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Zero-Order Markov Processes with Multiple Final Sequences of States

Alexandru Lazari

Abstract. A zero-order Markov process with multiple final sequences of states represents a stochastic system with independent transitions that stops its evolution as soon as one of the given final sequences of states is reached. The transition time of the system is unitary and the transition probability depends only on the destination state. It is proved that the distribution of the evolution time is a homogeneous linear recurrent sequence and a polynomial algorithm to determine the initial state and the generating vector of this recurrence is developed. Using the generating function, the main probabilistic characteristics are determined.

Mathematics subject classification: 65C40, 60J22, 90C39, 90C40.
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1 Introduction and Problem Formulation

Let L be a discrete stochastic system with finite set of states V, $|V| = \omega$. At every discrete moment of time $t \in \mathbb{N}$ the state of the system is $v(t) \in V$. The system L starts its evolution from the state v with the probability $p^*(v)$, for all $v \in V$, where $\sum\limits_{v \in V} p^*(v) = 1$.

Also, the transition from one state u to another state v is performed according to the same probability $p^*(v)$ that depends only on the destination state v, for every $u \in V$ and $v \in V$. Additionally, we assume that r different sequences of states $X^{(\ell)} = (x_1^{(\ell)}, x_2^{(\ell)}, \ldots, x_m^{(\ell)}) \in V^m, \ell = \overline{1,r}$, are given and the stochastic system stops transitions as soon as the states $x_1^{(\ell)}, x_2^{(\ell)}, \ldots, x_m^{(\ell)}$ are reached consecutively in given order for an arbitrary $\ell \in \{1, 2, \ldots, r\}$. The time T, when the system stops, is called evolution time of the stochastic system L with given final sequences of states $X = \{X^{(1)}, X^{(2)}, \ldots, X^{(r)}\}$.

The stochastic system L, described above, represents a zero-order Markov process with final sequences of states $X = \{X^{(1)}, X^{(2)}, \ldots, X^{(r)}\}$. For the particular case $r = 1$, several interpretations of these Markov processes were analyzed in [8] and [9]. Using these concepts, the zero-order Markov processes with final sequence of state continued to be deeply studied in [3], with several further generalizations for the games, compositions and optimization problems in [2], [4] and [6]. Also, the obtained results were extended for stochastic systems with final sequence of states

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and interdependent transitions in [1], [5] and [7]. Based on polynomial algorithms proposed in [3], the main probabilistic characteristics (expectation, variance, mean square deviation, n-order moments) of evolution time and game duration were efficiently determined.

Next, in this paper, the generalization of this problem for any $r \geq 1$ is considered. This generalized problem is a bit different than the parallel compositions, studied in [2], because the dynamics of the systems are performed in a mixed one and they are interdependent.

Our goal is to analyze the evolution time T of the stochastic system L. We prove that the distribution of the evolution time T is a homogeneous linear recurrent sequence, and a polynomial algorithm to determine the initial state and the generating vector of this recurrence is developed. Having the generating vector and the initial state of the recurrence, we can use the related algorithm from [3], which was mentioned above, for determining the main probabilistic characteristics of the evolution time.

2 Determining the Distribution of the Evolution Time

In this section we will determine the distribution law of the evolution time T. Initially, we consider the notations

$$\begin{aligned} &X_k^{(\ell)} = \{x_k^{(\ell)}\},\ \pi_k^{(\ell)} = p^*(x_k^{(\ell)}),\ w_k^{(\ell)} = \prod_{j=2}^{k} \pi_j^{(\ell)}, \\ &Y_k^{(\ell)} = (x_1^{(\ell)}, x_2^{(\ell)}, \ldots, x_k^{(\ell)}),\ Y_k = \{Y_k^{(1)}, Y_k^{(2)}, \ldots, Y_k^{(r)}\}, \end{aligned} \tag{1}$$

for each $k = \overline{1,m}$ and $\ell = \overline{1,r}$.

Let $a = (a_n)_{n=0}^{\infty}$ be the distribution of the evolution time T, i.e. $a_n = \mathbb{P}(T = n)$, $n = \overline{0,\infty}$. Since $T \geq m-1$, we have $a_n = 0$, $n = \overline{0, m-2}$. If $T = m-1$, then $\exists \ell \in \{1, 2, \ldots, r\}$ such that $v(j) = x_{j+1}^{(\ell)}$, $j = \overline{0, m-1}$, that implies

$$\begin{aligned} a_{m-1} = \mathbb{P}(T = m-1) = \sum_{\ell=1}^{r} \prod_{j=1}^{m} p^*(x_j^{(\ell)}) = \\ = \sum_{\ell=1}^{r} \left(\pi_1^{(\ell)} \pi_2^{(\ell)} \ldots \pi_m^{(\ell)}\right) = \sum_{\ell=1}^{r} \left(\pi_1^{(\ell)} w_m^{(\ell)}\right). \end{aligned} \tag{2}$$

We consider $\forall n \in \mathbb{Z}$. Let be $S(V) = \{A \mid A \subseteq V\}$. Denote by $P_{\Phi}^{(\ell)}(n)$ the probability that $T = n$ and $v(j) \in \Phi_j$, $j = \overline{0, t-1}$, for all $\Phi = (\Phi_j)_{j=0}^{t-1} \in (S(V))^t$, $t \in \mathbb{N}$ and $\ell = \overline{1,r}$. We introduce the following functions on $\mathbb{Z}$, $k = \overline{0,m}$, $\ell = \overline{1,r}$:

$$\begin{aligned} &\beta_k^{(\ell)}(n) = P_{(X_1^{(\ell)}, X_2^{(\ell)}, \ldots, X_k^{(\ell)})}(n), \\ &\gamma_k^{(\ell)}(n) = P_{(X_2^{(\ell)}, X_3^{(\ell)}, \ldots, X_k^{(\ell)})}(n). \end{aligned} \tag{3}$$

For $\forall n \geq m$, we have

$$\beta_k^{(\ell)}(n) = P_{(X_1^{(\ell)}, X_2^{(\ell)}, \ldots, X_k^{(\ell)})}(n) =$$

$$= \pi_1^{(\ell)} P_{(X_2^{(\ell)}, \ldots, X_k^{(\ell)})}(n-1) - \pi_1^{(\ell)} \sum_{j=1}^{r} u_{j,k}^{(\ell)} P_{(X_2^{(j)}, \ldots, X_m^{(j)})}(n-1) =$$

$$= \pi_1^{(\ell)} \left(\gamma_k^{(\ell)}(n-1) - \sum_{j=1}^{r} u_{j,k}^{(\ell)} \gamma_m^{(j)}(n-1) \right), \; k = \overline{0, m}, \ell = \overline{1, r}, \tag{4}$$

where

$$u_{j,k}^{(\ell)} = \begin{cases} 1, & k = 0 \text{ or } Y_k^{(j)} = Y_k^{(\ell)} \\ 0, & k \neq 0 \text{ and } Y_k^{(j)} \neq Y_k^{(\ell)} \end{cases}. \tag{5}$$

We consider the sets

$$T_s^{(\ell)} = \{s+1\} \cup \{t \in \{2, 3, \ldots, s\} \mid (x_t^{(\ell)}, x_{t+1}^{(\ell)}, \ldots, x_s^{(\ell)}) \in Y_{s+1-t}\},$$

for each $s = \overline{1, m}$ and $\ell = \overline{1, r}$. The minimal elements from these sets are

$$t_s^{(\ell)} = \min_{k \in T_s^{(\ell)}} k, \; s = \overline{1, m}, \; \ell = \overline{1, r}. \tag{6}$$

The value $t_s^{(\ell)}$ represents the position in the sequence $(x_1^{(\ell)}, x_2^{(\ell)}, \ldots, x_s^{(\ell)})$ starting with which, if we overlap a final sequence of states $X^{(\tau_s^{(\ell)})} \in X$, the superposed elements are equal. Here by $\tau_s^{(\ell)}$ we denote the minimal index from the set $\{1, 2, \ldots, r\}$ that satisfies given condition.

Next, for $s = \overline{1, m}$ and $\ell = \overline{1, r}$, we obtain

$$\gamma_s^{(\ell)}(n) = P_{(X_2^{(\ell)}, X_3^{(\ell)}, \ldots, X_s^{(\ell)})}(n) =$$

$$= \pi_2^{(\ell)} \pi_3^{(\ell)} \ldots \pi_{t_s^{(\ell)}-1}^{(\ell)} P_{(X_{t_s^{(\ell)}}, X_{t_s^{(\ell)}+1}, \ldots, X_s)}(n - t_s^{(\ell)} + 2) =$$

$$= w_{t_s^{(\ell)}-1}^{(\ell)} P_{(X_1^{(\tau_s^{(\ell)})}, X_2^{(\tau_s^{(\ell)})}, \ldots, X_{s+1-t_s^{(\ell)}}^{(\tau_s^{(\ell)})})}(n - t_s^{(\ell)} + 2) =$$

$$= w_{t_s^{(\ell)}-1}^{(\ell)} \beta_{s+1-t_s^{(\ell)}}^{(\tau_s^{(\ell)})}(n - t_s^{(\ell)} + 2). \tag{7}$$

Particularly, for $s = 0$, we have

$$\gamma_0^{(\ell)}(n) = a_n = \gamma_1^{(\ell)}(n) = w_1^{(\ell)} \beta_0^{(\tau_1^{(\ell)})}(n) = \beta_0^{(\ell)}(n),$$

which implies

$$\beta_0^{(\ell)}(n) = \pi_1^{(\ell)} \left(\gamma_0^{(\ell)}(n-1) - \sum_{j=1}^{r} u_{j,0}^{(\ell)} \gamma_m^{(j)}(n-1) \right)$$

$$= \pi_1^{(\ell)} \left(\beta_0^{(\ell)}(n-1) - \sum_{j=1}^{r} u_{j,0}^{(\ell)} w_{t_m^{(j)}-1}^{(j)} \beta_{m+1-t_m^{(j)}}^{(\tau_m^{(j)})}(n - t_m^{(j)} + 1) \right) \tag{8}$$

and, for $k = \overline{1, m}$,

$$\beta_k^{(\ell)}(n) = \pi_1^{(\ell)} \left(\gamma_k^{(\ell)}(n-1) - \sum_{j=1}^{r} u_{j,k}^{(\ell)} \gamma_m^{(j)}(n-1) \right) =$$

$$= \pi_1^{(\ell)} \left(w_{t_k^{(\ell)}-1}^{(\ell)} \beta_{k+1-t_k^{(\ell)}}^{(\tau_k^{(\ell)})}(n - t_k^{(\ell)} + 1) - \right.$$

$$\left. - \sum_{j=1}^{r} u_{j,k}^{(\ell)} w_{t_m^{(j)}-1}^{(j)} \beta_{m+1-t_m^{(j)}}^{(\tau_m^{(j)})}(n - t_m^{(j)} + 1) \right). \tag{9}$$

Since $2 \le t_s^{(\ell)} \le s+1 \le m+1, \ s = \overline{1, m}, \ \ell = \overline{1, r}$, there exist some real coefficients $v_{jks\ell}^{(i)}$, $k, j, s = \overline{0, m-1}$, $i, \ell = \overline{1, r}$, such that

$$\beta_k^{(\ell)}(n) = \sum_{i=1}^{r} \sum_{j=0}^{m-1} \sum_{s=0}^{m-1} v_{jks\ell}^{(i)} \ \beta_s^{(i)}(n-1-j), \ k = \overline{0, m-1}, \ \ell = \overline{1, r}, \ \forall n \ge m.$$

So, we have

$$\beta_k(n) = \sum_{j=0}^{m-1} \sum_{s=0}^{m-1} V_{jks} \ \beta_s(n-1-j), \ k = \overline{0, m-1}, \ \forall n \ge m,$$

where $V_{jks} = (v_{jks\ell}^{(i)})_{\ell, \ i=\overline{1,r}}$, $\beta_k(n) = (\beta_k^{(\ell)}(n))_{\ell=\overline{1,r}}$, $k, j, s = \overline{0, m-1}$. This recurrence relation can be written in the form

$$\beta(n) = \sum_{j=0}^{m-1} V_j \ \beta(n-1-j), \ \forall n \ge m,$$

where $V_j = (V_{jks})_{k,s=\overline{0,m-1}}$ and $\beta(n) = ((\beta_k(n))_{k=0}^{m-1})^T$, $j = \overline{0, m-1}$, $\forall n \in \mathbb{Z}$. From this relation, we obtain that $\beta = (\beta(n))_{n=0}^{\infty} \in Rol^*[\mathcal{M}_m(\mathcal{M}_r(\mathbb{R}))][m]$ with generating vector $V = (V_j)_{j=0}^{m-1} \in G^*[\mathcal{M}_m(\mathcal{M}_r(\mathbb{R}))][m](\beta)$. Using the results from [1], we have $\beta \in Rol^*[\mathbb{R}][m^2 r]$, which implies that also

$$(\beta_k^{(\ell)}(n))_{n=0}^{\infty} \in Rol^*[\mathbb{R}][m^2 r], \ k = \overline{0, m-1}, \ \ell = \overline{1, r},$$

with the same generating vector. Since

$$a_n = \beta_0^{(1)}(n), \ \forall n \ge 0, \tag{10}$$

we have

$$a = (a_n)_{n=0}^{\infty} \in Rol^*[\mathbb{R}][m^2 r].$$

Next, we will use only the relation $a \in Rol^*[\mathbb{C}][m^2r]$, the minimal generating vector being determined using the minimization method based on the matrix rank, described in [3]. So, according to this method, we have that the minimal generating vector $q = (q_0, q_1, \ldots, q_{R-1}) \in G^*[\mathbb{C}][R](a)$ is obtained from the unique solution $x = (q_{R-1}, q_{R-2}, \ldots, q_0)$ of the system

$$A_R^{[a]} x^T = (f_R^{[a]})^T, \tag{11}$$

where

$$f_R^{[a]} = (a_R, a_{R+1}, \ldots, a_{2R-1}),\ A_n^{[a]} = (a_{i+j})_{i,j=\overline{0,n-1}},\ \forall n \in \mathbb{N}^* \tag{12}$$

and R is the rank of the matrix $A_{m^2r}^{[a]}$.

In order to apply this minimization method, we need to have only the values a_k, $k = \overline{0, 2m^2r-1}$. These values can be determined using the recurrences (8) and (9) and the relations (1), (2), (5), (6) and (10).

3 Describing the developed algorithm

In previous section we theoretically grounded the following algorithm for determining the main probabilistic characteristics of the evolution time T: the distribution $(\mathbb{P}(T = n))_{n=0}^{\infty}$, the expectation $\mathbb{E}(T)$, the variance $\mathbb{V}(T)$, the mean square deviation $\sigma(T)$ and the k-order moments $\nu_k(T)$, $k = 1, 2, \ldots$.

Algorithm 1.

Input: $X^{(\ell)} = (x_1^{(\ell)}, x_2^{(\ell)}, \ldots, x_m^{(\ell)}) \in V^m$, $\pi_k^{(\ell)}$, $k = \overline{1,m}$, $\ell = \overline{1,r}$;
Output: $\mathbb{E}(T)$, $\mathbb{V}(T)$, $\sigma(T)$, $\nu_k(T)$, $k = \overline{1,t}$, $t \geq 2$.

1. Determine the values a_k, $k = \overline{0, 2m^2r-1}$, using the recurrences (8) and (9) and the relations (1), (2), (5), (6) and (10);

2. Find the minimal generating vector $q = (q_0, q_1, \ldots, q_{R-1}) \in G^*[\mathbb{R}][R](a)$ by solving the system (11), taking into account the relation (12);

3. Consider the distribution $a = (a_n)_{n=0}^{\infty} = (\mathbb{P}(T = n))_{n=0}^{\infty}$ of the evolution time T as a homogeneous linear recurrence with the initial state $I_R^{[a]} = (a_n)_{n=0}^{R-1}$ and the minimal generating vector $q = (q_k)_{k=0}^{R-1}$, determined at the steps 1 and 2;

4. Determine the expectation $\mathbb{E}(T)$, the variance $\mathbb{V}(T)$, the mean square deviation $\sigma(T)$ and the k-order moments $\nu_k(T)$, $k = \overline{1,t}$, of the evolution time T by using the corresponding algorithm from [3].

4 Conclusions

In this paper the zero-order Markov processes with multiple final sequences of states were studied and the evolution time of these stochastic systems was analyzed.

It was proved that the evolution time is a discrete random variable with homogeneous linear recurrent distribution. Based on this fact, the generating function is applied for determining the main probabilistic characteristics of the evolution time. The developed algorithm has polynomial complexity.

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Alexandru Lazari
Institute of Mathematics and Computer Science,
Moldova State University,
5 Academiei str., Chişinău, MD-2028, Moldova.
E-mail: *alexan.lazari@gmail.com*

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