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Relative Separation Axioms via Semi-Open Sets

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Abstract. The concept of relative topological properties was introduced by Arhangel'skii and Gennedi and was subsequently investigated by many authors for different notions of general topology. In this paper few semi-separation axioms in relative sense are introduced and studied by utilizing semi-open sets. Characterizations and preservation under mapping of these newly defined notions are provided. Relationship that exists between these notions, with some of the absolute properties and with the existing relative separation axioms are investigated.

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1 Introduction and Preliminaries

The notions of semi-open set and semi-continuity were introduced by N. Levine [14] and were subsequently utilized by several researchers in different settings. A set A is said to be semi-open in a topological space X if there is an open set U such that $U \subset A \subset cl(U)$, where $cl(U)$ is the closure of U in X. The condition of being semi-open is weaker than the condition of being open. A function $f : X \to Y$ is said to be semi-continuous if the inverse image of every open set is semi-open. Semi-closed sets, semi-interior and semi-closure were defined by S. Gene Crossley and S. K. Hildebrand in a manner analogous to the corresponding concepts of closed set, interior and closure [6]. Semi-open and semi-closed functions were defined by Biswas [5] and Noiri [19]. According to them a function $f : X \to Y$ is semi-open if the image of every open set is semi-open and $f : X \to Y$ is said to be semi-closed if the image of every closed set is semi-closed. Various separation axioms have been defined using semi-open sets. Maheshwari and Prasad in [15–17] defined semi-T_i, $i = 0, 1, 2$, s-regular, and s-normal spaces respectively just by replacing open sets by semi-open sets in definition of T_i, $i = 0, 1, 2$, regular, and normal space. Charles Dosett in [10] further investigated these separation axioms and established relationships with each other and with other notions. Crossley and Hildebrand gave the concept of semi-homeomorphism [7] and stated that a property of topological spaces is defined to be a semi-topological property if it is preserved by semi-homeomorphism. They showed that some of the topological properties like first category, Hausdorffness, separability and connectedness are semi-topological properties. Hamlett showed that the property of a topological space being a Baire space is semi-topological [11]. Nayar

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and Arya [18] developed techniques which help to establish whether a topological property is semi-topological or not. Till now a lot of work has been done in general topology using semi-open sets.

In this paper we study some relative versions of semi-separation axioms. We establish the relationship of relative semi-separation axioms with the absolute properties and with the existing relative separation axioms. Characterizations of relatively s-regular and relatively s-normal are also given. Behavior of these spaces under mapping is also studied. In this paper we proved that for $Y \subset X$, Y is relatively s-regular in X iff $\pi_R(Y)$ is relatively s-regular in X_R, where R is an equivalence relation on X, X_R denotes the quotient space X/R and $\pi_R : X \to X/R$ is canonical projection map defined by $\pi_R(x) = [x]$. This result is similar to Dosett's Theorem 3.3. [10] generalized in relative sense. A number of examples and counter examples are also provided in support of various statements.

Let $Y \subset X$. Y is said to be T_1 in X or *relatively* T_1 [2] if for every $y \in Y$, $\{y\}$ is closed in X. Y is said to be T_2 in X or *relatively* T_2 [2] if for every pair of distinct points in Y there exist disjoint open sets in X separating them. Y is said to be regular in X [2] or *relatively regular* if for every closed set A of X and a point $y \in Y$ such that $y \notin A$, there exist disjoint open subsets U and V of X such that $A \cap Y \subset U$ and $y \in V$. Y is said to be normal in X or *relatively normal* [2] if for every pair of disjoint closed sets A and B of X, there exist disjoint open subsets U and V of X such that $A \cap Y \subset U$ and $B \cap Y \in V$.

Throughout this paper the semi-closure of A in X is denoted by $scl(A)$ and the semi-interior of A in X is denoted by $sint(A)$.

2 Relative Semi-Separation Axioms

Semi-T_i, for $i = 0, 1, 2$, in relative sense can be defined in the same manner as relatively T_i just by replacing open sets by semi-open sets. It is clear from the definitions that the condition of being semi-T_i is stronger than the condition of being relatively semi-T_i. Also the condition of being relatively T_i is stronger than the condition of being relatively semi-T_i.

Definition 1. $Y \subset X$ is said to be relatively s-regular if for every closed set A in X and a point $y \in Y$ such that $y \notin A$, there exist disjoint semi-open sets U and V in X such that $y \in U$ and $A \cap Y \subset V$.

Definition 2. $Y \subset X$ is said to be relatively s-normal if for every pair of closed sets A and B in X, there exist disjoint semi-open sets U and V in X such that $A \cap Y \subset U$ and $B \cap Y \subset V$.

It is clear from the definitions that Y is relatively s-normal if X is s-normal and Y is relatively s-normal if Y is relatively normal. Similarly Y is relatively s-regular if X is s-regular and Y is relatively s-regular if Y is relatively regular. The following are some examples showing that none can be reversed.

Example 1. Let $X = \{a, b, c, d\}$ and $\tau = \{\phi, X, \{b, c, d\}, \{a, b, d\}, \{b, d\}, \{d\}, \{b\}\}$. Let $Y = \{a, c\}$. Semi-open sets of X other than all open sets includes $\{a, d\}$, $\{c, d\}$, $\{a, b\}$ and $\{a, c\}$. Clearly Y is relatively s-normal in X but Y is not relatively normal in X as $\{a\}$ and $\{c\}$ are closed in X which cannot be separated by disjoint semi-open sets in X.

Example 2. Let $X = \{a, b, c, \}$ and $\tau = \{\{a\}, \{a, b\}, \{a, c\}, X, \phi\}$. Let $Y = \{a\}$. The only semi-open sets of X are all open sets. X is not s-normal because $\{b\}$ and $\{c\}$ are disjoint closed sets in X which cannot be separated by disjoint semi-open sets in X. But Y is relatively s-normal.

Theorem 1. *$Y \subset X$ is said to be relatively s-regular if and only if for every $y \in Y$ and every open set O in X containing y, there exists semi-open set U in X such that $y \in U \subset scl(U) \subset O \cap X \setminus Y$.*

Proof. Let Y be a relatively s-regular space. Let $y \in Y$ and O be an open set in X such that $y \in O$. $X \setminus O$ is closed in X and $y \notin X \setminus O$. Since Y is relatively s-regular, there exist disjoint semi-open sets U and V in X such that $y \in U$ and $(X \setminus O) \cap Y \subset V$, thus $y \in U \subset X \setminus V \subset O \cup X \setminus Y$. $X \setminus V$ being semi-closed implies $y \in U \subset scl(U) \subset X \setminus V \subset O \cup X \setminus Y$.

Conversely let $y \in Y$ and A be a closed set in X such that $y \notin A$. $X \setminus A$ is open in X and $y \in X \setminus A$, there exists semi-open set U such that $y \in U \subset scl(U) \subset (X \setminus A) \cap X \setminus Y$, which implies $y \in U \subset scl(U) \subset X \setminus (A \cap Y)$. Now let $V = X \setminus scl(U) = sint(X \setminus U)$, therefore V is the largest semi-open set contained in $X \setminus U$. Also $scl(U) \subset X \setminus (A \cap Y)$ which implies $A \cap Y \subset X \setminus scl(U) = V$. Hence U and V are disjoint semi-open sets such that $y \in U$ and $A \cap Y \subset V$. Thus Y is relatively s-regular space. □

Theorem 2. *$Y \subset X$ is relatively s-normal if and only if for every closed set A of X and every open set B of X containing A, there exists semi-open set U in X such that $A \cap Y \subset U \subset scl(U) \subset B \cap X \setminus Y$.*

Proof. Let Y be a relatively s-normal space. Let A be a closed set in X and B be an open set in X containing A. Then A and $X \setminus B$ are disjoint closed sets in X. Since Y is relatively s-normal, there exist disjoint semi-open sets U and V in X such that $A \cap Y \subset U$ and $(X \setminus B) \cap Y \subset V$, thus $A \cap Y \subset U \subset X \setminus V \subset B \cup X \setminus Y$. $X \setminus V$ being semi-closed implies $A \cap Y \subset U \subset scl(U) \subset X \setminus V \subset B \cup X \setminus Y$.

Conversely let A and B be disjoint closed sets in X. Since $A \subset X \setminus B$ which is open in X, there exists a semi-open set U in X such that $A \cap Y \subset U \subset scl(U) \subset (X \setminus B) \cap X \setminus Y \subset X \setminus (B \cap Y)$. Now let $V = X \setminus scl(U) = sint(X \setminus U)$, therefore V is the largest semi-open set contained in $X \setminus U$. Also $scl(U) \subset X \setminus (B \cap Y)$ which implies $B \cap Y \subset X \setminus scl(U) = V$. Here U and V are disjoint semi-open sets such that $A \cap Y \subset U$ and $B \cap Y \subset V$. Hence Y is relatively s-normal space. □

Proof of the following theorem is obvious from definitions.

Theorem 3. *Every relatively T_0, relatively s-regular space is relatively semi-T_2.*

Definition 3. A topological space X is said to be R_0 if for every open set G in X, $x \in G$ implies $cl(\{x\}) \subset G$.

Theorem 4. *In an R_0 space every relatively s-normal subset is relatively s-regular.*

Proof. Let Y be relatively s-normal and X be an R_0 space. Let A be a closed set in X and $y \in Y$ be such that $y \notin A$. X being R_0, $cl\{y\} \cap A = \phi$. Now A and $cl\{y\}$ are two disjoint open sets in X and Y is relatively s-normal, there exist disjoint semi-open sets U and V in X such that $cl\{y\} \subset U$ and $A \cap Y \subset V$. $\square$

The following corollary follows from the fact that every T_1 space is R_0 space.

Corollary 1. *In a T_1 space any relatively s-normal space is relatively s-regular.*

Theorem 5. *Every relatively T_1, relatively s-normal space is relatively s-regular.*

In the above Corollary and Theorem, T_1 and relatively T_1 cannot be replaced by semi-T_2 and relatively semi-T_2 respectively as is evident from the following example.

Example 3. Let $X_1 = \{a, b, c\}$ and $X_2 = [0, 1]$, $T = \{X_1, \phi, \{a\}, \{a, b\}, \{c\}, \{a, c\}\}$ be a topology on X_1 and S be the usual topology on X_2. Then $(X_1 \times X_2, P)$, where P denotes the product topology on $X_1 \times X_2$, is s-normal, semi-T_2 [10]. Let $Y = \{a, b\} \times X_2$. Y is not relatively s-regular since $C = \{b\} \times X_2$ is closed in X and $y = (a, 1/2) \in Y$, $y \notin C$, and there do not exist disjoint semi-open sets containing y and C, respectively.

Definition 4. [20] $Y \subset X$ is said to be relatively almost normal if for any two disjoint closed subsets A and B of X such that one of them is regularly closed, there exist disjoint open sets U and V in X such that $A \cap Y \subset U$ and $B \cap Y \subset V$.

Definition 5. [20] $Y \subset X$ is said to be relatively κ-normal if for any two disjoint regularly closed subsets A and B of X, there exist disjoint open sets U and V in X such that $A \cap Y \subset U$ and $B \cap Y \subset V$.

Example 4. Relative almost normality does not imply relative s-normality.
Consider Example 2. Let $Y = \{b, c\}$. Y is relatively almost normal because the only regularly closed sets are ϕ and X. But Y is not relatively s-normal.

Example 5. Relative s-normality does not imply relative κ-normality.
Let X be the set of integers. Define a topology τ on X, where every odd integer is open and a set U is open if for even integer $p \in U$ the successor and the predecessor of p also belong to U. This topology is called odd-even topology. Let Y be the set of all even integers. $A = \{4, 5, 6\}$ and $B = \{8, 9, 10\}$ are regularly closed in X. But $A \cap Y = \{4, 6\}$ and $B \cap Y = \{8, 10\}$, which cannot be separated by disjoint open sets in X. Hence Y is not relatively κ-normal in X. If we denote any even integer by e and any odd integer by o, then the semi-open sets of X are of the form $\{o, o, ...e\}$, $\{e, o, o.., o\}$, $\{e, o, o, ...e\}$, $\{o, o, ..., e, o, ..., o\}$, $\{o, o, ..., o\}$ and the sets which are not semi-open are the sets containing two consecutive even numbers and no odd number between them, like $\{2, 4, 5\}$ is not semi-open. Here in this case we can easily check that Y is relatively s-normal.

From the above examples we conclude that the concept of relatively s-normal is independent of relatively almost normal and relatively κ-normal.

Definition 6. X is said to be *β-normal* [4] if for any two disjoint closed subsets A and B of X, there exist open subsets U and V of X such that $A \cap U$ is dense in A and $B \cap V$ is dense in B and $cl(U) \cap cl(V) = \phi$.

Definition 7. [9] $Y \subset X$ is said to be relatively super β-normal if for any two disjoint subsets A and B closed in X, there exist open subsets U and V of X such that $(A \cap Y) \cap U$ is dense in A and $(B \cap Y) \cap V$ is dense in B and $cl(U) \cap cl(V) = \phi$.

Definition 8. [9] $Y \subset X$ is said to be relatively strong by β-normal if for any two disjoint subsets A and B closed in Y, there exist open subsets U and V of X such that $A \cap U$ is dense in A and $B \cap V$ is dense in B and $cl(U) \cap cl(V) = \phi$.

Theorem 6. *[9] In the class of relatively super β-normal spaces, every κ-normal space is normal.*

Theorem 7. *[9] In the class of β-normality (relative β-normality or relative strong β-normality) every κ-normal space is relatively normal.*

From above results the following results are obvious.

Theorem 8. *In the class of relative super β-normality every κ-normal space is s-normal.*

Theorem 9. *In the class of β-normality (relative β-normality or relative strong β-normality) every κ-normal space is relatively s-normal.*

Definition 9. [3] $Y \subset X$ is said to be relatively superregular if for every closed set A in X and a point $y \in Y$ such that $y \notin A$, there exist disjoint open sets U and V in X such that $A \subset U$ and $y \in V$.

Theorem 10. *The image of a relatively superregular space under continuous, semi-closed, semi-open and onto map is relatively s-regular.*

Proof. Let $f : (X_1, T) \to (X_2, S)$ be a continuous, semi-open, semi-closed and onto map. Let $Y \subset X$ be relatively superregular. Let O be an open set in X_2 and $y_2 \in f(Y)$. Let $y_1 \in f^{-1}(y_2)$. Since f is continuous, $f^{-1}(O)$ is open in X_1 and $y_1 \in f^{-1}(O)$. Since Y is relatively superregular, there exists an open set U in X_1 such that $y_1 \in U \subset cl(U) \subset O$ which implies $y_2 \in f(U) \subset f(cl(U)) \subset f(O)$. Since f is semi-open, $cl(U)$ is closed in X_1, $f(cl(U))$ is semi-closed in X_2 and $scl(f(U))$ is the smallest semi-closed set containing $f(U)$. Therefore $y_2 \in f(U) \subset scl(f(U)) \subset f(cl(U)) \subset f(O) \subset f(O) \cup X_2 \setminus f(Y)$. Hence $f(Y)$ is relatively s-regular. □

Remark 1. Relative supperregularity cannot be replaced by relative s-regularity in the above theorem.

Let $X_1 = \{a, b, c, d\}$ with topology $T = \{\{a, b, d\}, \{b, c, d\}, \{b, d\}, \{d\}, \{b\}, X_1, \phi\}$ and

$X_2 = \{e, f, g\}$ with topology $S = \{\{e, f\}, \{e, g\}, \{e\}, X_2, \phi\}$. Define $f : X_1 \to X_2$ as $f(a) = f$, $f(b) = e$, $f(c) = g$ and $f(d) = e$. Then this map is continuous, onto, semi-open and semi-closed. Let $Y = \{a, c\}$. Clearly Y is relatively s-regular (not relatively superregular) and $f(Y)$ is not relatively s-regular.

Theorem 11. *$Y \subset X$ is relatively s-normal if for every pair of disjoint closed sets A and B in X, there exists a semi-continuous function $f : X \to [0, 1]$ such that $f(A) = \{0\}$ and $f(B) = \{1\}$.*

The Theorem stated above is actually one-sided Urysohn's Lemma type result and can be proved easily.

Let X be a topological space and R be an equivalence relation on X defined as xRy iff $cl(\{x\}) = cl(\{y\})$. The resulting quotient space X/R is actually T_0 and is called T_0-identification of X. Also this quotient space is a decomposition space as X/R is a set of equivalence classes of R which forms a partition of X. The canonical projection map $\pi_R : X \to X/R$ defined by $\pi_R(x) = [x]$ is the decomposition map. For simplicity we are using X_R instead of X/R.

Theorem 12. *$Y \subset X$ is relatively s-regular in X iff $\pi_R(Y)$ is relatively s-regular in X_R.*

Proof. Let $Y \subset X$ be relatively s-regular. Let $\mathcal{C}$ be a closed set in X_R and $C \in \pi_R$ such that $C \notin \mathcal{C}$. Let $y \in C$, then $[y] = C$. Since π_R is continuous, $\pi_R(\mathcal{C})$ is closed in X. Also $y \notin \pi_R^{-1}(\mathcal{C})$ and $y \in Y$. By relative s-regularity of Y in X, there exist disjoint semi-open sets A and B in X such that $y \in A$ and $\pi_R^{-1}(\mathcal{C}) \cap Y \subset B$. Since A and B are semi-open sets, there exist open sets U and V in X such that $U \subset A \subset cl(U)$ and $V \subset B \subset cl(V)$ which implies that $U \cup \{y\}$ and $V \cup \pi_R^{-1}(\mathcal{C}) \cap Y$ are disjoint semi-open sets in X. Now since π_R is open and continuous and $\pi_R^{-1}(\pi_R(O)) = O$ for all O open in X, $D = \pi_R(U \cup Y)$ and $E = \pi_R(V \cup \pi_R^{-1}(\mathcal{C}) \cap Y)$ are disjoint semi-open sets in X_R containing $[y]$ and $\mathcal{C} \cap \pi_R(Y)$. Hence $\pi_R(Y)$ is s-regular.

Conversely suppose that $\pi_R(Y)$ is relatively s-regular in X_R. Let C be a closed set in X and $y \in Y$ such that $y \notin C$. Then $[y] \cap \pi_R(C) = \phi$. Since π_R is closed, $\pi_R(C)$ is closed in X_R. By relative s-regularity of $\pi_R(Y)$ for $[y] \in \pi_R(Y)$ and as $\pi_R(C)$ is closed in X_R and $[y] \cap \pi_R(C)$, there exist semi-open sets U and V in X_R such that $[y] \in U$ and $\pi_R(C) \cap \pi_R(Y) \subset V$. Since π_R is continuous and open, $\pi_R^{-1}(U)$ and $\pi_R^{-1}(V)$ are disjoint semi-open sets in X containing y and $C \cap Y$. Hence Y is relatively s-regular in X. □

Remark 2. The space X_R in the above theorem is T_0, $\pi_R(Y)$ is relatively T_0. By Theorem 3 if $\pi_R(Y)$ is relatively s-regular then $\pi_R(Y)$ is relatively semi-T_2.

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