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The Second Hankel Determinant for k-symmetrical Functions

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Abstract. In this article, we find the upper bound of the second Hankel determinant $|a_2a_4 - a_3^2|$ for subclasses of starlike and convex functions with respect to k-symmetric points.

Mathematics subject classification: 30C45, 30C50.
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1 Introduction

The work here is considering the class $\mathcal{F}$ of functions analytic in $\widetilde{\mathcal{U}} = \{w : w \in \mathbb{C}, |w| < 1\}$, and of the form

$$f(w) = w + \sum_{n=2}^{\infty} a_n w^n, \tag{1}$$

and suppose $\widetilde{\mathcal{S}}$ denotes the subclass of $\mathcal{F}$ consisting of all functions that are univalent in $\widetilde{\mathcal{U}}$. For $f, g \in \mathcal{F}$, we say that f is subordinate to g written as $f \prec g$ if there exists a holomorphic map h of the unit disk $\widetilde{\mathcal{U}}$ into itself with $h(0) = 0$ such that $f = g \circ h$. Note that if $g \in \widetilde{\mathcal{S}}$, then $f \prec g$ is equivalent to the condition that $f(0) = g(0)$ and $f(\widetilde{\mathcal{U}}) \subset g(\widetilde{\mathcal{U}})$. Let $\mathcal{P}$ be the family of analytic functions p in $\widetilde{\mathcal{U}}$ with $\Re\{p(w)\} > 0$ which have the form $p(w) = 1+q_1w+q_2w^2+...$ $(w \in \widetilde{\mathcal{U}})$. The class $\mathcal{P}$ of functions with positive real part plays a crucial role in geometric function theory. Its significance can be seen from the fact that simple subclasses are class $\widetilde{\mathcal{S}}^*$ of starlike functions and class $\widetilde{\mathcal{K}}$ of convex functions.

Definition 1. [10] For $k \in \mathbb{N} = \{1, 2, ...\}$, let $\varepsilon = e^{\left(\frac{2\pi i}{k}\right)}$ denote the k^{th} root of unity for $f \in \mathcal{F}$. Its k-weighted mean function is

$$M_{f,k}(w) = \sum_{v=1}^{k-1} \varepsilon^{-v} f(\varepsilon^v w).\frac{1}{\sum_{v=1}^{k-1} \varepsilon^{-v}}.$$

A function f in $\mathcal{F}$ is called k-symmetrical function for each $w \in \widetilde{\mathcal{U}}$ if $f(\varepsilon w) = \varepsilon f(w)$. The family of all k-symmetrical functions will be denoted by $\mathcal{F}^k$.

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A function f in $\mathcal{F}$ is said to belong to the class $\widetilde{\mathcal{S}}_k^*$ of functions starlike with respect to k-symmetric points if for every r close to 1, $r<1$, the angular velocity of f about the point $M_{f_k}(w_0)$ is positive at $w=w_0$ as z traverses the circle $|w|=r$ in the positive direction, that is $\Re\left(\frac{zf'(w)}{f(w)-M_{f,k}(w_0)}\right)>0$ for $w=w_0$, $|w_0|=r$.

Definition 2. [27] For a positive integer k, let $\mathcal{S}_k^*$ denote the family of starlike functions with respect to k-symmetric points $f\in\mathcal{F}$ which satisfy

$$\Re\left\{\frac{wf'(w)}{f_k(w)}\right\}>0, \quad w\in\widetilde{\mathcal{U}}, \tag{2}$$

where

$$f_k(w)=\frac{1}{k}\left[f(w)-M_{f,k}(w)\right]. \tag{3}$$

Remark 1. Equivalently, (3) can be written as

$$f_k(w)=\frac{1}{k}\sum_{v=0}^{k-1}\varepsilon^{-v}f(\varepsilon^v w), \tag{4}$$

or

$$f_k(w)=w+\sum_{n=2}^{\infty}\psi_n a_n w^n \quad \text{where} \quad \psi_n=\begin{cases}1 & \text{if} \quad n=lk+1, \qquad l\in\mathbb{N}_0.\\ 0 & \text{if} \quad n\neq lk+1\end{cases}. \tag{5}$$

Let $\widetilde{\mathcal{K}}_k$ denote the subclass of functions $f\in\mathcal{F}$ which satisfies

$$f\in\widetilde{\mathcal{K}}_k \Leftrightarrow wf'\in\widetilde{\mathcal{S}}_k^*. \tag{6}$$

For more details, some interesting properties of the classes of functions with respect to k-symmetric points have been discussed by the authors in [1, 2].

One of the most fundamental problems in geometric function theory is to find the coefficient bounds for a certain class of functions. In this work, we study the Hankel determinant $\widetilde{\mathcal{H}}_{\vartheta,n}(f)$ $(\vartheta,n\in\mathbb{N})$ for the well-known class of starlike functions $\widetilde{\mathcal{S}}^*$ which was introduced by Pommerenke [23, 24], and is defined as follows:

$$\widetilde{\mathcal{H}}_{\vartheta,n}(f)=\begin{vmatrix} a_n & a_{n+1} & \cdots & a_{n+\vartheta-1}\\ a_{n+1} & \cdots & \cdots & \cdots\\ \cdots & \cdots & \cdots & \cdots\\ a_{n+\vartheta-1} & \cdots & \cdots & a_{n+2\vartheta-2}\end{vmatrix}.$$

We can easily note that $\widetilde{\mathcal{H}}_{2,1}(f)=a_3-a_2^2$, $\widetilde{\mathcal{H}}_{2,2}(f)=a_2a_4-a_3^2$ and

$$\widetilde{\mathcal{H}}_{3,1}(f)=2a_2a_3a_4-a_3^3-a_4^2+a_3a_5-a_2^2a_5.$$

Many authors have studied and investigated the Hankel determinants for various subclasses of $\mathcal{F}$. The famous problem solved by using the Loewner technique to determine the greatest value of the coefficient was investigated by Fekete and Szegö in [9], they generalized the estimate $|a_3 - \mu a_2^2|$ where μ is real and $f \in \widetilde{\mathcal{S}}^*$. Later, Jae Ho Choi et al.[6] provided a new method for solving the Fekete-Szegö problem which opened up a lot of new opportunities for research in the related fields. This determinant was studied for other classes of functions by many other authors like Noor [21], Ehrenborg [8], and Layman [15]. For some other related works for subclasses regarding symmetric points, one can look up Janteng et al.[11–13] who have considered the functional $|\widetilde{\mathcal{H}}_{2,2}(f)|$ and studied the second Hankel determinant and have shown that $|\widetilde{\mathcal{H}}_{2,2}(f)| \leq 4/9$, $|\widetilde{\mathcal{H}}_{2,2}(f)| \leq 1$, $|\widetilde{\mathcal{H}}_{2,2}(f)| \leq 1/8$ and $|\widetilde{\mathcal{H}}_{2,2}(f)| \leq 1$, $|\widetilde{\mathcal{H}}_{2,2}(f)| \leq 1/9$, respectively, for the classes of analytic, starlike, convex, close-to-starlike and close-to-convex functions concerning symmetric points.

The third-order Hankel determinant $|\widetilde{\mathcal{H}}_{3,1}(f)|$ for subclasses of $\mathcal{F}$ was studied for the first time by Babalola [3]. In 2017, Zaprawa [28] improved the results of Babalola [3] by proving $|\widetilde{\mathcal{H}}_{3,1}(f)| \leq 1$, $|\widetilde{\mathcal{H}}_{3,1}(f)| \leq 49/540$, $|\widetilde{\mathcal{H}}_{3,1}(f)| \leq 41/60$ for the classes of starlike, convex and bounded turning functions respectively.

The estimation of the fourth Hankel determinant $|\widetilde{\mathcal{H}}_{4,1}(f)|$ for the bounded turning functions has been obtained by Arif et al.[16] and they proved $|\widetilde{\mathcal{H}}_{4,1}(f)| \leq 0.78050$.

Recently, Barukab et al.[4] obtained the sharp bounds of $|\widetilde{\mathcal{H}}_{3,1}(f)|$ for a collection of bounded turning functions associated with the petal-shaped domain. Khan et al.[14] investigated the third Hankel determinant for a class of starlike functions with respect to two symmetric points with a sine function. Other interesting topics have been discussed in 2021 and 2022; see [25, 26].

The aim of the present work is to determine the upper bound of the Hankel determinants of order two for the functions belonging to the classes $\widetilde{\mathcal{S}}_k^*$ and $\widetilde{\mathcal{K}}_k$.

2 Preliminary Results

Lemma 1. [7] *If $p \in \mathcal{P}$, then $|q_n| \leq 2, (n = 1, 2, ...)$.*

Lemma 2. [17, 18] *If $p \in \mathcal{P}$, then*

$$2q_2 = q_1^2 + (4 - q_1^2)x,$$

$$4q_3 = q_1^3 + 2q_1(4 - q_1^2)x - q_1(4 - q_1^2)x^2 + 2(4 - q_1^2)(1 - |x|^2)w,$$

for some x and w satisfying $|x| \leq 1$, $|w| \leq 1$ and $p_1 \in [0, 2]$.

3 Main Results

Theorem 1. *Let $f \in \widetilde{\mathcal{S}}_k^*$, then*

$$|a_2a_4 - a_3^2| \leq \frac{4}{(3-\psi_3)^2}, \tag{7}$$

where ψ_n is defined by (5).

Proof. Since $f \in \widetilde{\mathcal{S}}_k^*$, then there exists $p \in \mathcal{P}$ such that

$$\frac{wf'(w)}{f_k(w)} = p(w),$$

or

$$\frac{1+\sum_{n=2}^{\infty} na_nw^{n-1}}{\sum_{n=1}^{\infty}\psi_na_nw^{n-1}} = 1 + q_1w + q_2w^2 + q_3w^3 + \ldots. \tag{8}$$

Equating coefficients in (8) yields

$$\psi_1 = 1, \quad a_2 = \frac{q_1}{2-\psi_2}, \quad a_3 = \frac{1}{3-\psi_3}\left[q_2 + \frac{\psi_2q_1^2}{2-\psi_2}\right], \tag{9}$$

$$a_4 = \frac{1}{4-\psi_4}\left[q_3 + \frac{\psi_2q_1q_2}{2-\psi_2} + \frac{\psi_3q_1}{3-\psi_3}\left(q_2 + \frac{\psi_2q_1^2}{2-\psi_2}\right)\right]. \tag{10}$$

By (9) and (10) we get

$$|a_2a_4 - a_3^2| =$$

$$\left|\frac{q_1}{(2-\psi_2)(4-\psi_4)}\left[q_3 + \frac{\psi_2q_1q_2}{2-\psi_2} + \frac{\psi_3q_1}{3-\psi_3}\left(q_2 + \frac{\psi_2q_1^2}{2-\psi_2}\right)\right] - \frac{1}{(3-\psi_3)^2}\left[q_2 + \frac{\psi_2q_1^2}{2-\psi_2}\right]^2\right|.$$

Using Lemma (1) and Lemma (2) in the above equation we get

$$|a_2a_4 - a_3^2| =$$

$$\left|\frac{q_1}{4(2-\psi_2)(4-\psi_4)}\left[q_1^3 + 2p_1(4-q_1^2)x - q_1(4-q_1^2)x^2 + 2(4-q_1^2)(1-|x|^2)w\right] + \right.$$

$$\frac{q_1}{(2-\psi_2)(4-\psi_4)}\left[\frac{\psi_2q_1}{2(2\psi_2)}\{q_1^2 + (4-q_1^2)x\} + \frac{\psi_3q_1}{2(3-\psi_3)}\{q_1^2 + (4-q_1^2)x + \frac{2\psi_2q_1^2}{2-\psi_2}\}\right]$$

$$-\frac{1}{(3-\psi_3)^2}\left[\frac{1}{4}\{q_1^4 + 2q_1^2(4-q_1^2)x + (4-q_1^2)^2x^2\}\right]$$

$$\left.-\frac{1}{(3-\psi_3)^2}\left[(q_1^2 + (4-q_1^2)x)\frac{\psi_2q_1^2}{2-\psi_2} + \frac{\psi_2^2q_1^4}{(2-\psi_2)^2}\right]\right|$$

$$= \left|\left[\frac{1}{2(2-\psi_2)(4-\psi_4)}\left\{1 + \frac{\psi_2}{2-\psi_2} + \frac{\psi_3}{3-\psi_3}\right\} - \frac{1}{2(3-\psi_3)^2} - \frac{\psi_2}{(2\psi_2)(3\psi_3)^2}\right]q_1^2(4-q_1^2)x\right.$$

$$-\left[\frac{q_1^2}{4(2-\psi_2)(4-\psi_4)}+\frac{(4-q_1^2)}{4(3-\psi_3)^2}\right](4-q_1^2)x^2+\frac{q_1(4-q_1^2)(1-|x|^2)w}{2(2-\psi_2)(4-\psi_4)}$$
$$+\frac{q_1^4}{(2-\psi_2)(4-\psi_4)}\left\{\frac{1}{4}+\frac{\psi_2}{2(2-\psi_2)}+\frac{\psi_3}{2(3-\psi_3)}+\frac{\psi_2\psi_3}{(2-\psi_2)(3-\psi_3)}\right\}$$
$$-\frac{q_1^4}{4((3-\psi_3)^2}-\frac{q_1^4\psi_2}{(2-\psi_2)(3-\psi_3)^2}-\frac{q_1^4\psi_2^2}{(2-\psi_2)^2(3-\psi_3)^2}\Bigg|.$$

Let $q_1=q$ and $0\le q\le 2$, and utilizing the assumption $|w|\le 1$, we obtain

$$|a_2a_4-a_3^2|\le \mathcal{R}_1(q)+\mathcal{R}_2(q)\mu+\mathcal{R}_3(q)\mu^2=G(q,\mu), \tag{11}$$

where $\mu=|x|\le 1$ with

$$\mathcal{R}_1(q)=q^4\left[\frac{1}{(2-\psi_2)(4-\psi_4)}\left\{\frac{1}{4}+\frac{\psi_2}{2(2-\psi_2)}+\frac{\psi_3}{2(3-\psi_3)}+\frac{\psi_2\psi_3}{(2-\psi_2)(3-\psi_3)}\right\}\right]$$
$$+q^4\left[\frac{\psi_2}{(2\psi_2)(3-\psi_3)^2}-\frac{1}{4(3-\psi_3)^2}-\frac{\psi_2^2}{(2-\psi_2)^2(3-\psi_3)^2}\right]+\frac{q(4-q^2)}{2(2-\psi_2)(4-\psi_4)},$$
$$\mathcal{R}_2(q)=\left[\frac{1}{2(2-\psi_2)(4-\psi_4)}\left\{1+\frac{\psi_2}{2-\psi_2}+\frac{\psi_3}{3-\psi_3}\right\}+\frac{\psi_2}{(2-\psi_2)(3-\psi_3)^2}-\frac{1}{2(3-\psi_3)^2}\right]q^2(4-q^2),$$
$$\mathcal{R}_3(q)=\left[\frac{q(q+2)}{4(2-\psi_2)(4-\psi_4)}+\frac{(4-q^2)}{4(3-\psi_3)^2}\right](4-q^2).$$

Now, we have to maximize $G(q,\mu)$ on the closed square $[0,2]\times[0,1]$.
By taking partial derivative of $G(q,\mu)$ in (11) with respect to μ, we get

$$\frac{\partial G(q,\mu)}{\partial\mu}=\left[\frac{q(q+2)}{4(2-\psi_2)(4-\psi_4)}+\frac{(4-q^2)}{4(3-\psi_3)^2}\right]2(4-q^2)\mu \tag{12}$$
$$+\frac{q^2(4-q^2)}{2(2-\psi_2)(4-\psi_4)}\left\{1+\frac{\psi_2}{2-\psi_2}+\frac{\psi_3}{3-\psi_3}\right\}$$
$$+\frac{q^2(4-q^2)\psi_2}{(2-\psi_2)(3-\psi_3)^2}-\frac{q^2(4-q^2)}{2(3-\psi_3)^2}.$$

For $\mu\in(0,1)$ and for fixed $q\in(0,1)$, from (12), we observe that $\frac{\partial G(q,\mu)}{\partial\mu}>0$, and then $G(q,\mu)$ is increasing in μ, for fixed $p\in[0,2]$, the maximum of $G(q,\mu)$ occurs at $\mu=1$ and

$$\max_{0\le\mu\le1}G(q,\mu)=G(q,1)=F(q). \tag{13}$$

From (11) and (13), upon simplification, we get

$$F(q)=G(q,1)=\frac{\psi_2\psi_3q^4}{(2-\psi_2)^2(3-\psi_3)(4-\psi_4)}+\frac{q^4}{2(3-\psi_3)^2}-\frac{\psi_2^2q^4}{(2-\psi_2)^2(3-\psi_3)^2}$$
$$-\frac{q^4}{2(2-\psi_2)(4-\psi_4)}-q^3+\frac{2q^2}{4-\psi_4}\left\{1+\frac{\psi_2}{2-\psi_2}+\frac{\psi_3}{3-\psi_3}\right\}-\frac{q^2}{(3-\psi_3)^2} \tag{14}$$

$$+\frac{4q^2\psi_2}{(2-\psi_2)(3-\psi_3)^2}+\frac{q^2}{(2-\psi_2)(4-\psi_4)}+\frac{4}{(2-\psi_2)(4-\psi_4)}q+\frac{4}{(3-\psi_3)^2}.$$

Suppose that $F(q)$ has a maximum value at $q \in (0,2)$. Now by differentiating with respect to q and after some simple calculations we find

$$F'(q)=\frac{4\psi_2\psi_3 q^3}{(2-\psi_2)^2(3-\psi_3)(4-\psi_4)}+\frac{4q^3}{2(3-\psi_3)^2}-\frac{4q^3\psi_2^2}{2-\psi_2)^2(3-\psi_3)^2}$$
$$-\frac{4q^3}{2(2-\psi_2)(4-\psi_4)}-3q^2+\frac{4q}{(2-\psi_2)(4-\psi_4)}\left\{1+\frac{\psi_2}{2-\psi_2}+\frac{\psi_3}{3-\psi_3}\right\}$$
$$-\frac{2q}{(3-\psi_3)^2}+\frac{8q\psi_2}{(2-\psi_2)(3-\psi_3)^2}+\frac{2q}{(2-\psi_2)(4-\psi_4)}+\frac{4}{(2-\psi_2)(4-\psi_4)}.$$

Clearly, $F'(q)=0$ has no optimal solutions in $(0,2)$. Thus, $F(q)$ achieves its maximum value outside the interval, which contradicts our assumption of having the maximum value at the interior point of $q \in [0,2]$. Thus any maximum point of F must be on the boundary of $[0,2]$.
It is clear that $F(0) > F(2)$. Hence the maximum is achieved at $q=0$. Therefore the upper bound for (11) corresponds to $\mu=1$ and $q=0$. Hence from (11) we obtain (7).

□

For $k=1$ in Theorem 7, we have the following result proved by Janteng [12].

Corollary 1. *If $f(w) \in \widetilde{\mathcal{S}}^*$, then*

$$|a_2a_4-a_3^2| \le 1.$$

We can prove on similar lines the following theorem.

Theorem 2. *Let $f \in \widetilde{\mathcal{K}}_k$, then*

$$|a_2a_4-a_3^2| \le \frac{4}{9(3-\psi_3)^2}, \tag{15}$$

where ψ_n is defined by (5).

Replacing k by 1 in Theorem 2, we have the following result proved by Janteng [12].

Corollary 2. *If $f(w) \in \widetilde{\mathcal{K}}$, then*

$$|a_2a_4-a_3^2| \le \frac{1}{9}.$$

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