=\overline{1,m}},\ \{\psi^i_\theta(\mathbf{s})\}_{i=\overline{1,m}}\ \rangle$ determined by the tuple $(\{X_i\}_{i=\overline{1,m}},\ \{A(x)\}_{x\in X},\ \{f^i(x,a)\}_{i=\overline{1,m}},\ p,\ \{\theta_y\}_{y\in X})$ where θ_y for $y\in X$ are given positive values such that $\sum_{y\in X}\theta_y=1$. If $\theta_y=0,\ \forall y\in X\setminus\{x_0\}$ and $\theta_{x_0}=1$, then we obtain an average stochastic game in normal form $\langle \{\mathbf{S}^i\}_{i=\overline{1,m}},\ \{\omega^i_{x_0}(\mathbf{s})\}_{i=\overline{1,m}}\ \rangle$ when the starting state x_0 is fixed, i.e. $\psi^i_\theta(s^1,s^2,\dots,\ s^m)=\omega^i_{x_0}(s^1,s^2,\ \dots,\ s^m),\ i=1,2,\dots,m$. So, in this case the game is determined by $(\{X_i\}_{i=\overline{1,m}},\ \{A(x)\}_{x\in X},\ \{f^i(x,a)\}_{i=\overline{1,m}},\ p,\ x_0)$.

In [8] it has been shown that if $\theta_x>0,\forall x\in X,\sum_{x\in X}=1$ then each payoff function $\psi^i_\theta(s),\ i\in\{1,2,\dots,m\}$ in the game $\langle \{\mathbf{S}^i\}_{i=\overline{1,m}},\ \{\psi^i_\theta(\mathbf{s})\}_{i=\overline{1,m}}\ \rangle$ is quasi-monotonic (quasi-convex and quasi-concave) with respect to $\mathbf{s}^i$ on a convex and compact set $\mathbf{S}^i$ for fixed $\mathbf{s}^1,\mathbf{s}^2,\dots,\mathbf{s}^{i-1},\mathbf{s}^{i+1},\dots,\mathbf{s}^m$. Moreover for the game $\langle \{\mathbf{S}^i\}_{i=\overline{1,m}},\ \{\psi^i_\theta(\mathbf{s})\}_{i=\overline{1,m}}\ \rangle$ it has been shown that each payoff function $\psi^i_\theta(s),\ i\in\{1,2,\dots,m\}$, is graph-continuous in the sense of Dasgupta and Maskin [2]. Based on these properties in [8] the following theorem is proved.

Theorem 1. *The game* $\langle\{\mathbf{S}^i\}_{i=\overline{1,m}}, \{\psi^i_\theta(s)\}_{i=\overline{1,m}}\rangle$ *with* $\theta_x > 0, \forall x \in X, \sum_{x\in X} = 1$ *possesses a Nash equilibrium* $\mathbf{s}^*= (s^{1*}, s^{2*}, \dots, s^{m*}) \in \mathbf{S}$ *which is a Nash equilibrium in mixed stationary strategies for the average stochastic positional game determined by the tuple* $(\{X_i\}_{i=\overline{1,m}},\ \{A(x)\}_{x\in X}, \{f^i(x,a)\}_{i=\overline{1,m}},\ p,\ \{\theta_y\}_{y\in X})$*. Moreover,* $\mathbf{s}^* = (s^{1*}, s^{2*}, \dots, s^{m*})$ *is a Nash equilibrium in mixed stationary strategies for the average stochastic positional game* $\langle\{\mathbf{S}^i\}_{i=\overline{1,m}}, \{\omega^i_y(\mathbf{s})\}_{i=\overline{1,m}}\rangle$ *with an arbitrary starting state* $y \in X$*.*

Thus, for an average stochastic positional game a Nash equilibrium in mixed stationary strategies can be found using the noncooperative static game model $\langle\{\mathbf{S}^i\}_{i=\overline{1,m}}, \{\psi^i_\theta(\mathbf{s})\}_{i=\overline{1,m}}\rangle$, where $\mathbf{S}^i$ and $\psi^i_\theta(\mathbf{s})$, $i = 1, 2, \dots, m$, are determined according to (5)–(7). In the case $m = 2$, $f(x,a) = f^1(x,a) = -f^2(x,a)$, $\forall x \in X$, $\forall a \in A(x)$ this game corresponds to a two-player zero-sum average stochastic positional game. In [7] it is shown that for a two-player zero-sum average stochastic game there exist pure stationary equilibria. The proof of this results is similar to the proof of the existence of pure stationary equilibria for two-player zero-sum mean payoff games from [5]. Algorithms for determining the optimal stationary strategies in such games are proposed in [5, 6, 9, 11].

4 Formulation of Mean Payoff Games in Mixed Stationary strategies

Let us consider an m-player mean payoff game determined by the tuple $(G, \{X_i\}_{i=\overline{1,m}}, \{c^i\}_{i=\overline{1,m}}, x_0)$, where $G = (X, E)$ is a finite directed graph with a vertex set X and an edge set E, $X = X_1 \cup X_2 \cup \dots \cup X_m$ $(X_i \cap Xj = \emptyset, i \neq j)$ is a partition of X that determines the corresponding position sets of players and $c^i : E \to R^1$, $i = 1, 2, \dots, m$, are the real functions that determine the rewards on edges of graph G and x_0 is the starting position of the game.

The pure and mixed stationary strategies in the mean payoff game on G can be defined in a similar way as for the average stochastic positional game. We identify the set of mixed stationary strategies S^i of player $i \in \{1, 2, \dots, m\}$ with the set of solutions of the system

$$\begin{cases} \sum\limits_{y\in X(x)} s^i_{x,y} = 1, \quad \forall x \in X_i; \\ \qquad s^i_{x,y} \geq 0, \quad \forall x \in X_i,\ y \in X(x) \end{cases} \tag{8}$$

where $X(x)$ represents the set of neighbor vertices for the vertex x, i.e. $X(x) = \{y \in X | e = (x, y) \in E\}$.

Let $\mathbf{s} = (s^1, s^2, \dots, s^m)$ be a profile of stationary strategies (pure or mixed strategies) of the players. This means that the moves in the mean payoff game from an arbitrary $x \in X$ to $y \in X$ induced by $\mathbf{s}$ are made according to probabilities of the stochastic matrix $P^s = (s_{x,y})$, where

$$\mathbf{s}_{x,y} = \begin{cases} s^i_{x,y} & \text{if } e = (x,y) \in E, x \in X_i, y \in X;\ i = 1, 2, \dots, m; \\ 0 & \text{if } e = (x,y) \notin E. \end{cases} \tag{9}$$

Thus, for a given profile $\mathbf{s}$ we obtain a Markov process with the probability transition matrix $P^{\mathbf{s}} = (\mathbf{s}_{x,y})$ and the corresponding rewards $c^i_{x,y},\ i = 1, 2, \dots, m,$ on edges $(x, y) \in E$. Therefore, if $Q^{\mathbf{s}} = (q^{\mathbf{s}}_{x,y})$ is the limiting probability matrix of $P^{\mathbf{s}}$ then the average rewards per transition $\omega^1_{x_0}(\mathbf{s}),\ \omega^2_{x_0}(\mathbf{s}),\ \dots, \omega^m_{x_0}(\mathbf{s})$ for the players can be determined as follows

$$\omega^i_{x_0}(\mathbf{s}) = \sum_{k=1}^{m} \sum_{y \in X_k} q^{\mathbf{s}}_{x_0,y} \mu^i(y, s^k), \quad i = 1, 2, \dots, m, \tag{10}$$

where

$$\mu^i(y, s^k) = \sum_{z \in X(y)} s^k_{y,z} c^i(y, z), \quad \text{for} \ \ y \in X_k,\ k \in \{1, 2, \dots, m\} \tag{11}$$

expresses the average step reward of player i in the state $y \in X_k$ when player k uses the mixed stationary strategy s^k. The functions $\omega^1_{x_0}(\mathbf{s}), \omega^2_{x_0}(\mathbf{s}), \dots, \omega^m_{x_0}(\mathbf{s})$ on $\mathbf{S} = \mathbf{S}^1 \times \mathbf{S}^2 \times \cdots \times \mathbf{S}^m$, defined according to (10), (11), determine a game in normal form that we denote by $\langle \{\mathbf{S}^i\}_{i=\overline{1,m}},\ \{\omega^i_{x_0}(\mathbf{s})\}_{i=\overline{1,m}} \rangle$. This game corresponds to the *mean payoff game in mixed stationary strategies* on G with a fixed starting position x_0. So this game is determined by the tuple $(G, \{X_i\}_{i=\overline{1,m}}, \{c^i\}_{i=\overline{1,m}}, x_0)$.

In a similar way as for an average stochastic game here we can consider the mean payoff game on G when the starting state is chosen randomly according to a given distribution $\{\theta_x\}$ on X. So, for such a game we will assume that the play starts in the states $x \in X$ with probabilities $\theta_x > 0$ where $\sum_{x \in X} \theta_x = 1$. If the players in such a game use mixed stationary strategies of moves in their positions then the payoff functions

$$\psi^i_\theta(\mathbf{s}) = \sum_{x \in X} \theta_x \omega^i_x(\mathbf{s}), \quad i = 1, 2, \dots, m$$

on $\mathbf{S}$ define a game in normal form $\langle \{\mathbf{S}^i\}_{i=\overline{1,m}},\ \{\psi^i_\theta(\mathbf{s})\}_{i=\overline{1,m}} \rangle$ that is determined by the following tuple $(G,\ \{X_i\}_{i=\overline{1,m}},\ \{c^i\}_{i=\overline{1,m}},\ \{\theta_x\}_{x \in X})$. In the case $\theta_x = 0$, $\forall x \in X \setminus \{v_0\}$, $\theta_{v_0} = 1$ this game becomes a mean payoff game with fixed starting state x_0.

5 Nash Equilibria in Mixed Stationary Strategies for Mean Payoff Games and Determining the Optimal Strategies of the Players

In this section we show how the results from the previous sections can be applied for determining Nash equilibria and the optimal mixed stationary strategies of the players for mean payoff games.

Let $\langle \{\mathbf{S}^i\}_{i=\overline{1,m}},\ \{\psi^i_\theta(s)\}_{i=\overline{1,m}} \rangle$ be the game in normal form for the mean payoff game determined by $(G,\ \{X_i\}_{i=\overline{1,m}},\ \{c^i\}_{i=\overline{1,m}},\ \{\theta_x\}_{x \in X})$. We show that $\mathbf{S}^i$ and $\psi^i_\theta(s)$ for $i \in \{1, 2, \dots, m\}$ can be defined as follows: $\mathbf{S}^i$ represents a set of the solutions of the system

$$\begin{cases} \displaystyle\sum_{y \in X(x)} s^i_{x,y} = 1, \quad \forall x \in X_i; \\ \qquad s^i_{x,y} \geq 0, \quad \forall x \in X_i,\ y \in X(x) \end{cases} \tag{12}$$

and

$$\psi_\theta^i(s^1, s^2, \dots, s^m) = \sum_{k=1}^{m} \sum_{y \in X_k} \sum_{y \in X(x)} s_{x,y}^k c^i(x,y) q_x, \tag{13}$$

where q_x for $x \in X$ are determined uniquely (via $s_{x,y}^k$) from the following system of equations

$$\begin{cases} q_y - \sum\limits_{k=1}^{m} \sum\limits_{x \in X_k} s_{x,y}^k \; q_x = 0, & \forall y \in X; \\ q_y + w_y - \sum\limits_{k=1}^{m} \sum\limits_{x \in X_k} s_{x,y}^k \; w_x = \theta_y, & \forall y \in X. \end{cases} \tag{14}$$

Here θ_y for $y \in X$ represent arbitrary fixed positive values where $\sum\limits_{y \in X} \theta_y = 1$.

Using Theorem 1 we can prove now the following result.

Theorem 2. *For a mean payoff game on graph G the corresponding game in normal form $\langle \{\mathbf{S}^i\}_{i=\overline{1,m}}, \{\psi_\theta^i(s)\}_{i=\overline{1,m}} \rangle$ possesses a Nash equilibrium $\mathbf{s}^* = (s^{1^*}, s^{2^*}, \dots, s^{m*}) \in \mathbf{S}$ which is a Nash equilibrium in mixed stationary strategies for the mean payoff game on G with an arbitrary starting position $x_0 \in X$.*

Proof. To prove the theorem it is sufficient to show that the functions $\psi_\theta^i(s)$, $i \in \{1, 2, \dots, m\}$, defined according to (13), (14) represent the payoff functions for the mean payoff game determined by $(G,\ \{X_i\}_{i=\overline{1,m}},\ \{c^i\}_{i=\overline{1,m}},\ \{\theta_x\}_{x \in X})$. This is easy to verify because if we replace in (6) the rewards $f^i(x,a)$ for $x \in X$ and $a \in A(x)$ by rewards $c_{x,y}^i$ for $(x,y) \in E$ and in (6), (7) we replace the probabilities $p_{x,y}^a$, $x \in X_k$, $a \in A(x)$ for the corresponding players $k = 1, 2, \dots, m$ by $p_{x,y}^k \in \{0, 1\}$ according to the structure of graph G then we obtain that (6), (7) are transformed into (13), (14). If we apply Theorem 1 after that then obtain the proof of the theorem. □

So, the optimal mixed stationary strategies of the players in a mean payoff game can be found if we determine the optimal stationary strategies of the players for the game $\langle \{\mathbf{S}^i\}_{i=\overline{1,m}},\ \{\psi_\theta^i(s)\}_{i=\overline{1,m}} \rangle$ where $\mathbf{S}^i$ and $\psi_\theta^i(s)$ for $i \in \{1, 2, \dots, m\}$ are defined according to (12)–(14). If $m = 2$, $c_{x,y} = c_{x,y}^1 = -c_{x,y}^2$, $\forall (x,y) \in E$ then we obtain a game-theoretic model in normal form for the zero-sum two-player mean payoff on graph G. In this case the equilibrium exists in pure stationary strategies and the considered game model allows us to determine the optimal pure stationary strategies of the players. The results from [9] related to antagonistic average stochastic positional games can be also extended to antagonistic mean payoff games on graphs if we take into account the transformations mentioned above in the proof of Theorem 2, i.e. we should change the rewards $f^i(x,a)$ for $x \in X$, $a \in A(x)$ by rewards $c_{x,y}^i$ for $(x,y) \in E$ and replace the probabilities $p_{x,y}^a$, $x \in X_k$, $a \in A(x)$, $k = 1, 2, \dots, m$ by probabilities $p_{x,y}^k \in \{0, 1\}$ according to the structure of the graph G.

6 Conclusion

The considered m-player non-zero mean payoff games on graphs generalize the zero-sum two-player mean payoff games on graphs considered in [3, 5, 11]. For zero-sum two-player mean payoff games on graphs there exist Nash equilibria in pure stationary strategies that can be determined based on results from [5, 11]. For the case of non-zero-sum mean payoff games on graphs Nash equilibria in pure stationary strategies may not exist, however there exist Nash equilibria in mixed stationary strategies. Such equilibria can be determined and characterized as Nash equilibria for the noncooperative static game models from Sections 5, 6.

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