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On T-nilpotence of a matrix set

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Abstract. Let R be a ring and I be an arbitrary right T-nilpotent subset of R. In the paper it is proved that in this case the set of all $n \times n$-matrices with entries in I is a right T-nilpotent subset of the ring of $n \times n$-matrices with entries in R, where $n \in \mathbb{N}$. It is also showed that it is impossible to generalize this result for rings of matrices of infinite dimension.

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Dedicated to the memory of Professor M. Ya. Komarnytskyi

1 Introduction

All rings are considered to be associative with $1 \neq 0$. The category of left K-modules is denoted by $K - Mod$. The set of all $n \times n$-matrices with entries in a set I will be denoted by $M_n(I)$, where $n \in \mathbb{N}$.

Definition 1. ([5, p. 291]) A set A of elements of a ring R is called left (resp. right) T-nilpotent, if for every family

$$(a_1, a_2, a_3, ...), a_i \in A$$

a $k \in \mathbb{N}$ exists with

$$a_k a_{k-1} ... a_1 = 0, (a_1 a_2 ... a_k = 0).$$

(See also [1, p. 313].)

The notion of T-nilpotence has the important applications in certain areas of the Ring and Module Theory, especially in theory of perfect and semiartinian rings, but not only (for example, see [8, p. 183–184, 189], [6, p. 60], [7, p. 67], [3, p. 86, 87]).

Recall the definition of the equivalence. Let C and D be categories. A functor $S : C \to D$ is an equivalence if there exist a functor $T : D \to C$ and natural equivalences $TS \to 1_C$ and $ST \to 1_D$. (See [8, p. 82].)

In the paper [4] the following corollaries are obtained:

Corollary 1. *(See Corollary 11 [4, p. 52]) Let R, S be equivalent rings, via an equivalence $F : R - Mod \to S - Mod$. If I is a right T-nilpotent two-sided ideal of R, then so is the two-sided ideal $\{s \in S \,|\forall x \in F(R/I) : sx = 0\}$ of S.*

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Corollary 2. *(See Corollary 12 [4, p. 52]) Let R be a ring and let $n \in \mathbb{N}$. If I is a right T-nilpotent ideal of R, then $M_n(I)$ is a right T-nilpotent ideal of $M_n(R)$.*

The aim of our paper is to obtain the stronger statement than Corollary 2. Indeed, in this corollary an arbitrary subset of a ring instead of a two-sided ideal can be considered.

2 Preliminaries

Lemma 1. *(König's Graph Lemma, [2, p. 40]) Start with a countable sequence $\{F_n \,|n = 1, 2, ...\}$ of finite sets, and for each n, assume that there is a map Φ_n of F_n into $Pow(F_{n+1})$. In order to simplify notation, denote Φ_n by Φ, $\forall n$, and the union of the given family of finite sets by F. A path in (the ordered pair) (F, Φ) is a finite or infinite sequence of elements $b_1, ..., b_n, ...$ of F such that $b_i \in F_i$ and $b_{i+1} \in \Phi(b_i)$, $i = 1, 2, ...$ The length of a finite path $b_1, b_2, ..., b_m$ is m; the length of the infinite path $b_1, b_2, ...$is infinite. Then if (F, Φ) has paths of ever greater length, then it has a path of infinite length.*

3 Main result

Theorem 1. *Let R be a ring and I be a right T-nilpotent subset of R. Then $M_n(I)$ is a right T-nilpotent subset of $M_n(R)$, where $n \in \mathbb{N}$.*

Proof. Let I be a right T-nilpotent subset of R and $n \in \mathbb{N}$.

Assume $M_n(I)$ is not right T-nilpotent. Then there is an infinite sequence of matrices

$$||a_{ij}^{(1)}||, ||a_{ij}^{(2)}||, \dots, ||a_{ij}^{(k)}||, \dots$$

belonging to $M_n(I)$ such that for each $k \in \mathbb{N}$

$$A_k \neq O, \tag{1}$$

where $A_k = ||a_{ij}^{(1)}||||a_{ij}^{(2)}||...||a_{ij}^{(k)}||$.

Let $A_k = ||A_{ij}^{(k)}||$ for each $k \in \mathbb{N}$.

Then it is obvious that

$$A_{ij}^{(1)} = a_{ij}^{(1)} \quad \text{and} \quad A_{ij}^{(k)} = \sum_{t_1=1}^{n} \sum_{t_2=1}^{n} ... \sum_{t_{k-1}=1}^{n} a_{it_1}^{(1)} a_{t_1 t_2}^{(2)} ... a_{t_{k-1} j}^{(k)} \quad \text{for} \quad k \geq 2. \tag{2}$$

Consider the sets $F_1, F_2, ..., F_k, ...$ defined as follows:

$$F_1 = \{(\lambda_1, \lambda_2) | \lambda_1, \lambda_2 \in \{1, 2, ..., n\}, a_{\lambda_1 \lambda_2}^{(1)} \neq 0\},$$

$$F_2 = \{(\lambda_1, \lambda_2, \lambda_3) | \lambda_1, \lambda_2, \lambda_3 \in \{1, 2, ..., n\}, a_{\lambda_1 \lambda_2}^{(1)} a_{\lambda_2 \lambda_3}^{(2)} \neq 0\},$$

$$\vdots \quad \vdots \quad \vdots$$

$$F_k = \{(\lambda_1, \lambda_2, ..., \lambda_{k+1}) | \lambda_1, \lambda_2, ..., \lambda_{k+1} \in \{1, 2, ..., n\}, a^{(1)}_{\lambda_1\lambda_2} a^{(2)}_{\lambda_2\lambda_3} ... a^{(k)}_{\lambda_k\lambda_{k+1}} \neq 0\},$$

$$\vdots \quad \vdots \quad \vdots$$

(1)-(2) imply

$$\forall k \in \mathbb{N} : F_k \neq \emptyset.$$

Hence for each $k \in \mathbb{N}$ it is possible to consider the following mapping:

$$\Phi_k : \begin{cases} F_k \to Pow(F_{k+1}), \\ (\lambda_1, \lambda_2, ..., \lambda_{k+1}) \mapsto \{(\lambda_1, \lambda_2, ..., \lambda_{k+1}, \lambda_{k+2}) | a^{(1)}_{\lambda_1\lambda_2} ... a^{(k+1)}_{\lambda_{k+1}\lambda_{k+2}} \neq 0\}. \end{cases}$$

Let u be an arbitrary integer greater than 0. Then $A_u \neq O$. It follows from this that for some $i, j \in \{1, 2, ..., n\}$ $A^{(u)}_{ij} \neq 0$. It follows from (2) that for some $t_1, ..., t_{u-1} \in \{1, 2, ..., n\}$

$$a^{(1)}_{it_1} a^{(2)}_{t_1t_2} ... a^{(u)}_{t_{u-1}j} \neq 0. \tag{3}$$

Whence

$$\begin{aligned} a^{(1)}_{it_1} &\neq 0, \\ a^{(1)}_{it_1} a^{(2)}_{t_1t_2} &\neq 0, \\ &\vdots \quad \vdots \quad \vdots \\ a^{(1)}_{it_1} a^{(2)}_{t_1t_2} ... a^{(u-1)}_{t_{u-2}t_{u-1}} &\neq 0. \end{aligned} \tag{4}$$

Put

$$\begin{aligned} b_1 &= (i, t_1), \\ b_2 &= (i, t_1, t_2), \\ &\vdots \quad \vdots \quad \vdots \\ b_{u-1} &= (i, t_1, t_2, ..., t_{u-1}), \\ b_u &= (i, t_1, t_2, ..., t_{u-1}, j). \end{aligned}$$

(3)-(4) imply that $b_1 \in F_1$, $b_2 \in F_2$, $\ldots$, $b_u \in F_u$.

It is clear that

$$\begin{aligned} b_2 &\in \Phi_1(b_1), \\ b_3 &\in \Phi_2(b_2), \\ &\vdots \quad \vdots \quad \vdots \\ b_u &\in \Phi_{u-1}(b_{u-1}). \end{aligned}$$

The length of the path $b_1, b_2, ..., b_u$ is u. Since u is an arbitrary integer greater than 0, we have paths of ever greater length. Therefore, by König's Graph Lemma, there exists a path of infinite length.

It means that there exists an infinite sequence of numbers $\lambda_1, \lambda_2, ..., \lambda_p, ...$ belonging to $\{1, 2, ..., n\}$ satisfying the following conditions:

$$\begin{aligned} (\lambda_1, \lambda_2) &\in F_1, \\ (\lambda_1, \lambda_2, \lambda_3) &\in F_2, \\ \vdots \; &\vdots \; \vdots \\ (\lambda_1, \lambda_2, ..., \lambda_{p+1}) &\in F_p, \\ \vdots \; &\vdots \; \vdots \end{aligned} \tag{5}$$

Consider the sequence

$$a^{(1)}_{\lambda_1\lambda_2}, a^{(2)}_{\lambda_2\lambda_3}, ..., a^{(s)}_{\lambda_s\lambda_{s+1}}, ... \; .$$

It follows from (5) that for an arbitrary $p \in \mathbb{N}$

$$a^{(1)}_{\lambda_1\lambda_2} a^{(2)}_{\lambda_2\lambda_3} ... a^{(p)}_{\lambda_p\lambda_{p+1}} \neq 0.$$

Hence I is not right T-nilpotent, which is a contradiction.

□

Now we will see that it is impossible to generalize our result for rings of matrices of infinite dimension.

Example 1. Let K be a ring and S be a subset of K. Let $\mathbb{RFM}_{\mathbb{N}}(S)$ be the set of all mappings $f : \mathbb{N} \times \mathbb{N} \to S$, where for each $\alpha \in \mathbb{N}$ the set $\{f(\alpha, \beta) \neq 0 | \beta \in \mathbb{N}\}$ is finite. Then $\mathbb{RFM}_{\mathbb{N}}(K)$ is a natural generalization of the matrix rings $M_n(K)$ (see [1, p. 19]).

Let k be a field. Consider the polynomial ring $k[x_1, x_2, ..., x_n, ...]$ in a countable number of variables. Let M be the ideal of this ring spanned by the following elements: x_1^2, x_2^3, ..., x_n^{n+1}, ..., $x_i x_j$, where $i \neq j$ and $i, j \in \mathbb{N}$. Denote the elements $a + M$ of the factor ring $K := k[x_1, x_2, ..., x_n, ...]/M$ by $\bar{a}$.

And now let I be the ideal of K spanned by the elements $\bar{x}_1$, $\bar{x}_2$,..., $\bar{x}_n$,... . It is obvious that I is right T-nilpotent.

Define a function $g : \mathbb{N} \times \mathbb{N} \to K$ as follows:

$$g(i, i) = \bar{x}_i, g(i, j) = \bar{0},$$

for all $i, j \in \mathbb{N}$, where $i \neq j$. It is clear that $g \in \mathbb{RFM}_{\mathbb{N}}(I)$, but g is not nilpotent.

Therefore $\mathbb{RFM}_{\mathbb{N}}(I)$ is not right T-nilpotent, although I is right T-nilpotent.

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