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Growth Properties of Solutions to Higher Order Complex Linear Differential Equations with Analytic Coefficients in the Annulus

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Abstract. In this paper, by using the Nevanlinna value distribution theory of meromorphic functions on an annulus, we deal with the growth properties of solutions of the linear differential equation $f^{(k)}+B_{k-1}(z)\,f^{(k-1)}+\cdots+B_1(z)\,f'+B_0(z)\,f=0$, where $k\geq 2$ is an integer and $B_{k-1}(z),...,B_1(z),B_0(z)$ are analytic on an annulus. Under some conditions on the coefficients, we obtain some results concerning the estimates of the order and the hyper-order of solutions of the above equation. The results obtained extend and improve those of Wu and Xuan in [16].

Mathematics subject classification: 30D10, 30D20, 30B10, 34M05.
Keywords and phrases: linear differential equations, analytic solutions, annulus, hyper order.

1 Introduction and results

Throughout this article, we shall assume that the reader is familiar with the standard notations and fundamental results of the Nevanlinna value distribution theory of meromorphic functions in the complex plane and in the unit disc $\mathbb{D}=\{z\in\mathbb{C}:|z|<1\}$, see [3, 4, 10, 14, 17].

Nevanlinna theory has appeared to be powerful tool in the field of complex differential equations in the complex plane and in the unit disc which are simple connected domains. In the year 2000, Heittokangas [5] firstly investigated the growth and oscillation theory of second and higher order linear differential equations when the coefficients are analytic functions in the unit disc $\mathbb{D}$, by introducing the definition of the function spaces. Recently, Wu [15], Long [11], Belaïdi [2], Zemirni and Belaïdi [18] have obtained some results about the growth of analytic solutions of higher order linear differential equations in a sector of the unit disc. It is well-known that Nevanlinna theory of meromorphic functions in the complex plane and in the unit disc can be extended in a modified form to multiply-connected plane domains, in particular in the annulus [6–9, 12, 13] which is a doubly-connected domain. In 2005, Khrystiyanyn and Kondratyuk [6, 7] gave an extension of the Nevanlinna value distribution theory for meromorphic functions in annuli. In their extension the main characteristics of meromorphic functions are one-parameter and possessing the same

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properties as in the classical case of a simply connected domain. From the doubly-connected mapping theorem [1], we can get that each doubly-connected domain is conformally equivalent to the annulus $\{z : r < |z| < R,\ 0 \leq r < R \leq +\infty\}$. We consider only two cases: $r = 0$, $R = +\infty$ simultaneously and $0 \leq r < R \leq +\infty$. In the latter case, the homothety $z \longmapsto \frac{z}{\sqrt{rR}}$ reduces the given domain to the annulus $\frac{1}{R_0} < |z| < R_0$, where $R_0 = \sqrt{\frac{R}{r}}$. Thus, every annulus is invariant with respect to the inversion $z \longmapsto \frac{1}{z}$ in two cases.

Before stating our main results, we give some notations and basic definitions of meromorphic functions in the annulus $\mathcal{A} = \left\{z : \frac{1}{R_0} < |z| < R_0\right\}$, where $1 < R_0 \leq +\infty$. Let f be a meromorphic function in the complex plane, we define

$$m(r, f) = \frac{1}{2\pi} \int_0^{2\pi} \log^+ \left|f\left(re^{i\varphi}\right)\right| d\varphi,$$

$$N(r, f) = \int_0^r \frac{n(t, f) - n(0, f)}{t} dt + n(0, f) \log r$$

and

$$T(r, f) = m(r, f) + N(r, f) \quad (r > 0)$$

is the Nevanlinna characteristic function of f, where $\log^+ x = \max(0, \log x)$ for $x \geq 0$, and $n(t,\ f)$ is the number of poles of f lying in $\{z : |z| \leq t\}$, counted according to their multiplicity. Now, we give the Nevanlinna theory in the annulus $\mathcal{A} = \left\{z : \frac{1}{R_0} < |z| < R_0\right\}$, where $1 < R_0 \leq +\infty$. Set

$$N_1(r, f) = \int_{\frac{1}{r}}^1 \frac{n_1(t, f)}{t} dt, \quad N_2(r, f) = \int_1^r \frac{n_2(t, f)}{t} dt,$$

$$m_0(r, f) = m(r, f) + m\left(\frac{1}{r}, f\right) - 2m(1, f),$$

$$N_0(r, f) = N_1(r, f) + N_2(r, f),$$

where $n_1(t, f)$ and $n_2(t, f)$ are the counting functions of poles of f lying in $\{z : t < |z| \leq 1\}$ and $\{z : 1 < |z| \leq t\}$ respectively, counted according to their multiplicity. The Nevanlinna characteristic of f in the annulus $\mathcal{A}$ is defined by

$$T_0(r, f) = m_0(r, f) + N_0(r, f).$$

Definition 1. ([16]) Let f be a nonconstant meromorphic function in the annulus $\mathcal{A} = \left\{z : \frac{1}{R_0} < |z| < R_0\right\}$, where $1 < R_0 \leq +\infty$. The function f is called a transcendental or admissible in $\mathcal{A}$ provided that

$$\limsup_{r \to +\infty} \frac{T_0(r, f)}{\log r} = +\infty \text{ if } 1 < r < R_0 = +\infty$$

or

$$\limsup_{r\to R_0^-}\frac{T_0(r,f)}{\log\frac{1}{R_0-r}}=+\infty \text{ if } 1<r<R_0<+\infty$$

respectively. The order of f is defined as

$$\rho_{\mathcal{A}}(f)=\limsup_{r\to+\infty}\frac{\log T_0(r,f)}{\log r} \text{ if } 1<r<R_0=+\infty$$

or

$$\rho_{\mathcal{A}}(f)=\limsup_{r\to R_0^-}\frac{\log T_0(r,f)}{\log\frac{1}{R_0-r}} \text{ if } 1<r<R_0<+\infty$$

respectively. The hyper-order of f is defined as

$$\rho_{2,\mathcal{A}}(f)=\limsup_{r\to+\infty}\frac{\log\log T_0(r,f)}{\log r} \text{ if } 1<r<R_0=+\infty$$

or

$$\rho_{2,\mathcal{A}}(f)=\limsup_{r\to R_0^-}\frac{\log\log T_0(r,f)}{\log\frac{1}{R_0-r}} \text{ if } 1<r<R_0<+\infty$$

respectively.

Now, we introduce the concepts of lower order, hyper lower order, type and lower type of a meromorphic function f in the annulus $\mathcal{A}$.

Definition 2. Let f be a meromorphic function in $\mathcal{A}$. The lower order of f is defined as

$$\mu_{\mathcal{A}}(f)=\liminf_{r\to+\infty}\frac{\log T_0(r,f)}{\log r} \text{ if } 1<r<R_0=+\infty$$

or

$$\mu_{\mathcal{A}}(f)=\liminf_{r\to R_0^-}\frac{\log T_0(r,f)}{\log\frac{1}{R_0-r}} \text{ if } 1<r<R_0<+\infty$$

respectively. The hyper lower order of f is defined as

$$\mu_{2,\mathcal{A}}(f)=\liminf_{r\to+\infty}\frac{\log\log T_0(r,f)}{\log r} \text{ if } 1<r<R_0=+\infty$$

or

$$\mu_{2,\mathcal{A}}(f)=\liminf_{r\to R_0^-}\frac{\log\log T_0(r,f)}{\log\frac{1}{R_0-r}} \text{ if } 1<r<R_0<+\infty$$

respectively.

Definition 3. Let f be a meromorphic function in $\mathcal{A}$ with order $0<\rho_{\mathcal{A}}(f)<+\infty$. Then, the type of f is defined by

$$\tau_{\mathcal{A}}(f)=\limsup_{r\to+\infty}\frac{T_0(r,f)}{r^{\rho_{\mathcal{A}}(f)}} \text{ if } 1<r<R_0=+\infty$$

or

$$\tau_{\mathcal{A}}(f) = \limsup_{r \to R_0^-} \frac{T_0(r, f)}{\left(\frac{1}{R_0 - r}\right)^{\rho_{\mathcal{A}}(f)}} \text{ if } 1 < r < R_0 < +\infty$$

respectively. Similarly, let f be a meromorphic function in $\mathcal{A}$ with lower order $0 < \mu_{\mathcal{A}}(f) < +\infty$. Then, the lower type of f is defined by

$$\underline{\tau}_{\mathcal{A}}(f) = \liminf_{r \to R_0^-} \frac{T_0(r, f)}{r^{\mu_{\mathcal{A}}(f)}} \text{ if } 1 < r < R_0 = +\infty$$

or

$$\underline{\tau}_{\mathcal{A}}(f) = \liminf_{r \to R_0^-} \frac{T_0(r, f)}{\left(\frac{1}{R_0 - r}\right)^{\mu_{\mathcal{A}}(f)}} \text{ if } 1 < r < R_0 = +\infty$$

respectively.

For $k \geq 2$, we consider the linear differential equation

$$f^{(k)} + B_{k-1}(z) f^{(k-1)} + \cdots + B_1(z) f' + B_0(z) f = 0, \quad (1)$$

where $B_{k-1}(z), ..., B_1(z)$ and $B_0(z)$ are analytic in the annulus

$$\mathcal{A} = \left\{ z : \frac{1}{R_0} < |z| < R_0 \right\} \ (1 < R_0 \leq +\infty).$$

Recently in [16], Wu and Xuan have studied the growth of solutions of higher order linear complex differential equations in $\mathcal{A}$ and obtained the following result.

Theorem 1. *([16]) Let $B_{k-1}(z), ..., B_1(z)$, $B_0(z)$ be analytic functions in the annulus $\mathcal{A} = \left\{ z : \frac{1}{R_0} < |z| < R_0 \right\}$ $(1 < R_0 \leq +\infty)$ such that*

$$\max\{\rho_{\mathcal{A}}(B_j) : j = 1, 2, ..., k-1\} < \rho_{\mathcal{A}}(B_0).$$

Then every solution $f \not\equiv 0$ of equation (1) satisfies $\rho_{\mathcal{A}}(f) = +\infty$ and $\rho_{2,\mathcal{A}}(f) \geq \rho_{\mathcal{A}}(B_0)$.

Note that the result of Theorem 1 occurs when there exists only one dominant coefficient. Thus, the following question arises naturally: Whether the results similar to Theorem 1 can be obtained in $\mathcal{A}$ if there are more than one dominant coefficients? In this paper, we give some answers to the above question. In fact, by using the concepts of the type and the lower type, we obtain some results which indicate growth estimate of every non-trivial analytic solution of equation (1) by the growth estimate of the coefficient $B_0(z)$. We mainly obtain the following results.

Theorem 2. *Let $B_{k-1}(z), ..., B_1(z),\ B_0(z)\ (k \geq 2)$ be analytic functions in the annulus $\mathcal{A} = \left\{z : \frac{1}{R_0} < |z| < R_0\right\}\ (1 < R_0 \leq +\infty)$. Suppose that there exist three positive real numbers α, β and μ with $0 \leq (k-1)\beta < \alpha$, $\mu > 0$, such that we have*

$$T_0(r, B_0) \geq \alpha r^{\mu} \tag{2}$$

and

$$T_0(r, B_j) \leq \beta r^{\mu},\ j = 1, \ldots, k-1 \tag{3}$$

if $1 < r < R_0 = +\infty$ as $|z| = r \to +\infty$ for $r \in E_r$ which satisfies $\int\limits_{E_r} \frac{dr}{r} = +\infty$, or

$$T_0(r, B_0) \geq \frac{\alpha}{(R_0 - r)^{\mu}} \tag{4}$$

and

$$T_0(r, B_j) \leq \frac{\beta}{(R_0 - r)^{\mu}}\ \ (j = 1, ..., k-1) \tag{5}$$

if $1 < r < R_0 < +\infty$ as $|z| = r \to R_0^-$ for $r \in F_r$ which satisfies $\int\limits_{F_r} \frac{dr}{R_0 - r} = +\infty$. Then every solution $f \not\equiv 0$ of equation (1) satisfies $\rho_{\mathcal{A}}(f) = +\infty$ and $\rho_{2,\mathcal{A}}(f) \geq \mu$.

Theorem 3. *Let $B_{k-1}(z), ..., B_1(z),\ B_0(z)\ (k \geq 2)$ be analytic functions in the annulus $\mathcal{A} = \left\{z : \frac{1}{R_0} < |z| < R_0\right\}\ (1 < R_0 \leq +\infty)$ such that*

$$\max\{\rho_{\mathcal{A}}(B_j) : j = 1, 2, ..., k-1\} \leq \rho_{\mathcal{A}}(B_0) = \rho\ \ (0 < \rho < \infty)$$

and

$$\sum_{\rho_{\mathcal{A}}(B_j) = \rho_{\mathcal{A}}(B_0)} \tau_{\mathcal{A}}(B_j) < \tau_{\mathcal{A}}(B_0) = \tau\ \ (\ 0 < \tau < \infty).$$

Then every solution $f \not\equiv 0$ of equation (1) satisfies $\rho_{\mathcal{A}}(f) = +\infty$ and $\rho_{2,\mathcal{A}}(f) \geq \rho_{\mathcal{A}}(B_0)$.

Theorem 4. *Let $B_{k-1}(z), ..., B_1(z),\ B_0(z)\ (k \geq 2)$ be analytic functions in the annulus $\mathcal{A} = \left\{z : \frac{1}{R_0} < |z| < R_0\right\}\ (1 < R_0 \leq +\infty)$ such that $0 < \mu_{\mathcal{A}}(B_0) = \mu \leq \rho_{\mathcal{A}}(B_0) < \infty$. Assume that*

$$\max\{\rho_{\mathcal{A}}(B_j) : j = 1, 2, ..., k-1\} \leq \mu_{\mathcal{A}}(B_0) = \mu$$

and

$$\sum_{\rho_{\mathcal{A}}(B_j) = \mu_{\mathcal{A}}(B_0)} \tau_{\mathcal{A}}(B_j) < \underline{\tau}_{\mathcal{A}}(B_0) = \underline{\tau}\ \ (\ 0 < \underline{\tau} < \infty).$$

Then every solution $f \not\equiv 0$ of equation (1) satisfies $\rho_{\mathcal{A}}(f) = \mu_{\mathcal{A}}(f) = +\infty$ and $\rho_{2,\mathcal{A}}(f) \geq \mu_{2,\mathcal{A}}(f) \geq \mu_{\mathcal{A}}(B_0)$.

Theorem 5. *Let* $B_{k-1}(z), ..., B_1(z),\ B_0(z)\ (k \geq 2)$ *be analytic functions in the annulus* $\mathcal{A} = \left\{z : \frac{1}{R_0} < |z| < R_0\right\}$ $(1 < R_0 \leq +\infty)$ *such that* $B_0(z)$ *is admissible in* $\mathcal{A}$ *and*

$$\limsup_{r\to+\infty} \frac{\sum_{j=1}^{k-1} m_0(r, B_j)}{m_0(r, B_0)} < 1 \text{ if } 1 < r < R_0 = +\infty$$

or

$$\limsup_{r\to R_0^-} \frac{\sum_{j=1}^{k-1} m_0(r, B_j)}{m_0(r, B_0)} < 1 \text{ if } 1 < r < R_0 < +\infty.$$

Then every solution $f \not\equiv 0$ *of equation* (1) *satisfies* $\rho_{\mathcal{A}}(f) = +\infty$ *and* $\rho_{2,\mathcal{A}}(f) \geq \rho_{\mathcal{A}}(B_0)$.

2 Some Preliminary Lemmas

We need the following lemmas to prove our results.

Lemma 1. *Let* f *be a meromorphic function with finite order* $0 < \rho_{\mathcal{A}}(f) < +\infty$ *and finite type* $0 < \tau_{\mathcal{A}}(f) < +\infty$. *Then for any given* $\eta < \tau_{\mathcal{A}}(f)$, *there exists a subset* E_r *of* $(1, +\infty)$ *with* $\int_{E_r} \frac{dr}{r} = +\infty$ *such that for all* $r \in E_r$

$$T_0(r, f) > \eta r^{\rho_{\mathcal{A}}(f)} \text{ if } 1 < r < R_0 = +\infty$$

holds or there exists a subset $E_r^{/}$ *of* $(1, R_0)$ *with* $\int_{E_r^{/}} \frac{dr}{R_0 - r} = +\infty$ *such that for all* $r \in E_r^{/}$ *holds*

$$T_0(r, f) > \frac{\eta}{(R_0 - r)^{\rho_{\mathcal{A}}(f)}} \text{ if } 1 < r < R_0 < +\infty.$$

Proof. Case $R_0 = +\infty$: By Definition 3, there exists an increasing sequence $\{r_m\}_{m=1}^{\infty}$ $(r_m \to +\infty,\ m \to +\infty)$ satisfying $(1 + \frac{1}{m})r_m < r_{m+1}$ and

$$\lim_{m\to+\infty} \frac{T_0(r_m, f)}{r_m^{\rho_{\mathcal{A}}(f)}} = \tau_{\mathcal{A}}(f).$$

So, there exists a positive integer m_0 such that for all $m \geq m_0$ and for any given $0 < \varepsilon < \tau_{\mathcal{A}}(f) - \eta$, we have

$$T_0(r_m, f) > (\tau_{\mathcal{A}}(f) - \varepsilon) r_m^{\rho_{\mathcal{A}}(f)}. \tag{6}$$

Since

$$\lim_{m\to+\infty} \left(\frac{m}{m+1}\right)^{\rho_{\mathcal{A}}(f)} = 1,$$

then for any given $\eta < \tau_{\mathcal{A}}(f) - \varepsilon$, there exists a positive integer m_1 such that for all $m \geq m_1$, we have

$$\left(\frac{m}{m+1}\right)^{\rho_{\mathcal{A}}(f)} > \frac{\eta}{\tau_{\mathcal{A}}(f) - \varepsilon}. \tag{7}$$

Take $m \geq m_2 = \max\{m_1, m_0\}$. By (6) and (7), for any $r \in \left[r_m, (1 + \frac{1}{m})r_m\right]$

$$T_0(r, f) \geq T_0(r_m, f) > (\tau_{\mathcal{A}}(f) - \varepsilon) r_m^{\rho_{\mathcal{A}}(f)}$$

$$\geq (\tau_{\mathcal{A}}(f) - \varepsilon) \left(\frac{m}{m+1} r\right)^{\rho_{\mathcal{A}}(f)} > \eta r^{\rho_{\mathcal{A}}(f)}.$$

Set $E_r = \bigcup\limits_{m=m_2}^{+\infty} [r_m, (1 + \frac{1}{m})r_m]$. Then there holds

$$\int\limits_{E_r} \frac{dr}{r} = \sum_{m=m_2}^{+\infty} \int\limits_{r_m}^{(1+\frac{1}{m})r_m} \frac{dt}{t} = \sum_{m=m_2}^{+\infty} \log(1 + \frac{1}{m}) = +\infty.$$

Case $R_0 < +\infty$: By Definition 3, there exists an increasing sequence $\{r_m\}_{m=1}^{\infty} \subset (1, R_0)$ $(r_m \to R_0^-,\, m \to +\infty)$ satisfying $R_0 - \left(1 - \frac{1}{m}\right)(R_0 - r_m) < r_{m+1}$ and

$$\lim_{m \to +\infty} \frac{T_0(r_m, f)}{\left(\frac{1}{R_0 - r_m}\right)^{\rho_{\mathcal{A}}(f)}} = \tau_{\mathcal{A}}(f).$$

So, there exists a positive integer m_3 such that for all $m \geq m_3$ and for any given $0 < \varepsilon < \tau_{\mathcal{A}}(f) - \eta$, we have

$$T_0(r_m, f) > (\tau_{\mathcal{A}}(f) - \varepsilon) \left(\frac{1}{R_0 - r_m}\right)^{\rho_{\mathcal{A}}(f)}. \tag{8}$$

Since

$$\lim_{m \to +\infty} \left(1 - \frac{1}{m}\right)^{\rho_{\mathcal{A}}(f)} = 1,$$

then for any given $\eta < \tau_{\mathcal{A}}(f) - \varepsilon$, there exists a positive integer m_4 such that for all $m \geq m_4$, we have

$$\left(1 - \frac{1}{m}\right)^{\rho_{\mathcal{A}}(f)} > \frac{\eta}{\tau_{\mathcal{A}}(f) - \varepsilon}. \tag{9}$$

Take $m \geq m_5 = \max\{m_3, m_4\}$. By (8) and (9), for any $r \in [r_m, R_0 - \left(1 - \frac{1}{m}\right)(R_0 - r_m)]$, we obtain

$$T_0(r, f) \geq T_0(r_m, f) > (\tau_{\mathcal{A}}(f) - \varepsilon) \left(\frac{1}{R_0 - r_m}\right)^{\rho_{\mathcal{A}}(f)}$$

$$\geq (\tau_{\mathcal{A}}(f)-\varepsilon)\left(\frac{1-\frac{1}{m}}{R_0-r}\right)^{\rho_{\mathcal{A}}(f)} > \frac{\eta}{(R_0-r)^{\rho_{\mathcal{A}}(f)}}.$$

Set $E_r^{/} = \bigcup\limits_{m=m_5}^{+\infty} [r_m, R_0 - \left(1-\frac{1}{m}\right)(R_0-r_m)]$. Then there holds

$$\int\limits_{E_r^{/}} \frac{dr}{R_0-r} = \sum_{m=m_5}^{+\infty} \int\limits_{r_m}^{R_0-\left(1-\frac{1}{m}\right)(R_0-r_m)} \frac{dt}{R_0-t} = \sum_{m=m_5}^{+\infty} \log\frac{m}{m-1} = +\infty.$$

□

Lemma 2. *([7],[16]) (The lemma of the logarithmic derivative). Let f be a non-constant meromorphic function in the annulus $\mathcal{A} = \left\{z : \frac{1}{R_0} < |z| < R_0\right\}$, where $1 < r < R_0 \leq +\infty$, and $k \geq 1$ be an integer. Then*

$$m_0\left(r, \frac{f^{(k)}}{f}\right) = \begin{cases} O(\log r), \ R_0 = +\infty \text{ and } \rho_{\mathcal{A}}(f) < +\infty, \\ O\left(\log\frac{1}{R_0-r}\right), \ R_0 < +\infty \text{ and } \rho_{\mathcal{A}}(f) < +\infty, \\ O(\log r + \log T_0(r,f)), \ r \notin \Delta_r, \ R_0 = +\infty, \\ O\left(\log\frac{1}{R_0-r} + \log T_0(r,f)\right), \ r \notin \Delta_r', \ R_0 < +\infty, \end{cases}$$

where Δ_r and Δ_r' are sets with $\int\limits_{\Delta_r} \frac{dr}{r} < +\infty$ and $\int\limits_{\Delta_r'} \frac{dr}{R_0-r} < +\infty$ respectively.

Lemma 3. *Let f be a meromorphic function with finite order $\rho_{\mathcal{A}}(f) < +\infty$. Then, there exists a subset E_r of $(1,+\infty)$ with $\int\limits_{E_r} \frac{dr}{r} = +\infty$ such that for all $r \in E_r$ holds*

$$\rho_{\mathcal{A}}(f) = \lim_{r\to+\infty} \frac{\log T_0(r,f)}{\log r} \ \text{if } 1 < r < R_0 = +\infty$$

or there exists a subset $E_r^{/}$ of $(1, R_0)$ with $\int\limits_{E_r^{/}} \frac{dr}{R_0-r} = +\infty$ such that for all $r \in E_r^{/}$ holds

$$\rho_{\mathcal{A}}(f) = \lim_{r\to R_0^-} \frac{\log T_0(r,f)}{\log\frac{1}{R_0-r}} \ \text{if } 1 < r < R_0 < +\infty.$$

Proof. Case $R_0 = +\infty$. The definition of $\rho_{\mathcal{A}}(f)$ implies that there exists a sequence $\{r_n\}_{n=1}^{\infty}$ $(r_n \to +\infty, n \to +\infty)$ satisfying $\left(1+\frac{1}{n}\right) r_n < r_{n+1}$ and

$$\lim_{n\to+\infty} \frac{\log T_0(r_n,f)}{\log r_n} = \rho_{\mathcal{A}}(f).$$

Then, there exists an integer number n_1 such that for all $n \geq n_1$ and for any $r \in \left[r_n, \left(1+\frac{1}{n}\right) r_n\right]$, we have

$$\frac{\log T_0(r_n,f)}{\log\left(1+\frac{1}{n}\right) r_n} = \frac{\log T_0(r_n,f)}{\log\left(1+\frac{1}{n}\right) + \log r_n} \leq \frac{\log T_0(r,f)}{\log r}$$

$$\leq \frac{\log T_0\left(\left(1+\frac{1}{n}\right)r_n, f\right)}{\log r_n} = \frac{\log T_0\left(\left(1+\frac{1}{n}\right)r_n, f\right)}{\log\left(1+\frac{1}{n}\right)r_n} \cdot \frac{\log\left(1+\frac{1}{n}\right)+\log r_n}{\log r_n}.$$

Setting $E_r = \bigcup\limits_{n=n_1}^{+\infty}\left[r_n, \left(1+\frac{1}{n}\right)r_n\right]$, then for any $r \in E_r$, we get

$$\lim_{r\to+\infty}\frac{\log T_0(r,f)}{\log r} = \lim_{n\to+\infty}\frac{\log T_0(r_n,f)}{\log r_n} = \rho_{\mathcal{A}}(f),$$

where

$$\int\limits_{E_r}\frac{dr}{r} = \sum_{n=n_1}^{+\infty}\int\limits_{r_n}^{\left(1+\frac{1}{n}\right)r_n}\frac{dt}{t} = \sum_{n=n_1}^{+\infty}\log\left(1+\frac{1}{n}\right) = +\infty.$$

Case $R_0 < +\infty$: By definition of $\rho_{\mathcal{A}}(f)$, there exists an increasing sequence $\{r_n\}_{n=1}^{\infty} \subset (1, R_0)$ ($r_n \to R_0^-$, $n \to +\infty$) satisfying $R_0 - \left(1-\frac{1}{n}\right)(R_0 - r_n) < r_{n+1}$ and

$$\lim_{n\to+\infty}\frac{\log T_0(r_n,f)}{\log\frac{1}{R_0-r_n}} = \rho_{\mathcal{A}}(f).$$

So, there exists a positive integer n_2 such that for all $n \geq n_2$ and for any $r \in \left[r_n, R_0 - \left(1-\frac{1}{n}\right)(R_0-r_n)\right]$, we have

$$\frac{\log T_0(r_n,f)}{\log\frac{1}{\left(1-\frac{1}{n}\right)(R_0-r_n)}} \leq \frac{\log T_0(r,f)}{\log\frac{1}{R_0-r}} \leq \frac{\log T_0\left(R_0-\left(1-\frac{1}{n}\right)(R_0-r_n), f\right)}{\log\frac{1}{R_0-r_n}}.$$

It follows that

$$\frac{\log T_0(r_n,f)}{\log\frac{n}{n-1}+\log\frac{1}{R_0-r_n}} \leq \frac{\log T_0(r,f)}{\log\frac{1}{R_0-r}}$$

$$\leq \frac{\log T_0\left(R_0-\left(1-\frac{1}{n}\right)(R_0-r_n), f\right)}{\log\frac{1}{\left(1-\frac{1}{n}\right)(R_0-r_n)}} \cdot \frac{\log\frac{1}{\left(1-\frac{1}{n}\right)(R_0-r_n)}}{\log\frac{1}{R_0-r_n}}.$$

Set $E_r^{/} = \bigcup\limits_{n=n_2}^{+\infty}\left[r_n, R_0-\left(1-\frac{1}{n}\right)(R_0-r_n)\right]$. Then for any $r \in E_r^{/}$, we get

$$\lim_{r\to R_0^-}\frac{\log T_0(r,f)}{\log\frac{1}{R_0-r}} = \lim_{n\to+\infty}\frac{\log T_0(r_n,f)}{\log\frac{1}{R_0-r_n}} = \rho_{\mathcal{A}}(f),$$

where

$$\int\limits_{E_r^{/}}\frac{dr}{R_0-r} = \sum_{n=n_2}^{+\infty}\int\limits_{r_n}^{R_0-\left(1-\frac{1}{n}\right)(R_0-r_n)}\frac{dt}{R_0-t} = \sum_{n=n_2}^{+\infty}\log\frac{n}{n-1} = +\infty.$$

$\square$

3 Proofs of the Theorems

Proof of Theorem 2

Proof. Let $f \not\equiv 0$ be a solution of (1). We divide through equation (1) by f to get

$$-B_0(z) = \frac{f^{(k)}(z)}{f(z)} + \sum_{j=1}^{k-1} B_j(z) \frac{f^{(j)}(z)}{f(z)}. \tag{10}$$

By (10) and Lemma 2, it follows that

$$m_0(r, B_0) = T_0(r, B_0) \leq \sum_{j=1}^{k-1} m_0(r, B_j) + \sum_{j=1}^{k} m_0\left(r, \frac{f^{(j)}}{f}\right) + O(1)$$

$$\leq \sum_{j=1}^{k-1} m_0(r, B_j) + \left\{ \begin{array}{c} O(\log r + \log T_0(r, f)), \ R_0 = +\infty, \ r \notin \Delta_r \\ O\left(\log \frac{1}{R_0 - r} + \log T_0(r, f)\right), \ R_0 < +\infty, \ r \notin \Delta'_r \end{array} \right.$$

$$= \sum_{j=1}^{k-1} T_0(r, B_j) + \left\{ \begin{array}{c} O(\log r + \log T_0(r, f)), \ R_0 = +\infty, \ r \notin \Delta_r, \\ O\left(\log \frac{1}{R_0 - r} + \log T_0(r, f)\right), \ R_0 < +\infty, \ r \notin \Delta'_r, \end{array} \right. \tag{11}$$

where Δ_r and Δ'_r are sets with $\int_{\Delta_r} \frac{dr}{r} < +\infty$ and $\int_{\Delta'_r} \frac{dr}{R_0 - r} < +\infty$ respectively.
Case $R_0 = +\infty$. By substituting (2) and (3) into (11), we conclude for $r \in E_r \backslash \Delta_r$ sufficiently large

$$\alpha r^{\mu} \leq (k-1)\beta r^{\mu} + O(\log r + \log T_0(r, f)). \tag{12}$$

From (12), we obtain

$$(\alpha - (k-1)\beta) r^{\mu} \leq O(\log r + \log T_0(r, f))$$

and since $\alpha > (k-1)\beta$, this leads to $\rho_{\mathcal{A}}(f) = +\infty$ and $\rho_{2,\mathcal{A}}(f) \geq \mu$.

Case $R_0 < +\infty$. Let $f \not\equiv 0$ be a solution of (1). By substituting (4) and (5) into (11), we conclude for $r \in F_r \backslash \Delta'_r$, $r \to R_0^-$

$$\frac{\alpha}{(R_0 - r)^{\mu}} \leq (k-1) \frac{\beta}{(R_0 - r)^{\mu}} + O(\log \frac{1}{R_0 - r} + \log T_0(r, f)). \tag{13}$$

Then by (13), we obtain

$$\frac{\alpha - (k-1)\beta}{(R_0 - r)^{\mu}} \leq O(\log \frac{1}{R_0 - r} + \log T_0(r, f))$$

and since $\alpha > (k-1)\beta$, this leads to $\rho_{\mathcal{A}}(f) = +\infty$ and $\rho_{2,\mathcal{A}}(f) \geq \mu$. $\square$

Proof of Theorem 3

Proof. Let $f \not\equiv 0$ be a solution of (1). If $\rho_{\mathcal{A}}(B_j) < \rho_{\mathcal{A}}(B_0)$ for all $1, 2, ..., k-1$, then Theorem 3 reduces to Theorem 1. Thus, we assume that at least one of B_j $(1, 2, ..., k-1)$ satisfies $\rho_{\mathcal{A}}(B_j) = \rho_{\mathcal{A}}(B_0) = \rho$. So, there exists a set $J \subseteq \{1, 2, ..., k-1\}$ such that for $j \in J$, we have $\rho_{\mathcal{A}}(B_j) = \rho_{\mathcal{A}}(B_0) = \rho$ with $\sum\limits_{j\in J} \tau_{\mathcal{A}}(B_j) < \tau_{\mathcal{A}}(B_0) = \tau$ and for $j \in \{1, 2, ..., k-1\} \backslash J$, we have $\rho_{\mathcal{A}}(B_j) < \rho_{\mathcal{A}}(B_0) = \rho$. Hence, we can choose α_1, α_2 satisfying $\sum\limits_{j\in J} \tau_{\mathcal{A}}(B_j) < \alpha_1 < \alpha_2 < \tau$ such that for sufficiently large r and any given ε $\left(0 < \varepsilon < \frac{\alpha_2 - \alpha_1}{k-1}\right)$, we have

$$T_0(r, B_j) = m_0(r, B_j) \leq (\tau_{\mathcal{A}}(B_j) + \varepsilon) r^{\rho_{\mathcal{A}}(B_j)} = (\tau_{\mathcal{A}}(B_j) + \varepsilon) r^{\rho}, \ j \in J \tag{14}$$

and

$$T_0(r, B_j) = m_0(r, B_j) \leq r^{\rho_0}, \ j \in \{1, 2, ..., k-1\} \backslash J, \tag{15}$$

where $0 < \rho_0 < \rho$. For $r \to R_0^-$ and any given ε $\left(0 < \varepsilon < \frac{\alpha_2 - \alpha_1}{k-1}\right)$, we obtain

$$\begin{aligned} T_0(r, B_j) = m_0(r, B_j) &\leq (\tau_{\mathcal{A}}(B_j) + \varepsilon) \left(\frac{1}{R_0 - r}\right)^{\rho_{\mathcal{A}}(B_j)} \\ &= (\tau_{\mathcal{A}}(B_j) + \varepsilon) \left(\frac{1}{R_0 - r}\right)^{\rho}, \ j \in J \end{aligned} \tag{16}$$

and

$$T_0(r, B_j) = m_0(r, B_j) \leq \left(\frac{1}{R_0 - r}\right)^{\rho_0}, \ j \in \{1, 2, ..., k-1\} \backslash J, \tag{17}$$

where $0 < \rho_0 < \rho$. By applying Lemma 1, there exists a subset E_r of $(1, \infty)$ with $\int\limits_{E_r} \frac{dr}{r} = +\infty$ such that for all $r \in E_r$, we have

$$T_0(r, B_0) = m_0(r, B_0) > \alpha_2 r^{\rho} \text{ if } 1 < r < R_0 = +\infty \tag{18}$$

or there exists a subset $E_r^{/}$ of $(1, R_0)$ with $\int\limits_{E_r^{/}} \frac{dr}{R_0 - r} = +\infty$ such that for all $r \in E_r^{/}$ holds

$$T_0(r, B_0) = m_0(r, B_0) > \alpha_2 \left(\frac{1}{R_0 - r}\right)^{\rho} \text{ if } 1 < r < R_0 < +\infty. \tag{19}$$

Case $R_0 = +\infty$: By substituting the assumptions (14), (15) and (18) into (11), for all sufficiently large $r \in E_r \backslash \Delta_r$ and any given ε $\left(0 < \varepsilon < \frac{\alpha_2 - \alpha_1}{k-1}\right)$, we obtain

$$\begin{aligned} \alpha_2 r^{\rho} &\leq \sum_{j\in J} (\tau_{\mathcal{A}}(B_j) + \varepsilon) r^{\rho} + \sum_{j\in\{1,...,k-1\}\backslash J} r^{\rho_0} + O(\log r + \log T_0(r, f)) \\ &\leq (\alpha_1 + (k-1)\varepsilon) r^{\rho} + (k-1) r^{\rho_0} + O(\log r + \log T_0(r, f)). \end{aligned}$$

It follows that

$$(\alpha_2 - \alpha_1 - (k-1)\varepsilon) r^{\rho} \leq (k-1) r^{\rho_0} + O(\log r + \log T_0(r,f)). \tag{20}$$

Since $\varepsilon \left(0 < \varepsilon < \frac{\alpha_2 - \alpha_1}{k-1}\right)$, then from (20), we get $\rho_{\mathcal{A}}(f) = +\infty$ and $\rho_{2,\mathcal{A}}(f) \geq \rho_{\mathcal{A}}(B_0) = \rho$.

Case $R_0 < +\infty$: By substituting the assumptions (16), (17) and (19) into (11), for all $r \in E'_r \backslash \Delta'_r$ with $r \to R_0^-$ and any given $\varepsilon \left(0 < \varepsilon < \frac{\alpha_2 - \alpha_1}{k-1}\right)$, we obtain

$$\begin{aligned}
\alpha_2 \left(\frac{1}{R_0 - r}\right)^{\rho} &\leq \sum_{j \in J} (\tau_{\mathcal{A}}(B_j) + \varepsilon) \left(\frac{1}{R_0 - r}\right)^{\rho} \\
&+ \sum_{j \in \{1,\ldots,k-1\} \backslash J} \left(\frac{1}{R_0 - r}\right)^{\rho_0} + O\left(\log \frac{1}{R_0 - r} + \log T_0(r,f)\right) \\
&\leq (\alpha_1 + (k-1)\varepsilon) \left(\frac{1}{R_0 - r}\right)^{\rho} + (k-1) \left(\frac{1}{R_0 - r}\right)^{\rho_0} \\
&+ O\left(\log \frac{1}{R_0 - r} + \log T_0(r,f)\right).
\end{aligned}$$

It follows that

$$\begin{aligned}
(\alpha_2 - \alpha_1 - (k-1)\varepsilon) \left(\frac{1}{R_0 - r}\right)^{\rho} &\leq (k-1) \left(\frac{1}{R_0 - r}\right)^{\rho_0} \\
&+ O\left(\log \frac{1}{R_0 - r} + \log T_0(r,f)\right).
\end{aligned} \tag{21}$$

Since $\varepsilon \left(0 < \varepsilon < \frac{\alpha_2 - \alpha_1}{k-1}\right)$, then from (21), we obtain $\rho_{\mathcal{A}}(f) = +\infty$ and $\rho_{2,\mathcal{A}}(f) \geq \rho_{\mathcal{A}}(B_0) = \rho$. $\square$

Proof of Theorem 4

Proof. Let $f \not\equiv 0$ be a solution of (1). First, we suppose that $b = \max\{\rho_{\mathcal{A}}(B_j) : j = 1, 2, \ldots, k-1\} < \mu_{\mathcal{A}}(B_0) = \mu$. Then, for any given ε $(0 < 2\varepsilon < \mu - b)$ and sufficiently large r, we have

$$T_0(r, B_0) = m_0(r, B_0) \geq r^{\mu - \varepsilon} \tag{22}$$

and

$$T_0(r, B_j) = m_0(r, B_j) \leq r^{b+\varepsilon}, \quad j = 1, 2, \ldots, k-1. \tag{23}$$

For $r \to R_0^-$ and any given ε $(0 < 2\varepsilon < \mu - b)$, we obtain

$$T_0(r, B_0) = m_0(r, B_0) \geq \left(\frac{1}{R_0 - r}\right)^{\mu - \varepsilon} \tag{24}$$

and

$$T_0(r,B_j)=m_0(r,B_j)\le\left(\frac{1}{R_0-r}\right)^{b+\varepsilon},\quad j=1,2,...,k-1. \tag{25}$$

Case $R_0=+\infty$: By substituting the assumptions (22) and (23) into (11), for any given ε $(0<2\varepsilon<\mu-b)$ and sufficiently large $r\notin\Delta_r$, we obtain

$$r^{\mu-\varepsilon}\le(k-1)\,r^{b+\varepsilon}+O\left(\log r+\log T_0(r,f)\right). \tag{26}$$

Since ε $\left(0<\varepsilon<\frac{\mu-b}{2}\right)$, then from (26) we get $\rho_{\mathcal{A}}(f)=\mu_{\mathcal{A}}(f)=+\infty$ and $\rho_{2,\mathcal{A}}(f)\ge\mu_{2,\mathcal{A}}(f)\ge\mu_{\mathcal{A}}(B_0)=\mu$.

Case $R_0<+\infty$: By substituting the assumptions (24) and (25) into (11), for any given ε $(0<2\varepsilon<\mu-b)$ and $r\to R_0^-$, $r\notin\Delta_r'$ we obtain

$$\left(\frac{1}{R_0-r}\right)^{\mu-\varepsilon}\le(k-1)\left(\frac{1}{R_0-r}\right)^{b+\varepsilon}+O\left(\log\frac{1}{R_0-r}+\log T_0(r,f)\right). \tag{27}$$

Since ε $\left(0<\varepsilon<\frac{\mu-b}{2}\right)$, then from (27) we have $\rho_{\mathcal{A}}(f)=\mu_{\mathcal{A}}(f)=+\infty$ and $\rho_{2,\mathcal{A}}(f)\ge\mu_{2,\mathcal{A}}(f)\ge\mu_{\mathcal{A}}(B_0)=\mu$.

Assume

$$\max\{\rho_{\mathcal{A}}(B_j):j=1,2,...,k-1\}=\mu_{\mathcal{A}}(B_0)=\mu$$

and $\tau_1=\sum\limits_{\rho_{\mathcal{A}}(B_j)=\mu_{\mathcal{A}}(B_0)}\tau_{\mathcal{A}}(B_j)<\underline{\tau}_{\mathcal{A}}(B_0)=\underline{\tau}$. Then, there exists a set $J\subseteq\{1,2,...,k-1\}$ such that for $j\in J$, we have $\rho_{\mathcal{A}}(B_j)=\mu_{\mathcal{A}}(B_0)=\mu$ with $\tau_1=\sum\limits_{j\in J}\tau_{\mathcal{A}}(B_j)<\underline{\tau}_{\mathcal{A}}(B_0)=\underline{\tau}$ and for $j\in\{1,2,...,k-1\}\backslash J$, we have $\rho_{\mathcal{A}}(B_j)<\mu_{\mathcal{A}}(B_0)=\mu$. Hence, we can choose β_1, β_2 satisfying $\sum\limits_{j\in J}\tau_{\mathcal{A}}(B_j)<\beta_1<\beta_2<\underline{\tau}$ such that for sufficiently large r and any given ε $\left(0<\varepsilon<\frac{\beta_2-\beta_1}{k-1}\right)$, we have

$$T_0(r,B_j)=m_0(r,B_j)\le(\tau_{\mathcal{A}}(B_j)+\varepsilon)\,r^{\rho_{\mathcal{A}}(B_j)}=(\tau_{\mathcal{A}}(B_j)+\varepsilon)\,r^{\mu},\ j\in J \tag{28}$$

and

$$T_0(r,B_j)=m_0(r,B_j)\le r^{\rho_1},\ j\in\{1,2,...,k-1\}\backslash J, \tag{29}$$

where $0<\rho_1<\mu$. By the definition of lower type for sufficiently large r, we have

$$T_0(r,B_0)=m_0(r,B_0)\ge\beta_2 r^{\mu}. \tag{30}$$

For $r\to R_0^-$ and any given ε $\left(0<\varepsilon<\frac{\beta_2-\beta_1}{k-1}\right)$, we obtain

$$T_0(r,B_j)=m_0(r,B_j)\le(\tau_{\mathcal{A}}(B_j)+\varepsilon)\left(\frac{1}{R_0-r}\right)^{\rho_{\mathcal{A}}(B_j)}$$

$$= (\tau_{\mathcal{A}}(B_j) + \varepsilon)\left(\frac{1}{R_0 - r}\right)^{\mu}, \ j \in J \tag{31}$$

and

$$T_0(r, B_j) = m_0(r, B_j) \le \left(\frac{1}{R_0 - r}\right)^{\rho_1}, \ j \in \{1, 2, ..., k-1\} \setminus J, \tag{32}$$

where $0 < \rho_1 < \mu$. By the definition of lower type, for $r \to R_0^-$, we have

$$T_0(r, B_0) = m_0(r, B_0) \ge \beta_2 \left(\frac{1}{R_0 - r}\right)^{\mu}. \tag{33}$$

Case $R_0 = +\infty$: By substituting the assumptions (28), (29) and (30) into (11), for all sufficiently large $r \notin \Delta_r$ any given $\varepsilon \left(0 < \varepsilon < \frac{\beta_2 - \beta_1}{k-1}\right)$, we obtain

$$\beta_2 r^{\mu} \le \sum_{j \in J} (\tau_{\mathcal{A}}(B_j) + \varepsilon) r^{\mu} + \sum_{j \in \{1,...,k-1\} \setminus J} r^{\rho_1} + O(\log r + \log T_0(r, f))$$
$$\le (\beta_1 + (k-1)\varepsilon) r^{\mu} + (k-1) r^{\rho_1} + O(\log r + \log T_0(r, f)).$$

It follows that

$$(\beta_2 - \beta_1 - (k-1)\varepsilon) r^{\mu} \le (k-1) r^{\rho_1} + O(\log r + \log T_0(r, f)). \tag{34}$$

From (34), since $\varepsilon \left(0 < \varepsilon < \frac{\beta_2 - \beta_1}{k-1}\right)$, we have $\rho_{\mathcal{A}}(f) = \mu_{\mathcal{A}}(f) = +\infty$ and $\rho_{2,\mathcal{A}}(f) \ge \mu_{2,\mathcal{A}}(f) \ge \mu_{\mathcal{A}}(B_0) = \mu$.

Case $R_0 < +\infty$: By substituting the assumptions (31), (32) and (33) into (11), for all $r \notin \Delta_r'$ with $r \to R_0^-$ and any given $\varepsilon \left(0 < \varepsilon < \frac{\beta_2 - \beta_1}{k-1}\right)$, we obtain

$$\beta_2 \left(\frac{1}{R_0 - r}\right)^{\mu} \le \sum_{j \in J} (\tau_{\mathcal{A}}(B_j) + \varepsilon) \left(\frac{1}{R_0 - r}\right)^{\mu}$$
$$+ \sum_{j \in \{1,...,k-1\} \setminus J} \left(\frac{1}{R_0 - r}\right)^{\rho_1} + O\left(\log \frac{1}{R_0 - r} + \log T_0(r, f)\right)$$
$$\le (\beta_1 + (k-1)\varepsilon) \left(\frac{1}{R_0 - r}\right)^{\mu} + (k-1)\left(\frac{1}{R_0 - r}\right)^{\rho_1}$$
$$+ O\left(\log \frac{1}{R_0 - r} + \log T_0(r, f)\right).$$

It follows that

$$(\beta_2 - \beta_1 - (k-1)\varepsilon)\left(\frac{1}{R_0 - r}\right)^{\mu} \le (k-1)\left(\frac{1}{R_0 - r}\right)^{\rho_1}$$
$$+ O\left(\log \frac{1}{R_0 - r} + \log T_0(r, f)\right). \tag{35}$$

From (35), since $\varepsilon \left(0 < \varepsilon < \frac{\beta_2 - \beta_1}{k-1}\right)$, we get $\rho_{\mathcal{A}}(f) = \mu_{\mathcal{A}}(f) = +\infty$ and $\rho_{2,\mathcal{A}}(f) \ge \mu_{2,\mathcal{A}}(f) \ge \mu_{\mathcal{A}}(B_0) = \mu$. $\square$

Proof of Theorem 5

Proof. Let $f \not\equiv 0$ be a solution of (1). Suppose that

$$\limsup_{r\to+\infty} \frac{\sum\limits_{j=1}^{k-1} m_0(r, B_j)}{m_0(r, B_0)} < 1 \text{ if } 1 < r < R_0 = +\infty \tag{36}$$

or

$$\limsup_{r\to R_0^-} \frac{\sum\limits_{j=1}^{k-1} m_0(r, B_j)}{m_0(r, B_0)} < 1 \text{ if } 1 < r < R_0 < +\infty. \tag{37}$$

Then for sufficiently large r or $r \to R_0^-$, we have

$$\sum_{j=1}^{k-1} m_0(r, B_j) < \gamma m_0(r, B_0), 0 < \gamma < 1. \tag{38}$$

Thus, by substituting (38) into (11), we obtain for sufficiently large r or $r \to R_0^-$

$$m_0(r, B_0) \le \gamma m_0(r, B_0) + \begin{cases} O(\log r + \log T_0(r, f)), \ R_0 = +\infty, \ r \notin \Delta_r, \\ O\left(\log \frac{1}{R_0 - r} + \log T_0(r, f)\right), \ R_0 < +\infty, \ r \notin \Delta_r'. \end{cases} \tag{39}$$

From (39), it follows that

$$(1-\gamma) m_0(r, B_0) = (1-\gamma) T_0(r, B_0)$$
$$\le \begin{cases} O(\log r + \log T_0(r, f)), \ R_0 = +\infty, \ r \notin \Delta_r, \\ O\left(\log \frac{1}{R_0 - r} + \log T_0(r, f)\right), \ R_0 < +\infty, \ r \notin \Delta_r'. \end{cases} \tag{40}$$

Case $R_0 = +\infty$: By (40), we obtain for r sufficiently large

$$(1-\gamma) \frac{T_0(r, B_0)}{\log r} \le O\left(1 + \frac{\log T_0(r, f)}{\log r}\right), \ r \notin \Delta_r \tag{41}$$

and

$$\frac{\log(1-\gamma)}{\log r} + \frac{\log T_0(r, B_0)}{\log r} \le \frac{\log\log r}{\log r} + \frac{\log\log T_0(r, f)}{\log r} + \frac{O(1)}{\log r}, \ r \notin \Delta_r. \tag{42}$$

Since $B_0(z)$ is an admissible analytic function in the annulus $\mathcal{A}$, then from (41) and (42), we get $\rho_{\mathcal{A}}(f) = +\infty$ and $\rho_{2,\mathcal{A}}(f) \ge \rho_{\mathcal{A}}(B_0)$.

Case $R_0 < +\infty$: By (40), we have for $r \to R_0^-$

$$(1-\gamma) \frac{T_0(r, B_0)}{\log \frac{1}{R_0 - r}} \le O\left(1 + \frac{\log T_0(r, f)}{\log \frac{1}{R_0 - r}}\right), \ r \notin \Delta_r' \tag{43}$$

and

$$\frac{\log(1-\gamma)}{\log\frac{1}{R_0-r}} + \frac{\log T_0(r,B_0)}{\log\frac{1}{R_0-r}} \leq \frac{\log\log\frac{1}{R_0-r}}{\log\frac{1}{R_0-r}} + \frac{\log\log T_0(r,f)}{\log\frac{1}{R_0-r}} + \frac{O(1)}{\log\frac{1}{R_0-r}},\ r \notin \Delta'_r. \tag{44}$$

Since $B_0(z)$ is an admissible analytic function in the annulus $\mathcal{A}$, then from (43) and (44), we obtain $\rho_{\mathcal{A}}(f) = +\infty$ and $\rho_{2,\mathcal{A}}(f) \geq \rho_{\mathcal{A}}(B_0)$. □

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