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Poisson Stable Motions and Global Attractors of Symmetric Monotone Nonautonomous Dynamical Systems

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Abstract. This paper is dedicated to the study of the problem of existence of Poisson stable (Bohr/Levitan almost periodic, almost automorphic, almost recurrent, recurrent, pseudo-periodic, pseudo-recurrent and Poisson stable) motions of symmetric monotone non-autonomous dynamical systems (NDS). It is proved that every precompact motion of such system is asymptotically Poisson stable. We give also the description of the structure of compact global attractor for monotone NDS with symmetry. We establish the main results in the framework of general non-autonomous (cocycle) dynamical systems. We apply our general results to the study of the problem of existence of different classes of Poisson stable solutions and global attractors for a chemical reaction network and nonautonomous translation-invariant difference equations.

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1 Introduction

This article continues the author's series of works [13]-[18] devoted to the study of Poisson stable motions and global attractors of monotone nonautonomous dynamical systems.

In present work we study a class of monotone nonautonomous dynamical systems with symmetry. The writing of this article was motivated by works D. Angeli and E. Sontag [1,2], D. Angeli, P. Leenheer and E. Sontag [3] (for autonomous systems), H. Hu and J. Jiang [22, 23] (for periodic and almost periodic systems) and Q. Liu and Y. Wang [28] (for almost periodic and almost automorphic systems). We study these problems within the framework of general non-autonomous dynamical systems (cocycles).

2 NDS: some general properties

In this section we collect some notions and facts for non-autonomous dynamical systems which we will use below; the reader may refer to [9],[12, Ch. IX],[31] for details.

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Throughout the paper, we assume that X and Y are metric spaces and for simplicity we use the same notation ρ to denote the metrics on them, which we think would not lead to confusion. Let $\mathbb{R} = (-\infty, +\infty)$, $\mathbb{Z} := \{0, \pm 1, \pm 2, \dots\}$, $\mathbb{S} = \mathbb{R}$ or $\mathbb{Z}$, $\mathbb{S}_+ := \{s \in \mathbb{S} | \ s \geq 0\}$, $\mathbb{S}_- := \{s \in \mathbb{S} | \ s \leq 0\}$ and $\mathbb{T} \subseteq \mathbb{S}$ be a sub-semigroup of $\mathbb{S}$ such that $\mathbb{S}_+ \subseteq \mathbb{T}$. For given dynamical system $(X, \mathbb{T}, \pi)$ and given point $x \in X$, we denote by Σ_x (respectively, Σ_x^+) its *trajectory* (respectively, *semi-trajectory*), i.e. $\Sigma_x := \{\pi(t, x) : t \in \mathbb{T}\}$ (respectively, $\Sigma_x^+ := \{\pi(t, x) : t \in \mathbb{T}_+\}$), and call the mapping $\pi(\cdot, x) : \mathbb{T} \to X$ the *motion* through x at the moment $t = 0$. For given set $A \subseteq X$, we denote $\Sigma_A := \{\pi(t, x) : t \in \mathbb{T}, x \in A\}$; Σ_A^+ is defined similarly. We denote the *hull* (respectively, *semi-hull*) of a point x by $H(x) := \overline{\Sigma}_x$ (respectively, $H^+(x) := \overline{\Sigma}_x^+$), where by bar we mean closure. A point $x \in X$ is called *Lagrange stable*, "st. L" in short, (respectively, *positively Lagrange stable*, "st. L^+" in short) if $H(x)$ (respectively, $H^+(x)$) is compact.

Let $(Y, \mathbb{S}, \sigma)$ be a two-sided dynamical system on Y and E be a metric space.

Definition 1. (Cocycle on the state space E with the base $(Y, \mathbb{S}, \sigma)$.). A triplet $\langle E, \phi, (Y, \mathbb{S}, \sigma)\rangle$ (or briefly ϕ if no confusion) is said to be a *cocycle* on state space (or fibre) E with base $(Y, \mathbb{S}, \sigma)$ (or driving system $(Y, \mathbb{S}, \sigma)$) if the mapping $\phi : \mathbb{S}_+ \times Y \times E \to E$ satisfies the following conditions:

1. $\phi(0, u, y) = u$ for all $u \in E$ and $y \in Y$;
2. $\phi(t + \tau, u, y) = \phi(t, \phi(\tau, u, y), \sigma(\tau, y))$ for all $t, \tau \in \mathbb{S}_+, u \in E$ and $y \in Y$;
3. the mapping ϕ is continuous.

Definition 2. (Skew-product dynamical system.) Let $\langle E, \phi, (Y, \mathbb{S}, \sigma)\rangle$ be a cocycle on E, $X := E \times Y$ and π be a mapping from $\mathbb{S}_+ \times X$ to X defined by $\pi := (\phi, \sigma)$, i.e., $\pi(t, (u, y)) = (\phi(t, u, y), \sigma(t, y))$ for all $t \in \mathbb{S}_+$ and $(u, y) \in E \times Y$. The triplet $(X, \mathbb{S}_+, \pi)$ is an autonomous dynamical system and is called *skew-product dynamical system.*

Definition 3. (Nonautonomous dynamical system.) Let $\mathbb{T}_1 \subseteq \mathbb{T}_2$ be two sub-semigroups of the group $\mathbb{S}$, $(X, \mathbb{T}_1, \pi)$ and $(Y, \mathbb{T}_2, \sigma)$ be two autonomous dynamical systems and $h : X \to Y$ be a homomorphism from $(X, \mathbb{T}_1, \pi)$ on $(Y, \mathbb{T}_2, \sigma)$ (i.e., $h(\pi(t, x)) = \sigma(t, h(x))$ for all $t \in \mathbb{T}_1$ and $x \in X$, and h is continuous and surjective), then the triplet $\langle (X, \mathbb{T}_1, \pi), (Y, \mathbb{T}_2, \sigma), h\rangle$ is called a *nonautonomous dynamical system* (NDS) with basis $(Y, \mathbb{T}_2, \sigma)$.

Example 1. (The nonautonomous dynamical system generated by cocycle ϕ.) An important class of NDS are generated from cocycles. Indeed, let $\langle E, \phi, (Y, \mathbb{S}, \sigma)\rangle$ be a cocycle, $(X, \mathbb{S}_+, \pi)$ be the associated skew-product dynamical system $(X = E \times Y, \pi = (\phi, \sigma))$ and $h = pr_2 : X \to Y$ (the natural projection mapping), then the triplet $\langle (X, \mathbb{S}_+, \pi), (Y, \mathbb{S}, \sigma), h\rangle$ is a NDS.

Lagrange stable (or called "compact") motions have been studied comprehensively, but it is not the case for non-Lagrange stable motions. The following concept of conditional compactness introduced in [9] is important for our study of noncompact motions and NDS with non-compact base (driving system).

Definition 4. Let (X, h, Y) be a fiber space [24]. A set $M \subseteq X$ is said to be *conditionally precompact* if its intersection with the preimage of any precompact subset $Y' \subseteq Y$, i.e. the set $h^{-1}(Y') \bigcap M$, is a precompact subset of X. A set M is called *conditionally compact* if it is closed and conditionally precompact.

Remark 1. 1. Let K be a compact space, Y is a noncompact metric space, $X := K \times Y$ and $h = pr_2 : X \to Y$. Then the triplet (X, h, Y) is a fiber space. The space X is conditionally compact, but it is not compact.

2. If Y is a compact set and $M \subseteq X$ is conditionally precompact, then M is a precompact set.

Let $x_0 \in X$. Denote by $\Sigma^+_{x_0} := \{\pi(t, x_0) : \ t \geq 0\}$ the positive semi-trajectory of point x_0 and $H^+(x_0) := \overline{\Sigma}^+_{x_0}$ the semi-hull of x_0, where by bar the closure of $\Sigma^+_{x_0}$ in X is denoted.

The following result provides a useful criterion for conditional compactness in applications.

Lemma 1 ([7])**.** *Let $\langle E, \phi, (Y, \mathbb{S}, \sigma)\rangle$ be a cocycle and $\langle (X, \mathbb{S}_+, \pi), (Y, \mathbb{R}S\sigma), h\rangle$ be the NDS generated by the cocycle ϕ (cf. Example 1).*
Assume that $x_0 := (u_0, y_0) \in X = E\times Y$ and the set $Q^+_{(u_0,y_0)} := \overline{\{\phi(t, u_0, y_0) : t \in \mathbb{S}_+\}}$ is compact. Then the semi-hull $H^+(x_0)$ is conditionally compact.

Denote by $C(\mathbb{T}, X)$ the family of all continuous functions $f : \mathbb{T} \to X$ equipped with the compact-open topology. This topology can be generated by Bebutov distance (see, e.g.[4],[41, ChIV])

$$d(f, g) := \sup_{L>0} \min\{\max_{|t|\leq L} \rho(f(t), g(t)), 1/L\}.$$

Denote by $(C(\mathbb{T}, X), \mathbb{T}, \sigma)$ the shift dynamical system (or called Bebutov dynamical system), i.e. $\sigma(\tau, f) := f^\tau$, where $f^\tau(t) := f(t+\tau)$ for $t \in \mathbb{T}$. Note that the function $f \in C(\mathbb{T}, X)$ is positively Lagrange stable (respectively, Lagrange stable) if and only if the function f is bounded and uniformly continuous on $\mathbb{T}$ (see, e.g.[34],[41, ChIV]).

Let $(Y, \mathbb{S}, \sigma)$ be a two-sided dynamical system.

Definition 5. A point $y \in Y$ is called *positively* (respectively, *negatively*) *Poisson stable* if there exists a sequence $t_n \to +\infty$ (respectively, $t_n \to -\infty$) such that $\sigma(t_n, y) \to y$ as $n \to \infty$. If y is Poisson stable in both directions, it is called *Poisson stable.*

Definition 6. Let $\langle E, \phi, (Y, \mathbb{S}, \sigma)\rangle$ (respectively, $(X, \mathbb{S}_+, \pi)$) be a cocycle (respectively, one-sided dynamical system). A continuous mapping $\nu : \mathbb{S} \to E$ (respectively, $\gamma : \mathbb{S} \to X$) is called an *entire trajectory* of cocycle ϕ (respectively, of dynamical system $(X, \mathbb{S}_+, \pi)$) passing through the point $(u, y) \in E \times Y$ (respectively, $x \in X$) at $t = 0$ if $\phi(t, \nu(s), \sigma(s, y)) = \nu(t + s)$ and $\nu(0) = u$ (respectively, $\pi(t, \gamma(s)) = \gamma(t + s)$ and $\gamma(0) = x$) for all $t \in \mathbb{S}_+$ and $s \in \mathbb{S}$.

Denote by

- $C(\mathbb{T}, X)$ the space of all continuous functions $f : \mathbb{T} \to X$ equipped with the compact-open topology;
- Φ_x the family of all entire trajectories of $(X, \mathbb{S}_+, \pi)$ passing through the point $x \in X$ at the initial moment $t = 0$ and $\Phi := \bigcup\{\Phi_x : \ x \in X\}$.

Remark 2. Note that:

1. if $\gamma \in \Phi_x$ then $\gamma^\tau \in \Phi_{\gamma(\tau)}$, where $\gamma^\tau(t) := \gamma(t+\tau)$ for $t \in \mathbb{T}$, and consequently Φ is a translation invariant subset of $C(\mathbb{T}, X)$;
2. if $\gamma_n \in \Phi_{x_n}$ and $\gamma_n \to \gamma$ in $C(\mathbb{T}, X)$ as $n \to \infty$, then $\gamma \in \Phi_x$ with $x := \lim\limits_{n\to\infty} x_n$ and consequently Φ is a closed subset of $C(\mathbb{T}, X)$.

By Remark 2 Φ is a closed and invariant (with respect to shifts) subset of $C(\mathbb{T}, X)$, and consequently on Φ a shift dynamical system $(\Phi, \mathbb{T}, \lambda)$ induced from $\big(C(\mathbb{T}, X), \mathbb{T}, \lambda\big)$ is defined.

Let M be a subset of X. We denote the ω-limit set of M by

$$\omega(M) := \bigcap_{t\ge 0} \overline{\bigcup\{\pi(\tau, M) : \ \tau \ge t\}};$$

for a singleton set, for simplicity we also write $\omega(x)$ or ω_x for $\omega(\{x\})$ and denote $\omega_q(M) := \omega(M) \bigcap h^{-1}(q)$. Note that $x \in \omega(M)$ if and only if there exist sequences $\{x_n\} \subset M$ and $\{t_n\} \subset \mathbb{R}$ such that $t_n \to +\infty$ as $n \to \infty$ and $\lim\limits_{n\to\infty} \pi(t_n, x_n) = x$.

Definition 7. Let $\langle (X, \mathbb{S}_+, \pi), (Y, \mathbb{S}, \sigma), h\rangle$ be an NDS. A subset $A \subseteq X$ is said to be *(positively) uniformly stable* if for arbitrary $\varepsilon > 0$ there exists $\delta = \delta(\varepsilon) > 0$ such that $\rho(x, a) < \delta$ $(a \in A,\ x \in X$ and $h(a) = h(x))$ implies $\rho(\pi(t, x), \pi(t, a)) < \varepsilon$ for any $t \ge 0$. In particular, a point $x_0 \in X$ is called uniformly stable if the singleton set $\{x_0\}$ is so.

Remark 3. Let $A \subseteq X$ be uniformly stable and $B \subseteq A$, then B is also uniformly stable.

Lemma 2. *([5, ChIV],[6]) If the set $A \subseteq X$ is uniformly stable and the mapping $h : X \to Y$ is open, then the closure $\overline{A}$ of A is uniformly stable.*

Corollary 1. *If $\Sigma^+_{x_0}$ is uniformly stable and h is open, then:*

1. *$H^+(x_0)$ is uniformly stable;*
2. *ω_{x_0} is uniformly stable, because $\omega_{x_0} \subseteq H^+(x_0)$.*

Remark 4. Note that if an NDS is generated by a skew-product dynamical system (or equivalently by a cocycle) in which case the homomorphism h is given by the natural projection mapping, then clearly h is open.

3 Poisson stable motions and their comparability by character of recurrence

3.1 Classes of Poisson stable motions

Let $(X, \mathbb{S}, \pi)$ be a dynamical system. Let us recall the classes of Poisson stable motions we study in this paper, see [31, 34, 37, 41] for details.

Definition 8. A point $x \in X$ is called *stationary* (respectively, *τ-periodic*) if $\pi(t, x) = x$ (respectively, $\pi(t + \tau, x) = \pi(t, x)$) for all $t \in \mathbb{S}$.

Definition 9. A point $x \in X$ is called *quasi-periodic* with the base of frequency $\nu = (\nu_1, \nu_2, \dots, \nu_k)$ if the associated function $f(\cdot) := \pi(\cdot, x) : \mathbb{S} \to X$ satisfies the following conditions:

1. the numbers $\nu_1, \nu_2, \dots, \nu_k$ are rationally independent;
2. there exists a continuous function $\Phi : \mathbb{R}^k \to X$ such that $\Phi(t_1 + 2\pi, t_2 + 2\pi, \dots, t_k + 2\pi) = \Phi(t_1, t_2, \dots, t_k)$ for all $(t_1, t_2, \dots, t_k) \in \mathbb{R}^k$;
3. $f(t) = \Phi(\nu_1 t, \nu_2 t, \dots, \nu_k t)$ for $t \in \mathbb{R}$.

Definition 10. For given $\varepsilon > 0$, a number $\tau \in \mathbb{R}$ is called an *ε-shift of x* (respectively, *ε-almost period of x*) if $\rho(\pi(\tau, x), x) < \varepsilon$ (respectively, $\rho(\pi(\tau+t, x), \pi(t, x)) < \varepsilon$ for all $t \in \mathbb{R}$).

Definition 11. A point $x \in X$ is called *almost recurrent* (respectively, *Bohr almost periodic*) if for any $\varepsilon > 0$ there exists a positive number l such that any segment of length l contains an ε-shift (respectively, ε-almost period) of x.

Definition 12. If a point $x \in X$ is almost recurrent and its trajectory Σ_x is precompact, then x is called *(Birkhoff) recurrent*.

Denote $\mathfrak{N}_y := \{\{t_n\} \subset \mathbb{S} : \sigma(t_n, y) \to y\}$, $\mathfrak{N}_y^{+\infty} := \{\{t_n\} \in \mathfrak{N}_y : t_n \to +\infty\}$, $\mathfrak{N}_y^{-\infty} := \{\{t_n\} \in \mathfrak{N}_y : t_n \to -\infty\}$, and $\mathfrak{N}_y^{\infty} := \{\{t_n\} \in \mathfrak{N}_y : t_n \to \infty\}$.

Definition 13. A point $x \in X$ is called *Levitan almost periodic* [27] (see also [5, 10, 26]) if there exists a dynamical system $(Y, \mathbb{T}, \sigma)$ and a Bohr almost periodic point $y \in Y$ such that $\mathfrak{N}_y \subseteq \mathfrak{N}_x$.

Definition 14. A point $x \in X$ is called *almost automorphic* if it is Lagrange stable and Levitan almost periodic.

Definition 15. A point $x \in X$ is said to be *uniformly Poisson stable* or *pseudo-periodic* in the positive (respectively, negative) direction if for arbitrary $\varepsilon > 0$ and $l > 0$ there exists an ε-almost period $\tau > l$ (respectively, $\tau < -l$) of x. The point x is said to be uniformly Poisson stable or pseudo-periodic if it is so in both directions.

Definition 16 ([32, 33]). A point $x \in X$ is said to be *pseudo-recurrent* if for any $\varepsilon > 0$, $p \in \Sigma_x$ and $t_0 \in \mathbb{R}$ there exists $L = L(\varepsilon, t_0) > 0$ such that

$$B(p, \varepsilon) \bigcap \pi([t_0, t_0 + L], p) \neq \emptyset,$$

where
$B(p, \varepsilon) := \{x \in X : \rho(p, x) < \varepsilon\}$ and $\pi([t_0, t_0 + L], p) := \{\pi(t, p) : t \in [t_0, t_0 + L]\}$.

Definition 17. A point $x \in X$ is said to be [16, ChI] *strongly Poisson stable* (in the positive direction) if $p \in \omega_p$ for any $p \in H(x)$.

Remark 5. It is known that:

1. a strongly Poisson stable point is Poisson stable, but the converse is not true in general;
2. all the motions introduced above (Definitions 8–16) are strongly Poisson stable.

Definition 18 ([11,38]). A point $x \in X$ is said to be asymptotically stationary (respectively, asymptotically τ-periodic, asymptotically Bohr almost periodic, asymptotically almost automorphic, asymptotic recurrent, asymptotically Levitan almost periodic, asymptotically almost recurrent, asymptotically pseudo-periodic, asymptotically pseudo-recurrent, asymptotically Poisson stable) if there exists a stationary (respectively, τ-periodic, Bohr almost periodic, almost automorphic, recurrent, Levitan almost periodic, almost recurrent, pseudo-periodic, pseudo-recurrent, Poisson stable) point $p \in X$ such that $\lim_{t \to +\infty} \rho(\pi(t, x), \pi(t, p)) = 0$.

3.2 Comparability of motions by their character of recurrence

In this subsection we present some notions and results stated and proved by B. A. Shcherbakov [34]–[37].

Let $(X, \mathbb{S}, \pi)$ and $(Y, \mathbb{S}, \sigma)$ be two dynamical systems.

Definition 19. A point $x \in X$ is said to be *comparable with* $y \in Y$ *by character of recurrence* if for any $\varepsilon > 0$ there exists a $\delta = \delta(\varepsilon) > 0$ such that every δ-shift of y is an ε-shift for x, i.e., $\rho(\sigma(\tau, y), y) < \delta$ implies $\rho(\pi(\tau, x), x) < \varepsilon$.

Theorem 1. *The following conditions are equivalent:*

1. *the point x is comparable with y by character of recurrence;*
2. $\mathfrak{N}_y \subseteq \mathfrak{N}_x$*;*
3. $\mathfrak{N}_y^{\infty} \subseteq \mathfrak{N}_x^{\infty}$*;*
4. *from any sequence $\{t_n\} \in \mathfrak{N}_y$ we can extract a subsequence $\{t_{n_k}\} \in \mathfrak{N}_x$;*
5. *from any sequence $\{t_n\} \in \mathfrak{N}_y^{\infty}$ we can extract a subsequence $\{t_{n_k}\} \in \mathfrak{N}_x^{\infty}$.*

Theorem 2. *Let $x \in X$ be comparable with $y \in Y$. If the point y is stationary (respectively, τ-periodic, Levitan almost periodic, almost recurrent, Poisson stable), then so is the point x.*

Definition 20. A point $x \in X$ is called *uniformly comparable with $y \in Y$ by character of recurrence* if for any $\varepsilon > 0$ there exists a $\delta = \delta(\varepsilon) > 0$ such that every δ-shift of $\sigma(t,y)$ is aN ε-shift of $\pi(t,x)$ for all $t \in \mathbb{S}$, i.e., $\rho(\sigma(t+\tau,y),\sigma(t,y)) < \delta$ implies $\rho(\pi(t+\tau,x),x) < \varepsilon$ for all $t \in \mathbb{R}$ (or equivalently: $\rho(\sigma(t_1,y),\sigma(t_2,y)) < \delta$ implies $\rho(\pi(t_1,x),\pi(t_2,x)) < \varepsilon$ for all $t_1,t_2 \in \mathbb{S}$).

Denote $\mathfrak{M}_x := \{\{t_n\} \subset \mathbb{S} : \ \{\pi(t_n,x)\} \text{ converges}\}$, $\mathfrak{M}_x^{+\infty} := \{\{t_n\} \in \mathfrak{M}_x : t_n \to +\infty \text{ as } n \to \infty\}$ and $\mathfrak{M}_x^{\infty} := \{\{t_n\} \in \mathfrak{M}_x : t_n \to \infty \text{ as } n \to \infty\}$.

Definition 21 ([8,11])**.** A point $x \in X$ is said to be *strongly comparable with $y \in Y$ by character of recurrence* if $\mathfrak{M}_y \subseteq \mathfrak{M}_x$.

Theorem 3. *(i) If $\mathfrak{M}_y \subseteq \mathfrak{M}_x$, then $\mathfrak{N}_y \subseteq \mathfrak{N}_x$, i.e. strong comparability implies comparability.*

(ii) Let X be a complete metric space. If the point x is uniformly comparable with y by character of recurrence, then $\mathfrak{M}_y \subseteq \mathfrak{M}_x$, i.e. uniform comparability implies strong comparability.

Theorem 4. *Let y be Lagrange stable. Then $\mathfrak{M}_y \subseteq \mathfrak{M}_x$ holds if and only if the point x is Lagrange stable and uniformly comparable with y by character of recurrence.*

Theorem 5. *Let X and Y be two complete metric spaces. Let the point $x \in X$ be uniformly comparable with $y \in Y$ by character of recurrence. If y is quasi-periodic (respectively, Bohr almost periodic, almost automorphic, Birkhoff recurrent, Lagrange stable, pseudo-periodic, pseudo-recurrent), then so is x.*

4 Global Attractors of Non-Autonomous Dynamical Systems

Definition 22. A family $\{A_y|\ y \in Y\}$ of subsets A_y of W indexed by $y \in Y$ is called a non-autonomous set.

Let $\{A_y|\ y \in Y\}$ be a non-autonomous set. Denote by $\mathbb{A}$ the subset of $X := W \times Y$ defined by equality

$$\mathbb{A} := \bigcup\{A_p \times \{y\}|\ y \in Y\}\} = \{(w,y) \in X|\ w \in A_y,\ y \in Y\}.$$

Remark 6. 1. Let $\mathcal{A}$ be a subset of $X = W \times Y$, $\mathcal{A}_y := \mathbb{A} \bigcap pr_2^{-1}(y)$ and $A_y := pr_1(\mathcal{A}_y)$, then $\{A_y|\ y \in Y\}$ is a non-autonomous set.

2. Denote by $\mathfrak{A} = \bigcup\{A_y \times \{y\}|\ y \in Y\}$, then $\mathcal{A} \subseteq \mathfrak{A}$.

Definition 23. A non-autonomous set $\{A_y|\ y \in Y\}$is said to be

1. precompact (respectively, uniformLY precompact) if for every $y \in Y$ the set A_y (respectively, $\bigcup\{A_y|\ y \in Y\}$) is a precompact subset of W;

2. bounded (respectively, uniformLY bounded) if for every $y \in Y$ the set A_y (respectively, $\bigcup\{A_y |\ y \in Y\}$) is a bounded subset of W.

Let W be a complete metric space.

Definition 24. A cocycle φ over $(Y, \mathbb{T}, \sigma)$ with the fiber W is said to be compactly dissipative if there exits a nonempty compact $K \subseteq W$ such that

$$\lim_{t\to+\infty} \sup\{\beta(U(t, y)M, K) \mid y \in Y\} = 0 \tag{1}$$

for any $M \in C(W)$, where $\beta(A, B) := \sup\{\rho(a, B) : a \in A\}$ is a semi-distance of Hausdorff.

Definition 25. The family $\{I_y \mid y \in Y\}(I_y \subset W)$ of nonempty compact subsets is called a compact (forward) global attractor of the cocycle φ if the following conditions are fulfilled:

1. the set $I := \bigcup\{I_y \mid y \in Y\}$ is relatively compact;
2. the family $\{I_y \mid y \in Y\}$ is invariant with respect to the cocycle φ;
3. the equality
$$\lim_{t\to+\infty} \sup_{y\in Y} \beta(\varphi(t, K, y), I) = 0$$
holds for every $K \in C(W)$.

Let $M \subseteq W$ and

$$\omega_y(M) := \bigcap_{t\geq 0} \overline{\bigcup_{\tau\geq t} \varphi(\tau, M, \sigma(-\tau, y))}$$

for any $y \in Y$.

Theorem 6. *[12, ChII] Let $\langle W, \varphi, (Y, \mathbb{S}, \sigma)\rangle$ be compactly dissipative and K be the nonempty compact subset of W appearing in the equality (1), then:*

1. *$I_y = \omega_y(K) \neq \emptyset$, is compact, $I_y \subseteq K$ and*
$$\lim_{t\to+\infty} \beta(U(t, \sigma(-t, y))K, I_y) = 0$$
for every $y \in Y$;
2. *$U(t, y)I_y = I_{yt}$ for all $y \in Y$ and $t \in \mathbb{T}_+$;*
3. $$\lim_{t\to+\infty} \beta(U(t, \sigma(-t, y))M, I_y) = 0$$
for all $M \in C(W)$ and $y \in Y$;
4. *the set I is relatively compact, where $I := \cup\{I_y \mid y \in Y\}$.*

Theorem 7. *[14] Let $\langle W, \varphi, (Y, \mathbb{S}, \sigma)\rangle$ be compactly dissipative and K be the nonempty compact subset of W appearing in the equality (1), then the family of subsets $\{I_y|\ y \in Y\}$ is a maximal family possessing the properties 2.–4.*

Definition 26. Let $\langle W, \varphi, (Y, \mathbb{S}, \sigma)\rangle$ be compactly dissipative, K be the nonempty compact subset of W appearing in the equality (1) and $I_y := \omega_y(K)$ for any $y \in Y$. The family of compact subsets $\{I_y|\ y \in Y\}$ is said to be a Levinson center (compact global attractor) of non-autonomous (cocycle) dynamical system $\langle W, \varphi, (Y, \mathbb{S}, \sigma)\rangle$.

Remark 7. According to Theorem 7 by Definition 26 the notion Levinson center (compact global attractor) for non-autonomous (cocycle) dynamical system $\langle W, \varphi, (Y, \mathbb{S}, \sigma)\rangle$ is well defined.

Corollary 2. *Let $\langle W, \varphi, (Y, \mathbb{S}, \sigma)\rangle$ be compactly dissipative non-autonomous dynamical system, $\{I_y|\ y \in Y\}$ be its Levinson center and $\nu : \mathbb{T} \mapsto W$ be a relatively compact full trajectory of φ (i.e., $\nu(\mathbb{S})$ is relatively compact and there exists a point $y_0 \in Y$ such that $\nu(t+s) = \varphi(t, \nu(s), \sigma(s, y_0))$ for any $t \geq 0$ and $s \in \mathbb{S}$), then $\nu(0) \in I_{y_0}$.*

Theorem 8. *[12, ChII] Under the conditions of Theorem 6 $w \in I_y$ $(y \in Y)$ if and only if there exits a whole trajectory $\nu : \mathbb{S} \to W$ of the cocycle φ, satisfying the following conditions: $\nu(0) = w$ and $\nu(\mathbb{S})$ is relatively compact.*

Definition 27. A family of subsets $\{I_y|\ y \in Y\}$ $(I_y \subseteq W$ for any $y \in Y)$ is said to be upper semicontinuous if for any $y_0 \in Y$ and $y_n \to y_0$ as $n \to \infty$ we have

$$\lim_{n\to\infty} \beta(I_{y_n}, I_{y_0}) = 0.$$

Lemma 3. *[14] The following statements hold:*

1. *the family $\{I_y|\ y \in Y\}$ is invariant if and only if the set $J := \bigcup\{J_y|\ y \in Y\}$, where $J_y := I_y \times \{y\}$, is invariant with respect to skew-product dynamical system $(X, \mathbb{S}_+, \pi)$ $(X := W \times Y$ and $\pi := (\varphi, \sigma))$;*
2. *if $\bigcup\{I_y|\ y \in Y\}$ is relatively compact, then the family $\{I_y|\ y \in Y\}$ is upper semicontinuous if and only if the set J is closed in X.*

Definition 28. A non-autonomous set $\mathbb{K} = \{K_y :\ y \in Y\}$ with $K_y \subseteq W$ for any $y \in Y$ is said to be positively Lyapunov stable (respectively, uniformly stable) if for arbitrary $\varepsilon > 0$ and $y \in Y$ there exists a positive number $\delta = \delta(\varepsilon, y, \mathbb{K}) > 0$ (respectively, $\delta = \delta(\varepsilon, \mathbb{K}) > 0$) such that $\rho(\varphi(t_0, u, y), \varphi(t_0, u_0, y)) < \delta$ $(u_0 \in K_y$ and $u \in W)$ implies $\rho(\varphi(t, u, y), \varphi(t, u_0, y)) < \varepsilon$ for any $t \geq t_0$ and $y \in Y$.

Definition 29. A trajectory $\varphi(t, u_0, y_0)$ of the point $(u_0, y_0) \in W \times Y$ is said to be positively uniformly Lyapunov stable if the set $K_0 := \varphi(\mathbb{T}_+, u_0, y_0) = \{\varphi(t, u_0, y_0)|\ t \in \mathbb{S}_+\}$ is uniformly Lyapunov stable.

Definition 30. A cocycle φ is said to be positively Lyapunov stable (respectively, uniformly stable) if for arbitrary $\varepsilon > 0$ and non-autonomous uniformly precompact and upper semicontinuous set $\mathbb{K} = \{K_y : \ y \in Y\}$ there exists a number $\delta = \delta(\varepsilon, y, \mathbb{K}) > 0$ (respectively, $\delta = \delta(\varepsilon, \mathbb{K}) > 0$) such that $\rho(u, u_0) < \delta$ with $u_0 \in K_y$ and $u \in W$ implies $\rho(\varphi(t, u, y), \varphi(t, u_0, y)) < \varepsilon$ for any $t \geq 0$.

Theorem 9. *[12, Ch.IX] Let $\langle W, \varphi, (Y, \mathbb{S}, \sigma)\rangle$ be a cocycle with the following properties:*

1. *It admits a conditionally relatively compact invariant set $\{I_y \mid y \in Y\}$ (i.e. $\bigcup\{I_y \mid y \in Y'\}$ is relatively compact subset of W for any relatively compact subset Y' of Y).*
2. *The cocycle φ is positively uniformly stable on $\{I_y | \quad y \in Y\}$.*

Then all motions on $J := \bigcup\{J_y | \quad y \in Y\} \quad (J_y := I_y \times \{y\})$ may be continued uniquely to the left and on J a two-sided dynamical system $(J, \mathbb{S}, \pi)$ is defined , i.e., the skew-product system $(X, \mathbb{S}_+, \pi)$ generates on J a two-sided dynamical system $(J, \mathbb{S}, \pi)$.

5 Monotone NDS: existence and convergence to Poisson stable motions

Let E be a real Banach space with a closed convex cone $P \subset E$ such that $P \bigcap(-P) = \{0\}$. Assume that $Int(P) \neq \emptyset$. For $u_1, u_2 \in E$, we write $u_1 \leq u_2$ if $u_2 - u_1 \in P$; $u_1 < u_2$ if $u_2 - u_1 \in P \setminus \{0\}$; $u_1 \ll u_2$ if $u_2 - u_1 \in Int(P)$.

Assume that E is an ordered space.

Definition 31. A subset U of E is said [25] to be order convex if for any $a, b \in U$ with $a < b$, the segment $\{a + s(b - a) : \ s \in [0, 1]\}$ is contained in U.

Let $V = [0, b]_E$ with $b \gg 0$ or $V = P$, or furthermore, V be an order convex subset of E.

Definition 32. A subset U of E is called lower-bounded (respectively, upper-bounded) if there exists an element $a \in E$ such that $a \leq U$ (respectively, $a \geq U$). Such an a is said to be a lower bound (respectively, upper bound) for U.

Definition 33. A lower bound α is said to be the *greatest lower bound* (g.l.b.) or *infimum*, if any other lower bound a satisfies $a \leq \alpha$. Similarly, we can define the *least upper bound* (l.u.b.) or *supremum*.

A bundle (X, h, Y) is said to be *ordered* if each fiber X_y is ordered. Note that only points on the same fiber may be order related: if $x_1 \leq x_2$ or $x_1 < x_2$, then it implies $h(x_1) = h(x_2)$. We assume that the order relation and the topology on X are compatible in the sense that $x \leq \tilde{x}$ if $x_n \leq \tilde{x}_n$ for all n and $x_n \to x$, $\tilde{x}_n \to \tilde{x}$ as $n \to \infty$.

Definition 34. For given bundle (X, h, Y), an NDS $\langle (X, \mathbb{S}_+, \pi), (Y, \mathbb{S}, \sigma), h\rangle$ defined on it is said to be *monotone* (respectively, *strictly monotone*) if $x_1 \le x_2$ (respectively, $x_1 < x_2$) implies $\pi(t, x_1) \le \pi(t, x_2)$ (respectively, $\pi(t, x_1) < \pi(t, x_2)$) for any $t > 0$.

For given NDS $\langle (X, \mathbb{S}_+, \pi), (Y, \mathbb{S}, \sigma), h\rangle$, let $\mathcal{S} \subseteq X$ be a nonempty closed ordered subset possessing the following properties:

1. $h(\mathcal{S}) = Y$;
2. $\mathcal{S}$ is positively invariant with respect to π, i.e., $\langle (\mathcal{S}, \mathbb{S}_+, \pi), (Y, \mathbb{S}, \sigma), h\rangle$ is an NDS.

Below we will use the following assumptions:

(**C1**) For every conditionally compact subset K of $\mathcal{S}$ and $y \in Y$ the set $K_y := h^{-1}(y) \bigcap K$ has both infimum $\alpha_y(K)$ and supremum $\beta_y(K)$.

(**C2**) For every $x \in \mathcal{S}$, the semi-trajectory Σ_x^+ is conditionally precompact, $\omega_x \neq \emptyset$ and the set ω_x is positively uniformly stable.

(**C2.1**) For every $x \in \mathcal{S}$, the semi-trajectory Σ_x^+ is conditionally precompact and $\omega_x \neq \emptyset$.

(**C3**) The NDS
$$\langle (\mathcal{S}, \mathbb{S}_+, \pi), (Y, \mathbb{S}, \sigma), h\rangle$$
is monotone.

Let X, Y be two complete metric spaces and
$$\langle (X, \mathbb{T}_1, \pi), (Y, \mathbb{T}_2, \sigma), h\rangle \tag{2}$$
be a non-autonomous dynamical system.

Definition 35. A closed subset M of X is said to be a minimal set of non-autonomous dynamical system (NDS) (2) if it possesses the following properties:

a. $h(M) = Y$;

b. M is positively invariant, i.e., $\pi(t, M) \subseteq M$ for any $t \in \mathbb{T}_1$;

c. M is a minimal subset of X possessing properties a. and b..

Theorem 10. *[16, Ch.IV] Let $\langle (X, \mathbb{T}_1, \pi), (Y, \mathbb{T}_2, \sigma), h\rangle$ be a non-autonomous dynamical system and $M \subset X$ be a nonempty, conditionally compact and positively invariant set. If the dynamical system $(Y, \mathbb{T}_2, \sigma)$ is minimal, then the subset M is a minimal subset of NDS (2) if and only if $H(x) = M$ for any $x \in M$.*

Theorem 11. *[16, Ch.IV] Suppose that $\langle (X, \mathbb{T}_1, \pi), (Y, \mathbb{T}_2, \sigma), h\rangle$ is a non-autonomous dynamical system, $(Y, \mathbb{T}_2, \sigma)$ is minimal and the space X is conditionally compact, then there exists a minimal subset M of NDS (2).*

Lemma 4. *Let $\langle (X, \mathbb{S}_+, \pi), (Y, \mathbb{S}, \sigma), h\rangle$ be an NDS with the following properties:*

a. there exists a point $x_0 \in X$ such that the positive semi-trajectory $\Sigma^+_{x_0}$ is conditionally precompact;

b. the point $y_0 := h(x_0)$ is Poisson stable, i.e., $y_0 \in \omega_{y_0}$.

Then the following statements hold:

1. there are a Poisson stable point $p \in \omega_{x_0}$ and a sequence $\{t_k\} \in \mathfrak{N}^{+\infty}_{y_0}$ such that

$$\lim_{k\to\infty} \rho(\pi(t_k, x_0), \pi(t_k, p)) = 0;$$

2. if the dynamical system $(Y, \mathbb{S}, \sigma)$ is minimal, then there are a minimal subset $M \subseteq \omega_{x_0}$ of non-autonomous dynamical system $\langle(\omega_{x_0}, \mathbb{S}_+, \pi), (Y, \mathbb{S}, \sigma), h\rangle$, a point $p \in M \bigcap X_{y_0}$ and a sequence $\{t_k\} \in \mathfrak{N}^{+\infty}_{y_0}$ such that (3) is fulfilled.

Corollary 3. *Let $\langle (X, \mathbb{S}_+, \pi), (Y, \mathbb{S}, \sigma), h\rangle$ be an NDS with the following properties:*

a. there exists a point $x_0 \in X$ such that the positive semi-trajectory $\Sigma^+_{x_0}$ is conditionally precompact;

b. the point $y_0 := h(x_0)$ is Poisson stable, i.e., $y_0 \in \omega_{y_0}$;

c. the set ω_{x_0} is positively uniformly stable.

Then the following statements hold:

1. there is a Poisson stable point $p \in \omega_{x_0}$ such that

$$\lim_{t\to+\infty} \rho(\pi(t, x_0), \pi(t, p)) = 0; \tag{3}$$

2. if the dynamical system $(Y, \mathbb{S}, \sigma)$ is minimal, then there are a minimal subset $M \subseteq \omega_{x_0}$ of non-autonomous dynamical system $\langle(\omega_{x_0}, \mathbb{S}_+, \pi), (Y, \mathbb{S}, \sigma), h\rangle$ and a point $p \in M \bigcap X_{y_0}$ such that (3) is fulfilled, and hence, ω_{x_0} is a minimal subset of non-autonomous dynamical system $\langle(X_0, \mathbb{S}_+, \pi), (Y, \mathbb{S}, \sigma), h\rangle$.

Now fix $(x_0, y_0) \in V \times Y$, then the set $\omega_{(x_0,y_0)}$ is a nonempty , conditionally compact, positively invariant set. Assume that $(Y, \mathbb{T}, \sigma)$ is minimal and $y \in \omega_y$ for any $y \in Y$, then $h(\omega_{(x_0,y_0)}) = Y$. According to Corollary 3 the set $\omega_{(x_0,y_0)}$ is a minimal set of non-autonomous dynamical system $\langle (V \times Y, \mathbb{T}_+, \pi), (Y, \mathbb{T}, \sigma), h\rangle$. We put $K := \omega_{(x_0,y_0)}$.

Let (E_i, P_i), $1 \le i \le n$, be ordered Banach spaces with $Int(P_i) \neq \emptyset$. For each $I = \{j_1, \dots, j_m\} \subseteq N := \{1, \dots, n\}$, we define

$$E_I := \prod_{k=1}^{m} E_{j_k}, \quad P_I := \prod_{k=1}^{m} P_{j_k}.$$

Then (E_I, P_I) is an ordered Banach space with

$$Int(P_I) = \prod_{k=1}^{m} Int(P_{j_k}) \neq \emptyset.$$

Let $\leq_I$ (respectively, $<_I$ and $\ll_I$) be the orders induced by P_I in E_I. In the case where $I = N$, we use (E, P) to denote the ordered Banach space (E_N, P_N), and omit the order subscripts to get the orders $\leq$, $<$ and $\ll$ in E, respectively. For each $1 \leq i \leq n$, let $Q_i : E \times Y :\to E_i$ be the projection mapping defined by $Q_i(x, y) = x_i$.

Condition **(C4)**. For any two bounded full orbits $\gamma_j \in \mathcal{F}_{(x_j,y)}$ $(j = 1, 2)$ with $\gamma_1(t) \leq \gamma_2(t)$ for any $t \in \mathbb{S}$, there exists $t_0 > 0$ such that whenever $Q_i\gamma_1(s) < Q_i\gamma_2(s)$ holds for some $i \in N$ and $s \in \mathbb{S}$, then $Q_i\gamma_1(t) \ll Q_i\gamma_2(t)$ for all $t \geq s + t_0$.

Definition 36. A skew-product semi-flow π on $V \times Y$ is said [25] to be componentwise strongly monotone if it is monotone and whenever $x_1 \leq x_2$ with $x_{1i} < x_{2i}$, one has $Q_i\pi(t, (x_1, y)) \ll Q_i\pi(t, (x_2, y))$ for all $t > 0$.

Remark 8. If the semi-flow π is componentwise strongly monotone, then it satisfies Condition (**C4**).

Theorem 12. *[20] Assume that the dynamical system $(Y, \mathbb{S}, \sigma)$ is minimal and $q \in \omega_q$ for any $q \in Y$. Under conditions **(C1)**-**(C4)** for any $(x_0, y_0) \in V \times Y$ the following statements hold:*

1. *for any $q \in Y$ the set*

$$\omega_{(x_0,y_0)} \bigcap X_q$$

consists of a single point $\{(x_q, q)\}$;

2. *the point (x_q, q) is strongly comparable by character of recurrence with the point $q \in Y$;*

3. $$\lim_{t \to +\infty} \rho(\varphi(t, x_0, y_0), \varphi(t, x_{y_0}, y_0)) = 0.$$

Corollary 4. *Under the conditions $(\boldsymbol{C1}) - (\boldsymbol{C4})$ if the point y_0 is τ-periodic (respectively, quasi-periodic, Bohr almost periodic, recurrent, strongly Poisson stable and $H(y_0)$ is a minimal set), then:*

1. *the point x_{y_0} is so;*

2. *the point x_0 is asymptotically τ-periodic (respectively, asymptotically quasi-periodic, asymptotically Bohr almost periodic, asymptotically recurrent, asymptotically strongly Poisson stable).*

6 Structure of the Levinson center for monotone non-autonomous dynamical systems

Lemma 5. *[16, Ch. V] Suppose that $\langle W, \varphi, (Y, \mathbb{S}, \sigma)\rangle$ is a cocycle under $(Y, \mathbb{S}, \sigma)$ with the fibre W. If Y is a compact space, then the following conditions are equivalent:*

a) the cocycle φ is positively uniformly Lyapunov stable;

b) every trajectory $\varphi(t, u_0, y_0)$ ($x_0 := (u_0, y_0) \in W \times Y$) of cocycle φ is positively uniformly Lyapunov stable.

Theorem 13. *[17] Assume that the cocycle $\langle E, \varphi, (Y, \mathbb{S}, \sigma)\rangle$*

1. *is monotone;*
2. *admits a compact global attractor $\boldsymbol{I} := \{I_y | \ y \in Y\}$;*
3. *is positively uniformly Lyapunov stable and denote by $\alpha(y)$ (respectively, by $\beta(y)$) the greatest lower bound of the set I_y (respectively, the least upper bound of I_y) and*
4. *the point $y \in Y$ is positively Poisson stable, i.e., $y \in \omega_y$.*

Then the following statements hold:

1. *$\alpha(y) \le u \le \beta(y)$ for any $u \in I_y$ and $y \in Y$;*
2. *$\alpha(y), \beta(y) \in I_y$ and, consequently, $I_y \subseteq [\alpha(y), \beta(y)]$;*
3. *$\varphi(t, \alpha(y), y) = \alpha(\sigma(t, y))$ (respectively, $\varphi(t, \beta(y), y) = \beta(\sigma(t, y))$) for any $t \ge 0$;*
4. *the point $\gamma_*(y) := (\alpha(y), y) \in X = E \times Y$ (respectively, $\gamma^*(y) := (\beta(y), y) \in X$) is comparable by character of recurrence with the point y;*
5. *if $u \in E$ and $u \le \alpha(y)$ (respectively, $u \ge \beta(y)$), then $\omega_x \bigcap X_y = \{\gamma_*(y)\}$ (respectively, $\omega_x \bigcap X_y = \{\gamma^*(y)\}$), where $x := (u, y)$;*
6. *if $u \le \alpha(y)$ (respectively, $u \ge \beta(y)$),then*

$$\lim_{t \to +\infty} \rho(\varphi(t, u, y).\gamma_*(\sigma(t, y))) = 0$$

(respectively,

$$\lim_{t \to +\infty} \rho(\varphi(t, u, y).\gamma^*(\sigma(t, y))) = 0);$$

7. *if y is strongly Poisson stable, then the point $\gamma_*(y) := (\alpha(y), y) \in X = E \times Y$ (respectively, $\gamma^*(y) := (\beta(y), y) \in X$) is strongly comparable by character of recurrence with the point y.*

Corollary 5. *Under the conditions of Theorem 13 the following statements take place:*

1. *if the point y is τ-periodic (respectively, Levitan almost periodic, almost recurrent, almost automorphic, recurrent, Poisson stable), then the full trajectory γ_y passing through the point $(\alpha(y), y)$ (respectively, through the point $(\beta(y), y)$) is so;*

2. *if the point y is quasi-periodic (respectively, Bohr almost periodic, almost automorphic, recurrent, pseudo-recurrent and Lagrange stable, uniformly Poisson stable and stable in the sense of Lagrange), then the full trajectory γ_y passing through the point $(\alpha(y), y)$ (respectively, through the point $(\beta(y), y)$) is so.*

Remark 9. Corollary 5 generalizes and refines the results of the work [14] which give as the positive answer for I. U. Bronshtein's conjecture [5, ChIV, p.273] for monotone Bohr almost periodic systems.

7 Translation-invariant monotone systems

Definition 37. Let E be a real Banach space. A cone P is said to be normal if the norm $|\cdot|$ in E is semi-monotone, i.e., there exists a constant $k > 0$ such that the property $0 \le u \le v$ implies that $|u| \le k|v|$.

Assume that E is a strongly ordered Banach space with normal cone P.

Fix $v \in Int(P)$ with $|v| = 1$. Let G be the group of phase-translations $T_a : E \to E$; $T_a(u) := u + av$, by a scalar $a \in \mathbb{R}$.

Definition 38. The phase-translation group $G = \{T_a |\ a \in \mathbb{R}\}$ commutes with the skew-product dynamical system $(X, \mathbb{T}, \pi)$ ($X = E \times Y$, $\pi = (\varphi, \sigma)$) if

$$\pi(t, (T_a(u), y)) = (T_a(\varphi(t, u, y)), \sigma(t, y))$$

for any $x = (u, y) \in X = E \times Y$, $t \in \mathbb{T}$ and $T_a \in G$.

For such v above, the Banach space E has a direct sum decomposition

$$E = E_0 \oplus Span(v),$$

where E_0 is the null space of a bounded linear functional f on E with $\langle f, v\rangle = 1$.

Let $v \in P$ be a strongly positive unit vector, i.e., $v \in Int(P)$ and $|v| = 1$. Define the v-norm as follows

$$||u||_v := \inf\{\alpha > 0 :\ -\alpha v \le u \le \alpha v\}.$$

Remark 10. If the cone P is normal, then the norms $|\cdot|$ and $||\cdot||_v$ are equivalent.

Let V be an ordered convex subset of E.

Definition 39. A cocycle $\langle V, \varphi, (Y, \mathbb{S}, \sigma)\rangle$ is said to be translation invariant with respect to v if

$$\varphi(t, u + \lambda v, y) = \varphi(t, u, y) + \lambda v$$

for any $(t, u, y, \lambda) \in \mathbb{S}_+ \times V \times Y \times \mathbb{R}$.

Lemma 6. *[22, 28] Assume that the following conditions are fulfilled:*

1. *$(X, \mathbb{S}_+, \pi)$ is the skew-product dynamical system on $X := V \times Y$ generated by cocycle φ;*
2. *the subset $\mathcal{E} \subset V$ is invariant with respect to translation by a strongly positive vector v, i.e., $v \gg 0$ and $u + \lambda v \in \mathcal{E}$ for any $u \in \mathcal{E}$ and $\lambda \in \mathbb{R}$;*
3. *the set $\mathcal{E} \times Y$ is invariant with respect to skew-product dynamical system $(X, \mathbb{S}_+, \pi)$.*

Then

1. *the cocycle φ is positively uniformly stable;*
2. *every positive semitrajectory of the skew-product dynamical system $(\mathcal{E} \times Y, \mathbb{S}_+, \pi)$ is uniformly positively stable;*
3. *$\|\varphi(t, u_1, y) - \varphi(t, u_2, y)\|_v \le \|u_1 - u_2\|_v$ for any $u_1, u_2 \in E$, $y \in Y$ and $t \ge 0$.*

Proof. Let $(u_1, y) \in X \times Y$. Then we shall prove that

$$\|\varphi(t, u_1, y) - \varphi(t, u_2, y)\|_v \le \|u_1 - u_2\|_v, \tag{4}$$

for all $t >$. By the definition of v-norm,

$$-\|u_1 - u_2\|_v v \le u_1 - u_2 \le \|u_1 - u_2\|_v v,$$

for all $u_1, u_2 \in E$, that is,

$$u_2 - \|u_1 - u_2\|_v v \le u_1 \le u_2 + \|u_1 - u_2\|_v v,$$

for all $u_1, u_2 \in E$. This inequality implies together with monotonicity and positive translation invariance that for all $u_1, u_2 \in E, t > 0$,

$$\varphi(t, u_2, y) - \|u_1 - u_2\|_v v \le \varphi(t, u_1, y) \le \varphi(t, u_2, y) + \|u_1 - u_2\|_v v.$$

Equivalently, for all $u_1, u_2 \in E, t > 0$,

$$-\|u_1 - u_2\|_v v \le (\varphi(t, u_1, y) - \varphi(t, u_2, y) \le \|u_1 - u_2|_v v. \tag{5}$$

(4) immediately follows from (5) and the definition of v-norm and, consequently, φ is positively uniformly stable. By the cocyle property, we have that

$$\varphi(t + \tau, u, y) = \varphi(t, \varphi(\tau, u, y), \sigma(\tau, y)) \text{ for all } u \in E,\ y \in Y,\quad t, \tau > 0.$$

From this cocyle property together with (4), we conclude that

$$\|\varphi(t + \tau, u_1, y) - \varphi(t + \tau, u_2, y)\|_v \le \|\varphi(\tau, u_1, y) - \varphi(\tau, u_2, y)\|_v, \quad \text{for all } \ t, \tau > 0.$$

This proves that every forward orbit of $(X, \mathbb{R}_+, \pi)$ is uniformly stable in the order-norm. The normality of the cone P implies that every forward orbit of $(X, \mathbb{R}_+, \pi)$ is uniformly stable. □

Theorem 14. *Let $\langle W, \varphi, (Y, \mathbb{S}, \sigma)\rangle$ be a cocycle over dynamical system $(Y, \mathbb{S}, \sigma)$ with the fiber W. Assume that the dynamical system $(Y, \mathbb{S}, \sigma)$ is minimal, $q \in \omega_q$ for any $q \in Y$ and the cocycle φ is translation invariant with respect to $v \in Int(P)$.*

*Under the conditions **(C1)**, **(C2.1)** and **(C3)**-**(C4)** for any $(x_0, y_0) \in V \times Y$ the following statements hold:*

1. *for any $q \in Y$ the set*
$$\omega_{(x_0,y_0)} \bigcap X_q$$
consists of a single point $\{(x_q, q)\}$;
2. *the point (x_q, q) is strongly comparable by character of recurrence with the point $q \in Y$;*
3. $$\lim_{t\to+\infty} \rho(\varphi(t, x_0, y_0), \varphi(t, x_{y_0}, y_0)) = 0.$$

Proof. Since the cocycle φ is monotone and translation invariant with respect to $v \in Int(P)$, then by Lemma 6

1. every trajectory $\varphi(t, u, y)$ ($(u, y) \in W \times Y$) is positively uniformly Lyapunov stable;
2. every semi-trajectory Σ^+_x of skew-product dynamical system $(X, \mathbb{S}_+, \pi)$ ($X := W \times Y$ and $\pi := (\varphi, \sigma)$) is conditionally precompact and $\omega_x \neq \emptyset$.

Now to finish the proof of Theorem it is sufficient to apply Theorem 12. □

Corollary 6. *Under the conditions of Theorem 14 if the point y_0 is τ-periodic (respectively, quasi-periodic, Bohr almost periodic, recurrent, strongly Poisson stable and $H(y_0)$ is a minimal set), then:*

1. *the point x_{y_0} is so;*
2. *the point x_0 is asymptotically τ-periodic (respectively, asymptotically quasi-periodic, asymptotically Bohr almost periodic, asymptotically recurrent, asymptotically strongly Poisson stable).*

Definition 40. A non-autonomous set $\{A_y |\ y \in Y\}$ is said to be

1. positively (respectively, negatively) invariant (with respect to cocycle φ) if $\varphi(t, A_y, y) \subseteq A_{\sigma(t,y)}$ (respectively, $\varphi(t, A_y, y) \supseteq A_{\sigma(t,y)}$) for any $y \in Y$ and $t \geq 0$;
2. invariant if it is positively and negatively invariant.

Lemma 7. *[19] Assume that the set Y is invariant, that is, $\sigma(t, Y) = Y$ for any $t \in \mathbb{T}$. The non-autonomous set $\{A_y |\ y \in Y\}$ is positively invariant (respectively, negatively invariant or invariant) if and only if the set $\mathbb{A}$ is a positively invariant (respectively, negatively invariant or invariant) subset of skew-product dynamical system $(X, \mathbb{T}, \pi)$.*

Lemma 8. *[19] The following statements are equivalent:*

1. *for any compact subset $K \subseteq Y$ the set $\bigcup\{A_y|\ y \in K\}$ is precompact in W;*
2. *the set $\mathbb{A} \subseteq X$ is conditionally precompact in (X, h, Y) ($X = W \times Y$ and $h := pr_2 : X \to Y$).*

Corollary 7. *Let $\{A_y|\ y \in Y\}$ be a uniformly precompact non-autonomous set, then the set $\mathbb{A}$ is a conditionally compact subset of X with respect to (X, h, Y), where $h = pr_2$.*

Lemma 9. *[19] Let $\{I_y|\ y \in Y\}$ be a non-autonomous set. Assume that the set $\mathbb{J} = \bigcup\{J_y = I_y \times \{y\}|\ y \in Y\}$ is conditionally precompact, then the following statements are equivalent:*

1. *the mapping $y \to I_y$ is upper semicontinuous;*
2. *the set $\mathbb{J}$ is closed in X.*

Definition 41. A trajectory $\varphi(t, u_0, y_0)$ ($x_0 := (u_0, y_0) \in W \times Y$) of cocycle φ is said to be precompact if $Q_{(u_0,y)} := \overline{\varphi(\mathbb{T}_+, u_0, y_0)}$ is a compact subset of W.

Lemma 10. *Suppose that $\langle W, \varphi, (Y, \mathbb{S}, \sigma)\rangle$ is a cocycle under $(Y, \mathbb{S}, \sigma)$ with the fibre W and $\mathbb{K} = \{K_y :\ y \in Y\}$ is a non-autonomous set with $K_y \subset W$ for any $y \in Y$. Assume that the following conditions hold:*

1. *Y is a compact space;*
2. *$\mathbb{K} = \{K_y :\ y \in Y\}$ is uniformly precompact;*
3. *$\mathbb{K} = \{K_y :\ y \in Y\}$ is upper semicontinuous.*

Then the following conditions are equivalent:

a) *the non-autonomous set $\mathbb{K} = \{K_y :\ y \in Y\}$ is positively uniformly Lyapunov stable;*

b) *every precompact trajectory $\varphi(t, u_0, y_0)$ ($x_0 := (u_0, y_0) \in \mathcal{K} := \bigcup\{K_y \times \{y\} :\ y \in Y\}$) of cocycle φ is positively uniformly Lyapunov stable.*

Proof. Taking into consideration that the implication $a) \Rightarrow b)$ is evident it is sufficient to show that b) implies a). If we suppose that it is not true then there are a positive number ε_0, sequences $\{y_k\} \subseteq Y$, $\{u_k^0\}$ ($u_k^0 \in K_{y_k}$), and $\{u_k\}$ ($u_k \in W$), $\{t_k\} \subset \mathbb{S}_+$ such that

$$\rho(u_k, u_k^0) < \delta_k \quad \text{and} \quad \rho(\varphi(t_k, u_k, y_k), \varphi(t_k, u_k^0, y_k)) \geq \varepsilon_0. \tag{6}$$

Since $\mathbb{K}$ is uniformly precompact and the space Y is compact, then without loss of generality we can suppose that the sequences $\{u_k^0\}$ $\{u_k\}$ and $\{y_k\}$ are convergent.

Denote by $u_0 := \lim\limits_{k\to\infty} u_k^0 = \lim\limits_{k\to\infty} u_k$ and $y_0 := \lim\limits_{k\to\infty} y_k$. Since $\{K_y :\ y \in Y\}$ is upper semicontinuous, then $u_0 \in K_{y_0}$. By condition b) for $y_0 \in Y$, $u_0 \in K_{y_0}$ and ε_0 there exists a positive number $\delta_0 := \delta(\varepsilon_0/3, u_0, y_0) > 0$ such that

$$\rho(\varphi(t_0, u, y_0), \varphi(t_0, u_0, y_0)) < \delta$$

implies

$$\rho(\varphi(t, u_0, y_0), \varphi(t, u_0, y_0)) < \varepsilon_0/3$$

for any $t \geq t_0 \geq 0$. Let $k_0 = k_0(\varepsilon_0/3)$ be a natural number such that $\rho(u_k, u_0) < \delta_0$ for any $k \geq k_0$ and, consequently,

$$\rho(\varphi(t, u_k, y_0), \varphi(t, u_0, y_0)) < \varepsilon_0/3 \tag{7}$$

for $t \geq 0$. Then from (7) we obtain

$$\begin{aligned} \rho(\varphi(t, u_k, y_0), \varphi(t, u_k^0, y_0)) \leq \rho(\varphi(t, u_k, y_0), \varphi(t, u_0, y_0)) + \\ \rho(\varphi(t, u_0, y_0), \varphi(t, u_k^0, y_0)) < \tfrac{\varepsilon_0}{3} + \tfrac{\varepsilon_0}{3} < \varepsilon_0 \end{aligned} \tag{8}$$

for any $t \geq 0$. Inequalities (6) and (8) are contradictory. The obtained contradiction proves our statement. □

Let E be a real Banach space and $P \subset E$ be a cone in E with $Int(P) \neq \emptyset$ and $W \subseteq E$.

Theorem 15. *Let $\langle W, \varphi, (Y, \mathbb{S}, \sigma)\rangle$ (shortly φ) be a cocycle over dynamical system $(Y, \mathbb{T}, \sigma)$ with the fiber W. Assume that the following conditions are fulfilled:*

1. *the cone P is normal;*
2. *the space Y is compact and $(Y, \mathbb{T}, \sigma)$ is minimal;*
3. *the cocycle φ is monotone and translation invariant with respect to $v \in Int(P)$;*
4. *the cocycle φ satisfies (**C1**);*
5. *the cocycle φ admits a uniformly precompact global attractor $\boldsymbol{I} = \{I_y :\ y \in Y\}$.*

Then the following statements hold:

1. *$\alpha(y) \leq u \leq \beta(y)$ for any $u \in I_y$ and $y \in Y$;*
2. *$\alpha(y), \beta(y) \in I_y$ and, consequently, $I_y \subseteq [\alpha(y), \beta(y)]$;*
3. *$\varphi(t, \alpha(y), y) = \alpha(\sigma(t, y))$ (respectively, $\varphi(t, \beta(y), y) = \beta(\sigma(t, y))$) for any $t \geq 0$;*
4. *the point $\gamma_*(y) := (\alpha(y), y) \in X = W \times Y$ (respectively, $\gamma^*(y) := (\beta(y), y) \in X$) is strongly comparable by character of recurrence with the point y;*

5. *if* $u \in W$ *and* $u \le \alpha(y)$ *(respectively,* $u \ge \beta(y)$*), then* $\omega_x \bigcap X_y = \{\gamma_*(y)\}$ *(respectively,* $\omega_x \bigcap X_y = \{\gamma^*(y)\}$*), where* $x := (u, y)$*;*

6. *if* $u \le \alpha(y)$ *(respectively,* $u \ge \beta(y)$*),then*

$$\lim_{t\to+\infty} \rho(\varphi(t,u,y).\gamma_*(\sigma(t,y))) = 0$$

(respectively,

$$\lim_{t\to+\infty} \rho(\varphi(t,u,y).\gamma^*(\sigma(t,y))) = 0).$$

Proof. Since the cocycle φ is monotone and translation invariant with respect to $v \in Int(P)$, then by Lemma 6

1. the cocycle φ is positively uniformly stable;

2. every trajectory $\varphi(t,u,y)$ $((u,y) \in W \times Y)$ is positively uniformly Lyapunov stable;

3. the uniformly compact global attractor $\mathbf{I} = \{I_y : \ y \in Y\}$ is positively uniformly Lyapunov stable.

Now to finish the proof of Theorem it is sufficient to apply Theorem 13. □

Corollary 8. *Under the conditions of Theorem 15 the following statements take place:*

1. *if the point* y *is* τ*-periodic (respectively, quasi-periodic, Bohr almost periodic, almost automorphic, recurrent), then the full trajectory* γ_y *passing through the point* $(\alpha(y), y)$ *(respectively, through the point* $(\beta(y), y)$*) is so.*

Proof. This statement follows from Theorem 15 and Corollary 5. □

8 Application

8.1 Time-dependent chemical reaction networks

In the works of Angeli and Sontag [1, 2] and Angeli, Leenheer and Sontag [3] the authors have contributed a new type of global convergence condition, named positive translation invariance, which is motivated by a chemical reaction network. A standard form for representing (well-mixed and isothermal) chemical reactions by ordinary differential equations is

$$S' = \Gamma R(S), \tag{9}$$

evolving on the nonnegative orthant $\mathbb{R}^m_+$, where S is an m-vector specifying the concentrations of m chemical species, $\Gamma : \mathbb{R}^n \to \mathbb{R}^m$ is the stoichiometry matrix, and

$R : \mathbb{R}^m_+ \to \mathbb{R}^n$ is a function which provides the vector of reaction rates for any given vector of concentrations. Choosing $\mu \in \mathbb{R}^m_+$ and using the reaction coordinates x,

$$S = \mu + \Gamma x,$$

instead of the traditional species coordinates S, the authors of [1]–[3] have investigated the monotonicity and global behavior of systems in the reaction coordinates,

$$x' = f_\mu(x) = R(\mu + \Gamma x), \tag{10}$$

evolving on the state space $X_\mu := \{x \in \mathbb{R}^n |\ \mu + \Gamma x \geq 0\}$.

Suppose that the matrix Γ has rank exactly $n-1$ and its kernel is spanned by a strongly positive vector v. Then the state space is invariant with respect to translation by v, namely,

$$x \in X_\mu \Rightarrow x + \lambda v \in X,\ \forall \lambda \in \mathbb{R},$$

and the solution $\varphi(t,\xi)$ generated by (10) enjoys positive translation invariance:

$$\varphi(t, \xi + \lambda v) = \varphi(t,\xi) + \lambda v,\ \forall\ x \in X_\mu \text{ and } \lambda \in \mathbb{R}.$$

Motivated by the study of Angeli and Sontag [1,2], Angeli, Leenheer and Sontag [3] and Hongxiao Hu and Jifa Jiang [22], we shall investigate the nonautonomous chemical reaction network. Suppose that the reaction rates depend on time

$$S' = \Gamma R(t, S), \tag{11}$$

where $R(t,S)$ is almost periodic (respectively, quasi-periodic, Bohr almost periodic, automorphic, Birkhoff recurrent, Levitan almost periodic, Bebutov almost recurrent, Poisson stable) in t. Choosing $\mu \in \mathbb{R}^m_+$ and using the reaction coordinates $x : S = \mu + \Gamma x$, we transform (11) into a system in the reaction coordinates:

$$x' = F_\mu(t,x) := R(t, \mu + \Gamma x) \tag{12}$$

evolving on the state space X_μ.

Remark 11. Note that X_μ is an ordered convex closed subset of $\mathbb{R}^n$.

Let U be a subset of $\mathbb{R}^m$. Denote by $C(\mathbb{T}\times U, \mathbb{R}^n)$ the space of all continuous functions $F : \mathbb{T}\times U \to \mathbb{R}^n$ equipped with the compact-open topology. This topology can be generated, for example, by the following distance d:

$$d(F,G) := \sum_{k=1}^{\infty} \frac{1}{2^k} \frac{d_k(F,G)}{1 + d_k(F,G)},$$

where $d_k(F,G) := \max\{(t,x) \in \mathbb{T}\times U |\ |t| \leq k,\ |x| \leq k\}$ $(k \in \mathbb{N})$. For $F \in C(\mathbb{T}\times U, \mathbb{R}^n)$ and $\tau \in \mathbb{T}$ we denote by F^τ the τ-translation of F with respect to time t, i.e., $F^\tau(t,x) := F(t+\tau, x)$ for any $(t,x) \in \mathbb{T}\times U$ and by $(C(\mathbb{T}\times U, \mathbb{R}^n), \mathbb{T}, \lambda)$ the shift (Bebutov's) dynamical system on the space $C(\mathbb{T}\times U, \mathbb{R}^n)$. Let $f \in C(\mathbb{T}\times U, \mathbb{R}^n)$

and denote by $H(f)$ its hull, i.e., $H(f) := \overline{\{f^\tau | \ \tau \in \mathbb{T}\}}$, where by bar the closure in the space $C(\mathbb{T} \times U, \mathbb{R}^n)$ is denoted.

Condition (**A1**). A function $F_\mu \in C(\mathbb{R} \times X_\mu, \mathbb{R}^n)$ is regular, that is, for any $u \in X_\mu$ and $G \in H(F_\mu)$ there exists a unique solution $\varphi(t, u, G)$ of equation

$$u' = G(t, u) \tag{13}$$

defined on $\mathbb{R}_+$.

Assume that the function $F_\mu \in C(\mathbb{R} \times X_\mu, \mathbb{R}^n)$ is regular. Let $\phi(t, v, G)$ denote the solution of (13) passing through v at $t = 0$. Then it enjoys positive translation invariance:

$$\phi(t, \xi + \lambda v, G) = \varphi(t, v, G) + \lambda v, \ \forall \ \xi \in X_\mu, \ \lambda \in \mathbb{R} \text{ and } G \in H(F_\mu), \tag{14}$$

where $H(F_\mu)$ is the hull of F_μ. So the skew-product flow induced by H-class

$$v' = G(t, v) \ \ (G \in H(F_\mu)) \tag{15}$$

of system (12) has positive translation invariance.

Lemma 11. *[22] Suppose that the function F_μ is regular. If the positive orthant $\mathbb{R}^m_+$ is positively invariant for (11), then X_μ is invariant under(12).*

Recall that the set $\mathbb{R}^m_+$ (respectively, the set X_μ) is positively invariant with respect to (11) (respectively, with respect to (12)) if for any $(S_0, \tilde{R}) \in \mathbb{R}^m_+ \times H(R)$ (respectively, $(u_0, G) \in X_\mu \times H(F_\mu)$) $\phi(t, S_0, \tilde{R}) \in \mathbb{R}^m_+$
(respectively, $\phi(t, u_0, G) \in X_\mu$) for any $t \in \mathbb{R}_+$.

The purpose of this paper is to study the periodic (respectively, quasi-periodic, Bohr almost periodic, automorphic, recurrent in the sense of Birkhoff, Levitan almost periodic, Bebutov almost recurrent, Poisson stable) solutions and compact global attractors of monotone equation (12) (respectively, (15)) possessing a positively translation-invariant property (14).

With the above notation, a chemical reaction network is described by the following differential equations:

$$S' = \Gamma R(t, S), \ t > 0, \ S(0) = S_0 \in \mathbb{R}^m_+, \tag{16}$$

where $\mathbb{R}^m_+ := \{S \in \mathbb{R}^m | \ S_i \geq 0\}$. Of course, S_0 is the initial concentration of all species and R(t,S) is a time-dependent vector-valued function. Given a chemical reaction network (16), following [28] we introduce the so-called associate system in reaction coordinates. For any $\mu \in \mathbb{R}^m_+$, such a system in reaction coordinates is defined as the following nonautonomous system:

$$u' = F_\mu(t, u), \ t > 0, \ u(0) = u_0 \in X_\mu, \tag{17}$$

where $F_\mu(t, u) := f(t, \mu + \Gamma u)$. Here $u = (u_1, \dots, u_n)$ is called the extent of the reaction [29]. For systems (16) and (17), let $H(F_\mu)$ and $H(f)$ be the hull of F_μ and f, respectively.

Denote by $X_{(\mu,\Gamma)} := \{\mu + \Gamma(u) | \ u \in X_\mu\}$.

Lemma 12. *If $rank(\Gamma) = n - 1$ and its kernel is spanned by a strongly positive vector v, then $X_{(\mu,\Gamma)}$ is a closed subset of $\mathbb{R}^m_+$.*

Proof. To prove this statement it is sufficient to show that $\overline{X_{(\mu,\Gamma)}} \subseteq X_{(\mu,\Gamma)}$, where by bar the closure of the set $X_{(\mu,\Gamma)}$ in the space $\mathbb{R}^m$ is denoted.

Let $U \in \overline{X_{(\mu,\Gamma)}}$, then there exists a sequence $\{U_k\} \subset X_{(\mu,\Gamma)}$ such that $U_k \to U$ as $k \to \infty$. Since $U_k \in X_{(\mu,\Gamma)}$, then there exists an element $u_k \in X_\mu$ such that $U_k = \mu + \Gamma u_k$. On the other hand we have $\mathbb{R}^n = \mathfrak{B}_1 \bigoplus \mathfrak{B}_2$, where $\mathfrak{B}_2 = Span\{v\}$. Since $rank(\Gamma) = n - 1$ and $\mathfrak{B}_2 = Span\{v\}$, then the subspaces $rank(\Gamma)$ and $\mathfrak{B}_1$ are isomorphic. Thus there exists a unique element $u^i_k \in \mathfrak{B}_i$ $(i = 1, 2)$ such that $u_k = u^1_k + u^2_k$. Note that $\Gamma(u_k) = \Gamma(u^1_k) + \Gamma(u^2_k) = \Gamma(u^1_k)$ and, consequently, $\mu+\Gamma(u^1_k) = \mu+\Gamma(u_k) \geq 0$, i.e., $u^1_k \in X_\mu$. Since $U_k-\mu = \Gamma(u^1_k) \to U-\mu$ as $k \to \infty$, then the sequence $\{u^1_k\}$ is convergent in $\mathbb{R}^n$. Denote by $u^1 = \lim\limits_{k\to\infty} u^1_k$, then $u^1 \in X_\mu$ and $U = \mu + \Gamma(u^1) \in X_{(\mu,\Gamma)}$. Lemma is proved. □

Let W be a subset of $\mathbb{R}^m_+$.

Definition 42. A function $f \in C(\mathbb{T} \times W, \mathbb{R}^n)$ is said to be Lagrange stable if the set $\Sigma_f := \bigcup\{f^\tau |\ \tau \in \mathbb{T}\}$ is precompact.

Let Φ be the mapping from $C(\mathbb{T} \times X_{(\mu,\Gamma)}, \mathbb{R}^n)$ into $C(\mathbb{T} \times X_\mu, \mathbb{R}^n)$ defined by equality

$$\Phi(f) := F_\mu,$$

where $F_\mu(t, u) := f(t, \mu + \Gamma u)$ for any $(t, u) \in \mathbb{T} \times X_\mu$.

Lemma 13. *The following statements hold:*

1. *the mapping Φ is continuous;*
2. *$\Phi(f_1) \neq \Phi(f_2)$ for any $f_1, f_2 \in C(\mathbb{T} \times X_{(\mu,\Gamma)}, \mathbb{R}^n)$ with $f_1 \neq f_2$;*
3. *$\Phi(\lambda(\tau, f)) = \lambda(\tau, \Phi(f))$ for any $(\tau, f) \in \mathbb{T} \times C(\mathbb{T} \times X_{(\mu,\Gamma)}, \mathbb{R}^n)$, i.e., Φ is a homomorphism of dynamical system $(C(\mathbb{T} \times X_{(\mu,\Gamma)}, \mathbb{R}^n), \mathbb{T}, \lambda)$ into $(C(\mathbb{T} \times X_\mu, \mathbb{R}^n), \mathbb{T}, \lambda)$;*
4. *$\Phi(H(f)) \subseteq H(\Phi(f))$ for any $f \in C(\mathbb{T} \times X_{(\mu,\Gamma)}, \mathbb{R}^n)$;*
5. *if the function f is Lagrange stable, then $\Phi(H(f)) = H(\Phi(f))$.*

Proof. The first three statements of Lemma are evident.

Let now $g \in H(f)$, then there is a sequence $\{\tau_n\} \subset \mathbb{T}$ such that $f^{\tau_n} \to g$ in $C(\mathbb{T} \times X_{(\mu,\Gamma)}, \mathbb{R}^n)$ as $n \to \infty$. Then taking into consideration the second and third statements of Lemma 13 we obtain

$$\Phi(g) = \Phi(\lim_{n\to\infty} f^{\tau_n}) = \lim_{n\to\infty} \Phi(f^{\tau_n}) = \lim_{n\to\infty} \Phi^{\tau_n}(f) := G \in H(\Phi(f)),$$

i.e., $\Phi(H(f)) \subseteq H(\Phi(f))$.

If the function f is Lagrange stable, then we will establish that the converse inclusion $H(\Phi(f)) \subseteq \Phi(H(f))$ is also true. In fact if $G \in H(\Phi(f))$, then there exists a sequence $\{\tau_n\} \subseteq \mathbb{T}$ such that $\Phi^{\tau_n}(f) \to G$ as $n \to \infty$. Note that $\Phi^{\tau_k}(t,u) = f(t+\tau_k, \mu + \Gamma u)$ for any $(t,u) \in \mathbb{T} \times X_\mu$. Since f is Lagrange stable, then there exists a subsequence $\{\tau_{k_m}\} \subset \{\tau_k\}$ such that the sequence $\{f^{\tau_{k_m}}\}$ converges in the space $C(\mathbb{T} \times X_{(\mu,\Gamma)}, \mathbb{R}^n)$. Denote by $h(t,x) = \lim_{m\to\infty} f(t+\tau_{k_m}, x)$ for any $(t,x) \in \mathbb{T} \times X_{(\mu,\Gamma)}$, then $h \in H(f)$ and $\Phi(h) = \lim_{n\to\infty} \Phi^{\tau_{k_m}}(f)$ and, consequently, $\Phi(h) \in H(\Phi(f))$. Notice that

$$G := \lim_{m\to\infty} \Phi^{\tau_{k_m}}(f) = \lim_{m\to\infty} \Phi(f^{\tau_{k_m}}) = \Phi(\lim_{m\to\infty} f^{\tau_{k_m}}) = \Phi(h) \in H(\Phi(f)),$$

i.e., $H(\Phi(f)) \subseteq \Phi(H(f))$. Lemma is completely proved. □

Corollary 9. *The following statements hold:*

1. $\mathfrak{N}_f \subseteq \mathfrak{N}_{\Phi(f)}$ *and, consequently if the function f is stationary (respectively, τ-periodic, Levitan almost periodic, almost recurrent, Poisson stable) in $t \in \mathbb{T}$ uniformly with respect to x on every compact subset from $X_{(\mu,\Gamma)}$ then the function $\Phi(f)$ is so;*

2. $\mathfrak{M}_f \subseteq \mathfrak{M}_{\Phi(f)}$ *and, consequently if the function f is stationary (respectively, τ-periodic, quasi-periodic, Bohr almost periodic, almost automorphic, recurrent, Poisson stable) in $t \in \mathbb{T}$ uniformly with respect to x on every compact subset from $X_{(\mu,\Gamma)}$ then the function $\Phi(f)$ is so;*

3. *If the function f is Lagrange stable, then Φ is a homomorphism of dynamical systems $(H(f), \mathbb{T}, \lambda)$ onto $(H(\Phi(f)), \mathbb{T}, \lambda)$ and, consequently, $\mathfrak{M}_f = \mathfrak{M}_{\Phi(f)}$.*

Proof. This statement follows from Lemma 13. □

Remark 12. According to Corollary 9 (item ii) the function $\Phi(f)$ is strongly comparable by character of recurrence with the function f. Moreover, there is a stronger statement. Namely, the function $\Phi(f)$ is uniformly comparable with the function f, i.e., for any $\varepsilon > 0$ there exists a positive number $\delta = \delta(\varepsilon)$ such that $d(\lambda(\tau_1, f), \lambda(\tau_2, f)) < \delta$ implies $d(\lambda(\tau_1, \Phi(f)), \lambda(\tau_2, \Phi(f))) < \varepsilon$.

As a consequence from Lemma 13 (item (v)) if the function f is Lagrange stable, then for any $G \in H(F_\sigma)$ there exists a unique $h \in H(f)$ $(G = \Phi(h))$ such that

$$G(t,u) = h(t, \mu + \Gamma u).$$

In particular, $G = F_\sigma$ if and only if $h = f$. For every $G \in H(F_\mu)$ and $h \in H(f)$ in (18), let $\varphi(t, x_0, h)$ and $\phi(t, u_0, G)$ be the solutions of

$$x' = \Gamma h(t,x),\ t > 0,\ x(0) = x_0 \in \mathbb{R}^m_+ \tag{18}$$

and

$$u' = G(t,u),\ t > 0,\ u(0) = u_0 \in X_\mu, \tag{19}$$

respectively.

Lemma 14. *[28] Let the function $f \in C(\mathbb{T} \times X_{(\mu,\Gamma)}, \mathbb{R}^n)$ be Lagrange stable and $\varphi(t, x_0, h), \phi(t, u_0, G)$ be the solutions of (18) and (19), respectively. Then we have*

$$\varphi(t, \mu + \Gamma u_0, h) = \mu + \Gamma\phi(t, u_0, \Phi(h))$$

for any $t \in \mathbb{T}$, $u_0 \in X_\mu$ and $h \in H(f)$.

Condition (**A2**). Equation (17) is monotone (respectively, strongly monotone). This means that the cocycle $\langle \mathbb{R}^n, \varphi, (H(F_\mu), \mathbb{R}, \sigma)\rangle$ generated by (17) is monotone (respectively, strongly monotone), i.e. if $u, v \in \mathbb{R}^d$ and $u \leq v$ (respectively, $u < v$) then $\varphi(t, u, G) \leq \varphi(t, v, G)$ (respectively, $\varphi(t, u, G) \ll \varphi(t, v, G)$) for all $t \geq 0$ and $G \in H(F_\mu)$.

Definition 43. Let $f \in C(\mathbb{R} \times W, \mathbb{R}^n)$. The set $H(f)$ is said to be minimal if it is a minimal set of shift dynamical system $(C(\mathbb{R} \times W, \mathbb{R}^n), \mathbb{R}, \sigma)$.

Definition 44. A function $f \in C(\mathbb{R} \times W, \mathbb{R}^n)$ is said to be strongly Poisson stable if every function $g \in H(f)$ is Poisson stable.

Remark 13. If the function $f \in C(\mathbb{R} \times W, \mathbb{R}^n)$ is time almost periodic, then

1. the set $H(f)$ is minimal;
2. every function $h \in H(f)$ is almost recurrent and, consequently, f is strongly Poisson stable.

Theorem 16. *Suppose that the following conditions hold:*

1. *$\mu \in \mathbb{R}^m$ is such that the system (17) is strongly monotone;*
2. *the set $H(F_\mu)$ is minimal and F_μ is strongly Poisson stable;*
3. *the matrix Γ has rank exactly $n - 1$ whose kernel is spanned by a strongly positive vector v;*
4. *for any $G \in H(F_\mu)$ all forward solutions of equation (19) are bounded.*

Then for any $U_0 \in X_{(\mu,\Gamma)}$ the following statements hold:

1. *the set $\omega_{(U_0,f)} \bigcap X_f$ consists of a single point $p_0 = (V_0, f)$, where $\omega_{(U_0,f)}$ is the ω-limit set of the motion $\pi(t, (U_0, f))$ of the skew-product dynamical system $(X, \mathbb{R}_+, \pi)$ ($X := X_{(\mu,\Gamma)} \times H(f)$, $\pi := (\varphi, \sigma)$) and $X_f := X_{(\mu,\Gamma)} \times \{f\}$;*
2. *the solution $\varphi(t, V_0, f)$ of equation (18) is defined on $\mathbb{R}$, $\overline{\varphi(\mathbb{R}, v_0, f)} \subseteq Q_+^{(U_0,f)}$ and it is strongly compatible;*
3. $$\lim_{t\to\infty} |\varphi(t, U_0, f) - \varphi(t, V_0, f)| = 0.$$

Proof. Let $(H(F_\mu),\mathbb{R},\sigma)$ (respectively, $(H(f),\mathbb{R},\sigma)$) be the shift dynamical system on $H(F_\mu)$ (respectively, on $H(f)$). Denote by $\langle X_\mu,\phi,(H(F_\mu),\mathbb{R},\sigma)\rangle$ (respectively, $\langle X_{(\mu,\Gamma)},\varphi,(H(f),\mathbb{R},\sigma)\rangle$, where $X_{(\mu,\Gamma)} = \mu+\Gamma(X_\mu) := \{\mu+\Gamma u|\ u\in X_\mu\}$) the cocycle generated by family of equations (19) (respectively, (18)). Note that under the conditions of Theorem 16 Conditions (**C1**), (**C3**) and (**C4**) are fulfilled for the cocycle $\langle X_\mu,\phi,(H(F_\mu),\mathbb{R},\sigma)\rangle$. Let $U_0\in X_{(\mu,\Gamma)}$, then there exists a point $u_0\in X_\mu$ such that $U_0=\mu+\Gamma(u_0)$. By equality (14) the cocycle ϕ is translation invariant with respect to vector $v\gg 0$. According to Theorem 14 for given $u_0\in X_\mu$ the following statements are fulfilled:

a. the set $\omega_{(u_0,F_\mu)}\bigcap X_{F_\mu}$ consists of a single point $q_0=(v_0,F_\mu)$, where $\omega_{(u_0,F_\mu)}$ is the ω-limit set of the motion $\pi(t,(u_0,F_\mu))$ of the skew-product dynamical system $(X,\mathbb{R}_+,\pi)$ ($X:=X_\mu\times H(F_\mu)$, $\pi:=(\phi,\sigma)$) and $X_{F_\mu}:=X_{(\mu,\Gamma)}\times\{F_\mu\}$;

b. the solution $\phi(t,V_0,F_\mu)$ of equation (17) is defined on $\mathbb{R}$, $\overline{\phi(\mathbb{R},v_0,F_\mu)}\subseteq Q_+^{(u_0,F_\mu)}$ and it is strongly compatible;

c.
$$\lim_{t\to\infty}|\phi(t,u_0,F_\mu)-\phi(t,v_0,F_\mu)|=0.$$

Denote by $V_0:=\mu+\Gamma(v_0)\in X_{(\mu,\Gamma)}$ and consider the solutions $\varphi(t,U_0,f)$ and $\varphi(t,V_0,f)$ of equation (18) ($h=f$). Since $\phi(\cdot,v_0,F_\mu)$ is a strongly compatible solution of equation (19) ($G=F_\mu$), then

$$\mathfrak{M}_{F_\mu}\subseteq\mathfrak{M}_{\phi(\cdot,v_0,F_\mu)}. \tag{20}$$

By Lemma 14 we have $\varphi(t,V_0,f)=\mu+\phi(t,v_0,F_\mu)$ for any $t\in\mathbb{R}$. Note that

$$\mathfrak{M}_{\phi(\cdot,v_0,F_\mu)}\subseteq\mathfrak{M}_{\varphi(\cdot,V_0,f)}. \tag{21}$$

Indeed if $\{t_k\}\in\mathfrak{M}_{\phi(\cdot,v_0,F_\mu)}$ then we have

$$\varphi(t+t_k,V_0,f)-\bar{\varphi}(t)=\mu+\Gamma\phi(t+t_k,v_0,F_\mu)-(\mu+\Gamma\bar{\phi}(t))=$$
$$\Gamma(\phi(t+t_k,v_0,F_\mu)-\bar{\phi}(t))\to 0$$

as $k\to\infty$ uniformly with respect to t on every compact subset from $\mathbb{R}$, where $\bar{\phi}=\lim\phi(\cdot+t_k,v_0,F_\mu)$ in the space $C(\mathbb{R},\mathbb{R}^n)$. This means that $\{t_k\}\in\mathfrak{M}_{\varphi(\cdot,V_0,f)}$.

From (20) and (21) we have

$$\mathfrak{M}_{F_\mu}\subseteq\mathfrak{M}_{\varphi(\cdot,V_0,f)}. \tag{22}$$

Finally, from Corollary 9 (item (ii)) we have

$$\mathfrak{M}_f\subseteq\mathfrak{M}_{\Phi(f)}. \tag{23}$$

In virtue of (22)–(23) and taking into consideration the equality $\Phi(f)=F_\mu$ we obtain

$$\mathfrak{M}_f\subseteq\mathfrak{M}_{\varphi(\cdot,V_0,f)},$$

i.e., $\varphi(t, V_0, f)$ is a strongly compatible solution of equation (18) (for $h = f$).

To finish the proof of Theorem it is sufficient to note that

$$|\varphi(t, U_0, f) - \varphi(t, V_0, f)| = |(\mu + \Gamma\phi(t, u_0, F_\mu)) - (\mu + \Gamma\phi(t, v_0, F_\mu))| = |\Gamma(\phi(t, u_0, F_\mu) - \phi(t, v_0, F_\mu))| \le \|\Gamma\| |\phi(t, u_0, F_\mu) - \phi(t, v_0, F_\mu)| \to 0$$

as $t \to \infty$. □

Corollary 10. *Under the conditions of Theorem 16 if the function $f \in C(\mathbb{R} \times X_{(\mu,\Gamma)}, \mathbb{R}^n)$ is stationary (respectively, τ-periodic, quasi-periodic, Bohr almost periodic, almost automorphic, recurrent in the sense of Birkhoff, strongly Poisson stable) in time $t \in \mathbb{R}$, then for any $U_0 \in X_{(\mu,\Gamma)}$ the following statements hold:*

1. *the set $\omega_{(U_0,f)} \bigcap X_f$ consists of a single point $p_0 = (V_0, f)$;*
2. *$\varphi(t, V_0, f)$ is a stationary (respectively, τ-periodic, quasi-periodic, Bohr almost periodic, almost automorphic, recurrent in the sense of Birkhoff, strongly Poisson stable) solution of equation (17);*
3. *$\lim\limits_{t\to+\infty} |\varphi(t, U_0, f) - \varphi(t, V_0, f)| = 0$, i.e., $\varphi(t, u_0, f)$ is asymptotically stationary (respectively, asymptotically τ-periodic, asymptotically quasi-periodic, asymptotically Bohr almost periodic, asymptotically almost automorphic, asymptotically recurrent in the sense of Birkhoff, asymptotically strongly Poisson stable).*

Proof. This statement follows from Theorem 16 and Corollary 6. □

Let Y be a compact metric space, $\langle \mathbb{R}^n, \varphi, (Y, \mathbb{S}, \sigma)\rangle$ be a cocycle on the state space $\mathbb{R}^n$ and $(X, \mathbb{S}_+, \pi)$ be the corresponding skew-product dynamical system, where $X := \mathbb{R}^n \times Y$ and $\pi := (\varphi, \sigma)$.

Definition 45. The cocycle $\langle \mathbb{R}^n \varphi, (Y, \mathbb{S}, \sigma)\rangle$ is said to be dissipative if for any $y \in Y$ there is a positive number r_y such that

$$\limsup_{t\to+\infty} |\varphi(t, u, y)| < r_y$$

for any $y \in Y$ and $u \in \mathbb{R}^n$, i.e., for all $u \in \mathbb{R}^n$ and $y \in Y$ there exists a positive number $L(u, y)$ such that $|\varphi(t, u, y)| < r_y$ for any $t \ge L(u, y)$.

Theorem 17. *[12, ChIII] Let $\langle \mathbb{R}^n, \varphi, (Y, \mathbb{S}, \sigma)\rangle$ be a cocycle over the dynamical system $(Y, \mathbb{S}, \sigma)$ with the fiber $\mathbb{R}^n$. Then the following statements are equivalent:*

1. *There exists a positive number R such that*

$$\limsup_{t\to+\infty} |\varphi(t, u, y)| < R$$

for all $u \in \mathbb{R}^n$ and $y \in Y$.

2. There is a positive number r_1 such that for all $u \in \mathbb{R}^n$ and $y \in Y$ there exists $\tau = \tau(u, y) > 0$ for which $|\varphi(\tau, u, y)| < r_1$.

3. There is a positive number r_2 such that

$$\liminf_{t \to +\infty} |\varphi(t, u, y)| < r_2$$

for all $u \in \mathbb{R}^n$ and $y \in Y$.

4. There exists a positive number R_0 and for all $R > 0$ there is $l(R) > 0$ such that $|\varphi(t, u, y)| \le R_0$ for all $t \ge l(R)$, $u \in \mathbb{R}^n$, $|u| \le R$ and $y \in Y$.

Remark 14. 1. Note that every condition 1.-4. that figures in Theorem 17 is equivalent to the (compact) dissipativity of the non-autonomous dynamical system $\langle (X, \mathbb{S}_+, \pi), (Y, \mathbb{S}, \sigma), h \rangle$ associated by the cocycle $\langle \mathbb{R}^n, \varphi, (Y, \mathbb{S}, \sigma) \rangle$ over $(Y, \mathbb{S}, \sigma)$ with the fiber $\mathbb{R}^n$.

2. Note that Theorem 17 remains true if we replace the space $\mathbb{R}^n$ by a closed subset W of $\mathbb{R}^n$.

Consider the differential equation

$$u' = f(t, u), \tag{24}$$

where $f \in C(\mathbb{R} \times \mathbb{R}^n, \mathbb{R}^n)$. Along with the equation (24) we consider its H-class [5, 21, 27, 34, 37], i.e., the family of the equations

$$v' = g(t, v), \tag{25}$$

where $g \in H(f) = \overline{\{f_\tau : \tau \in \mathbb{R}\}}$ and $f_\tau(t, u) = f(t + \tau, u)$, with the bar indicating closure in the compact-open topology.

We will suppose that the function f is regular. Denote by $\varphi(\cdot, v, g)$ the solution of (25) passing through the point $v \in \mathbb{R}^n$ for $t = 0$. Then the mapping $\varphi : \mathbb{R}_+ \times \mathbb{R}^n \times H(f) \to \mathbb{R}^n$ satisfies the following conditions (see, for example,[5, 31]):

1) $\varphi(0, v, g) = v$ for all $v \in \mathbb{R}^n$ and $g \in H(f)$;

2) $\varphi(t, \varphi(\tau, v, g), g_\tau) = \varphi(t + \tau, v, g)$ for each $v \in \mathbb{R}^n$, $g \in H(f)$ and $t, \tau \in \mathbb{R}_+$;

3) $\varphi : \mathbb{R}_+ \times \mathbb{R}^n \times H(f) \to \mathbb{R}^n$ is continuous.

Denote by $Y := H(f)$ and $(Y, \mathbb{R}, \sigma)$ a dynamical system of translations on Y, induced by the dynamical system of translations $(C(\mathbb{R} \times \mathbb{R}^n, \mathbb{R}^n), \mathbb{R}, \sigma)$. The triple $\langle \mathbb{R}^n, \varphi, (Y, \mathbb{R}, \sigma) \rangle$ is a cocycle over $(Y, \mathbb{R}_+, \sigma)$ with the fiber $\mathbb{R}^n$. Hence, the equation (24) generates a cocycle $\langle \mathbb{R}^n, \varphi, (Y, \mathbb{R}, \sigma) \rangle$ and the non-autonomous dynamical system $\langle (X, \mathbb{R}_+, \pi), (Y, \mathbb{R}, \sigma), h \rangle$, where $X := \mathbb{R}^n \times Y$, $\pi := (\varphi, \sigma)$ and $h := pr_2 : X \to Y$.

Definition 46. Recall that the equation (24) is called dissipative [21, 30, 39, 40] if for all $t_0 \in \mathbb{R}$ and $u_0 \in \mathbb{R}^n$ there exists a unique solution $x(t; u_0, t_0)$ of the equation (24) passing through the point (u_0, t_0) and if there exists a number $R > 0$ such that

$\lim_{t\to+\infty}\sup|x(t;u_0,t_0)| < R$ for all $u_0 \in \mathbb{R}^n$ and $t_0 \in \mathbb{R}$. In other words, for every solution $x(t;u_0,t_0)$ there is an instant $t_1 = t_0 + l(t_0,u_0)$ such that $|x(t;u_0,t_0)| < R$ for any $t \ge t_1$. If for any $r > 0$ the number $l(t_0,u_0)$ can be chosen independently on t_0 and u_0 with $|u_0| \le r$, then the equation (24) is called uniformly dissipative [21].

Lemma 15. *[12, ChIII] Let $W \subseteq \mathbb{R}^m$ and $f \in C(\mathbb{R}\times\mathbb{R}^n,\mathbb{R}^n)$ be regular. If $H(f)$ is compact, then equation (24) is uniformly dissipative if and only if there is a positive number r such that*

$$\limsup_{t\to+\infty}|\varphi(t,u_0,g)| < r \quad (u_0 \in W,\ g \in H(f)) \ .$$

Remark 15. If $f \in C(\mathbb{R}\times W,\mathbb{R}^n)$ is regular, $H(f)$ is compact and then equation (24) is uniformly dissipative, then the cocycle φ generated by equation (24) admits a compact global attractor.

Theorem 18. *Suppose that the following assumptions are fulfilled:*

- *$\mu \in \mathbb{R}^m$ is such that the system (17) is monotone;*
- *the matrix Γ has rank exactly $n-1$ whose kernel is spanned by a strongly positive vector v;*
- *the function $F_\mu \in C(\mathbb{R}\times X_\mu,\mathbb{R}^n)$ is recurrent in $t\in\mathbb{R}$ uniformly with respect to u on every compact subset from X_μ;*
- *the cocycle ϕ generated by equation (17) admits a compact global attractor and $\boldsymbol{I} := \{I_G|\ G \in H(F_\mu)\}$ is its Levinson center.*

*Then under conditions (**A1**) – (**A2**) the following statements hold:*

1. *$\alpha(G),\beta(G) \in I_G$ for any $G \in H(F_\mu)$ and, consequently, $I_G \subseteq [\alpha(G),\beta(G)]$;*
2. *$\phi(t,\alpha(G),G) = \alpha(\sigma(t,G))$ (respectively, $\phi(t,\beta(G),G) = \beta(\sigma(t,G))$) for any $t\ge 0$ and $G \in H(F_\mu)$;*
3. *the point $\gamma_*(F_\mu) := (\alpha(F_\mu),F_\mu) \in X = X_\mu \times Y$ (respectively, $\gamma^*(F_\mu) := (\beta(F_\mu),F_\mu) \in X$) is strongly comparable by character of recurrence with the point F_μ;*
4. *for any $h \in H(f)$ equation (19) has at least two solutions $\varphi(t,U_0(h),h)$ ($U_0(h) = \mu + \Gamma\alpha(\Phi(h))$) and $\varphi(t,V_0(h),h)$ ($V_0(h) = \mu+\Gamma\beta(\Phi(h))$) defined and bounded on $\mathbb{R}$ which are strongly compatible and belong to Levinson center of (17);*
5. *if the function $F_\mu \in C(\mathbb{R}\times X_{(\mu,\Gamma)},\mathbb{R}^n)$ is quasi-periodic (respectively, Bohr almost periodic, almost automorphic, recurrent, pseudo-recurrent and Lagrange stable, uniformly Poisson stable and stable in the sense of Lagrange) in $t\in\mathbb{R}$ uniformly with respect to u on every compact subset from $X_{(\mu,\Gamma)}$,*

then $\varphi(t, U_0(f), f)$ and $\varphi(t, V_0(f), f)$ are quasi-periodic (respectively, Bohr almost periodic, almost automorphic, recurrent, pseudo-recurrent and Lagrange stable, uniformly Poisson stable and stable in the sense of Lagrange).

Proof. Let $(H(F_\mu), \mathbb{R}, \sigma)$ (respectively, $(H(f), \mathbb{R}, \sigma)$) be the shift dynamical system on $H(F_\mu)$ (respectively, on $H(f)$). Denote by $\langle X_\mu, \phi, (H(F_\mu), \mathbb{R}, \sigma)\rangle$ (respectively, $\langle X_{(\mu,\Gamma)}, \varphi, (H(f), \mathbb{R}, \sigma)\rangle$, where $X_{(\mu,\Gamma)} = \mu+\Gamma(X_\mu) := \{\mu+\Gamma u|\ u \in X_\mu\}$) the cocycle generated by family of equations (19) (respectively, (18)). By equality (14) the cocycle ϕ is translation invariant with respect to vector $v \gg 0$. Applying Theorem 15 to nonautonomous dynamical system $\langle X_\mu, \phi, (H(F_\mu), \mathbb{R}, \sigma)\rangle$ we obtain the following statements:

1. $\alpha(G), \beta(G) \in I_G$ for any $G \in H(F_\mu)$ and, consequently, $I_G \subseteq [\alpha(G), \beta(G)]$, where $\alpha(G) := \inf I_G$ (respectively, $\beta(G) := \sup I_G$);

2. $\phi(t, \alpha(G), G) = \alpha(\sigma(t, G))$ (respectively, $\phi(t, \beta(G), G) = \beta(\sigma(t, G))$) for any $t \geq 0$ and $G \in H(F_\sigma)$;

3. the point $\gamma_*(F_\mu) := (\alpha(F_\mu), F_\mu) \in X = X_\mu \times Y$ (respectively, $\gamma^*(F_\mu) := (\beta(F_\mu), F_\mu) \in X$) is strongly comparable by character of recurrence with the point F_μ.

Note that the nonautonomous dynamical system $\langle X_{(\mu,\Gamma)}, \varphi, (H(f), \mathbb{R}, \sigma)\rangle$ is compactly dissipative because $\varphi(t, U, h) = \mu + \Gamma\phi(t, u, \Phi(h))$ for any $h \in H(f)$ $(U = \mu + \Gamma u)$ and the cocycle ϕ is so. Let $\mathbf{A} = \{A_h|\ h \in H(f)\}$ be the Levinson center for the compact dissipative cocycle φ generated by equation (18). Denote by $U(h) := \mu + \Gamma\alpha(\Phi(h))$ and $V(h) := \mu + \Gamma\beta(\Phi(h))$. Then by Lemma 14 for any $h \in H(f)$

$$\varphi(t, U(h), h) = \mu + \Gamma\phi(t, \alpha(\Phi(h)), \Phi(h)) \tag{26}$$

(respectively,

$$\varphi(t, V(h), h) = \mu + \Gamma\phi(t, \beta(\Phi(h)), \Phi(h))) \tag{27}$$

is a bounded on $\mathbb{R}$ solution of equation (19). By Theorem 8 we have $U(h), V(h) \in A_h$, i.e., $U(h)$ and $V(h)$ belong to the Levinson center of the cocycle φ. Finally, from (26) (respectively, (27)) it follows that $\varphi(t, U(f), f)$ (respectively, $\varphi(t, V(f), f)$) is a strongly compatible solution of equation (18) for $h = f$, because $\phi(t, \alpha(\Phi(h), \Phi(h))$ (respectively, $\phi(t, \beta(\Phi(h), \Phi(h)))$ is a strongly compatible solution of equation (19), $\Phi : H(f) \to H(F_\mu)$ is a homeomorphism and $\Phi(f) = F_\mu$. Theorem is proved. □

8.2 Translation-Invariant Discrete Monotone Systems

Consider the discrete version of chemical reactions by ordinary differential equations (9), i.e.,

$$\Delta S(k) = \Gamma R(S(k)),\ (\Delta S(k) := S(k+1) - S(k)\ \forall\ t \in \mathbb{T})$$

evolving on the nonnegative orthant $\mathbb{R}^m_+$. Choosing $\mu \in \mathbb{R}^m_+$ and using the reaction coordinates x,

$$S = \mu + \Gamma u,$$

instead of the traditional species coordinates S, we will have investigated the monotonicity and global behavior of systems in the reaction coordinates,

$$\Delta u(k) = f_\mu(u(k)) = R(\mu + \Gamma u(k)), \tag{28}$$

evolving on the state space $X_\mu := \{u \in \mathbb{R}^n |\ \mu + \Gamma u \geq 0\}$.

Suppose that the matrix Γ has rank exactly $n-1$ and its kernel is spanned by a strongly positive vector v. Then the state space is invariant with respect to translation by v, namely,

$$u \in X \Rightarrow u + \lambda v \in X,\ \forall\ \lambda \in \mathbb{R},$$

and the solution $\varphi(t, \xi)$ generated by (28) enjoys positive translation invariance:

$$\varphi(k, \xi + \lambda v) = \varphi(k, \xi) + \lambda v,\ \forall\ \xi \in X_\mu \text{ and } \lambda \in \mathbb{R}.$$

Suppose that the reaction rates depend on time

$$\Delta S(k) = \Gamma R(k, S(k)), \tag{29}$$

where $R(k, S)$ is almost periodic (respectively, Bohr almost periodic, almost automorphic, Birkhoff recurrent, Levitan almost periodic, Bebutov almost recurrent, Poisson stable) in k. Choosing $\mu \in \mathbb{R}^m_+$ and using the reaction coordinates $x : S = \mu + \Gamma x$, we transform (29) into a system in the reaction coordinates:

$$\Delta u(k) = F_\mu(k, u(k)) := R(k, \mu + \Gamma u(k)) \tag{30}$$

evolving on the state space $X_\mu := \{u \in \mathbb{R}^n |\ \mu + \Gamma u \geq 0\}$.

Let $\varphi(k, \xi, f)$ denote the unique solution of (30) passing through ξ at $k = 0$. Then it enjoys positive translation invariance:

$$\varphi(k, \xi + \lambda v, f) = \varphi(k, \xi, f) + \lambda v,\ \forall\ \xi \in X_\mu,\ \lambda \in \mathbb{R}. \tag{31}$$

It can be checked that the solution for every $h \in H(f)$ possesses the positive translation invariance property (31), where $H(f)$ is the hull of f. So the skew-product flow induced by H-class

$$\Delta u(k) = h(k, u(k)) \quad (h \in H(f))$$

of system (30) has positive translation invariance.

According to Corollary 9 (item (iii)) for any $G \in H(F_\mu)$ there exists a unique $h \in H(f)$ $(G = \Phi(h))$ such that

$$G(k, u) = h(k, \mu + \Gamma u), \tag{32}$$

for any $(k,u) \in \mathbb{Z} \times X_\mu$. In particular, $G = F_\mu$ if and only if $h = f$. For every $G \in H(F_\mu)$ and $h \in H(f)$ in (32), let $\varphi(t,x_0,h)$ and $\phi(t,u_0,G)$ be the solutions of

$$\Delta U(k) = \Gamma h(k, U(k)),\ k > 0,\ U(0) = U_0 \in \mathbb{R}^m_+ \tag{33}$$

and

$$\Delta u(k) = G(k, u(k)),\ k > 0,\ u(0) = u_0 \in X_\mu, \tag{34}$$

respectively.

Lemma 16. *Let $\varphi(k,x_0,h)$ and $\phi(k,u_0,G)$ be the solutions of (33) and (34), respectively. Then we have*

$$\varphi(k, \mu + \Gamma u_0, h) = \mu + \Gamma \phi(k, u_0, \Phi(h))$$

for any $k \in \mathbb{Z}_+$, $u_0 \in X_\mu$ and $h \in H(f)$.

Proof. Denote by

$$\psi(k) := \mu + \Gamma\phi(k, u_0, G)\ (G = \Phi(h)) \tag{35}$$

for any $(k,u_0,G) \in \mathbb{Z}_+ \times X_\mu \times H(F_\mu)$. Then from (32), (34) and (35) we obtain

$$\Delta\psi(k) = \Gamma\Delta\phi(k,u_0,G) = \Gamma G(k, \phi(k,u_0,G)) =$$
$$\Gamma h(k, \mu + \Gamma\phi(k,u_0,G)) = \Gamma h(k, \psi(k))$$

for any $k \in \mathbb{Z}_+$. Thus ψ is a solution of equation (33). Taking into consideration that $\psi(0) = \mu + \Gamma u_0$, then we will have $\psi(k) = \varphi(k, \mu + \Gamma u_0, h)$ and, consequently, $\varphi(k, \mu+\Gamma u_0, h) = \mu + \Gamma\phi(k, u_0, \Phi(h))$ for any $(k,u_0,h) \in \mathbb{Z}_+ \times X_\mu \times H(f)$. Lemma is proved. □

Condition **(D).** Equation (34) is monotone (respectively, strongly monotone). This means that the cocycle $\langle \mathbb{X}_\mu, \phi, (H(F_\mu), \mathbb{Z}, \sigma)\rangle$ generated by (34) is monotone (respectively, strongly monotone), i.e. if $u,v \in \mathbb{X}_\mu$ and $u \le v$ (respectively, $u < v$) then $\phi(k,u,g) \le \phi(k,v,g)$ (respectively, $\phi(k,u,g) \ll \phi(k,v,g)$) for all $k \ge 0$ and $G \in H(F_\mu)$.

Definition 47. Let $F_\mu \in C(\mathbb{Z} \times X_\mu, \mathbb{R}^n)$. The set $H(F_\mu)$ is said to be minimal if it is a minimal set of shift dynamical system $(C(\mathbb{Z} \times X_\mu, \mathbb{R}^n), \mathbb{Z}, \sigma)$.

Definition 48. A function $F_\mu \in C(\mathbb{Z} \times X_\mu, \mathbb{R}^n)$ is said to be strongly Poisson stable if every function $G \in H(F_\mu)$ is Poisson stable.

Theorem 19. *Suppose that the following conditions hold:*

1. *$\mu \in \mathbb{R}^m$ is such that the system (30) is strongly monotone;*
2. *the set $H(F_\mu)$ is minimal and F_μ is strongly Poisson stable;*
3. *the matrix Γ has rank exactly $n-1$ whose kernel is spanned by a strongly positive vector v;*

4. for any $G \in H(F_\mu)$ all forward solutions of equation (30) are bounded.

Then for any $U_0 \in X_{(\mu,\Gamma)}$ the following statements hold:

1. *the set $\omega_{(U_0,f)} \bigcap X_f$ consists of a single point $p_0 = (V_0, f)$, where $\omega_{(U_0,f)}$ is the ω-limit set of the motion $\pi(k,(U_0,f))$ of the skew-product dynamical system $(X, \mathbb{Z}_+, \pi)$ ($X := X_{(\mu,\Gamma)} \times H(f)$, $\pi := (\varphi, \sigma)$) and $X_f := X_{(\mu,\Gamma)} \times \{f\}$;*

2. *the solution $\varphi(k, V_0, f)$ of equation (30) is defined on $\mathbb{Z}$, $\overline{\varphi(\mathbb{Z}, v_0, f)} \subseteq Q_+^{(U_0,f)}$ and it is strongly compatible;*

3. $$\lim_{k\to\infty} |\varphi(k, U_0, f) - \varphi(k, V_0, f)| = 0.$$

Proof. Let $(H(F_\mu), \mathbb{Z}, \sigma)$ (respectively, $(H(f), \mathbb{Z}, \sigma)$) be the shift dynamical system on $H(F_\mu)$ (respectively, on $H(f)$). Denote by $\langle X_\mu, \phi, (H(F_\mu), \mathbb{Z}, \sigma)\rangle$ (respectively, $\langle X_{(\mu,\Gamma)}, \varphi, (H(f), \mathbb{Z}, \sigma)\rangle$, where $X_{(\mu,\Gamma)} = \mu + \Gamma(X_\mu) := \{\mu + \Gamma u|\ u \in X_\mu\}$) the cocycle generated by family of equations (34) (respectively, (33)). Note that under the conditions of Theorem 19 Conditions (**C1**), (**C3**) and (**C4**) are fulfilled for the cocycle $\langle X_\mu, \phi, (H(F_\mu), \mathbb{R}, \sigma)\rangle$. Let $U_0 \in X_{(\mu,\Gamma)}$, then there exists a point $u_0 \in X_\mu$ such that $U_0 = \mu + \Gamma(u_0)$. By equality (30) the cocycle ϕ is translation invariant with respect to vector $v \gg 0$. According to Theorem 14 for given $u_0 \in X_\mu$ the following statements are fulfilled:

a. the set $\omega_{(u_0,F_\mu)} \bigcap X_{F_\mu}$ consists of a single point $q_0 = (v_0, F_\mu)$, where $\omega_{(u_0,F_\mu)}$ is the ω-limit set of the motion $\pi(k,(u_0,F_\mu))$ of the skew-product dynamical system $(X, \mathbb{Z}_+, \pi)$ ($X := X_\mu \times H(F_\mu)$, $\pi := (\phi, \sigma)$) and $X_{F_\mu} := X_{(\mu,\Gamma)} \times \{F_\mu\}$;

b. the solution $\phi(k, V_0, F_\mu)$ of equation (30) is defined on $\mathbb{Z}$, $\overline{\phi(\mathbb{Z}, v_0, F_\mu)} \subseteq Q_+^{(u_0,F_\mu)}$ and it is strongly compatible;

c. $$\lim_{k\to\infty} |\phi(k, u_0, F_\mu) - \phi(k, v_0, F_\mu)| = 0.$$

Denote by $V_0 := \mu + \Gamma(v_0) \in X_{(\mu,\Gamma)}$ and consider the solutions $\varphi(k, U_0, f)$ and $\varphi(k, V_0, f)$ of equation (33) ($h = f$). Since $\phi(\cdot, v_0, F_\mu)$ is a strongly compatible solution of equation (34) ($G = F_\mu$), then

$$\mathfrak{M}_{F_\mu} \subseteq \mathfrak{M}_{\phi(\cdot,v_0,F_\mu)}. \tag{36}$$

By Lemma 16 we have $\varphi(k, V_0, f) = \mu + \phi(k, v_0, F_\mu)$ for any $k \in \mathbb{Z}$. Note that

$$\mathfrak{M}_{\phi(\cdot,v_0,F_\mu)} \subseteq \mathfrak{M}_{\varphi(\cdot,V_0,f)}. \tag{37}$$

Indeed if $\{k_m\} \in \mathfrak{M}_{\phi(\cdot,v_0,F_\mu)}$ then we have

$$\varphi(k + k_m, V_0, f) - \bar{\varphi}(k) = \mu + \Gamma\phi(k + k_m, v_0, F_\mu) - (\mu + \Gamma\bar{\phi}(k)) =$$

$$\Gamma(\phi(k+k_m,v_0,F_\mu)-\bar{\phi}(k))\to 0$$

as $m\to\infty$ uniformly with respect to k on every compact subset from $\mathbb{Z}$, where $\bar{\phi}=\lim_{m\to\infty}\phi(\cdot+k_m,v_0,F_\mu)$ in the space $C(\mathbb{Z},\mathbb{R}^n)$. This means that $\{k_m\}\in\mathfrak{M}_{\varphi(\cdot,V_0,f)}$.

From (36) and (37) we have

$$\mathfrak{M}_{F_\mu}\subseteq\mathfrak{M}_{\varphi(\cdot,V_0,f)}. \tag{38}$$

Finally, from Corollary 9 (item (ii)) we have

$$\mathfrak{M}_f\subseteq\mathfrak{M}_{\Phi(f)}. \tag{39}$$

In virtue of (38)–(39) and taking into consideration the equality $\Phi(f)=F_\mu$ we obtain

$$\mathfrak{M}_f\subseteq\mathfrak{M}_{\varphi(\cdot,V_0,f)},$$

i.e., $\varphi(k,V_0,f)$ is a strongly compatible solution of equation (33) (for $h=f$).

To finish the proof of Theorem it is sufficient to note that

$$|\varphi(k,U_0,f)-\varphi(k,V_0,f)|=|(\mu+\Gamma\phi(k,u_0,F_\mu))-(\mu+\Gamma\phi(k,v_0,F_\mu))|=$$
$$|\Gamma(\phi(k,u_0,F_\mu)-\phi(k,v_0,F_\mu))|\le\|\Gamma\||\phi(k,u_0,F_\mu)-\phi(k,v_0,F_\mu)|\to 0$$

as $k\to\infty$. □

Corollary 11. *Under the conditions of Theorem 19 if the function $f\in C(\mathbb{Z}\times X_{(\mu,\Gamma)},\mathbb{R}^n)$ is stationary (respectively, τ-periodic, quasi-periodic, Bohr almost periodic, almost automorphic, recurrent in the sense of Birkhoff, strongly Poisson stable) in time, then for any $U_0\in X_{(\mu,\Gamma)}$ the following statements hold:*

1. *the set $\omega_{(U_0,f)}\bigcap X_f$ consists of a single point $p_0=(V_0,f)$;*
2. *$\varphi(k,V_0,f)$ is a stationary (respectively, τ-periodic, quasi-periodic, Bohr almost periodic, almost automorphic, recurrent in the sense of Birkhoff, strongly Poisson stable) solution of equation (30);*
3. *$\lim_{k\to+\infty}|\varphi(k,U_0,f)-\varphi(k,V_0,f)|=0$, i.e., $\varphi(k,u_0,f)$ is asymptotically stationary (respectively, asymptotically τ-periodic, asymptotically quasi-periodic, asymptotically Bohr almost periodic, asymptotically almost automorphic, asymptotically recurrent in the sense of Birkhoff, asymptotically strongly Poisson stable).*

Proof. This statement follows from Theorem 19 and Corollary 6. □

Consider the difference equation

$$\Delta u(k)=f(k,u(k)), \tag{40}$$

where $f \in C(\mathbb{Z} \times \mathbb{R}^n, \mathbb{R}^n)$. Along with the equation (40) we consider its H-class [5,21,27,34,37], i.e., the family of the equations

$$\Delta v(k) = g(k, v(k)), \tag{41}$$

where $g \in H(f) = \overline{\{f_\tau : \tau \in \mathbb{Z}\}}$ and $f_\tau(k, u) = f(k + \tau, u)$, with the bar indicating closure in the compact-open topology.

We will suppose that the function f is regular. Denote by $\varphi(\cdot, v, g)$ the solution of (41) passing through the point $v \in \mathbb{R}^n$ for $k = 0$. Then the mapping $\varphi : \mathbb{Z}_+ \times \mathbb{R}^n \times H(f) \to \mathbb{R}^n$ satisfies the following conditions (see, for example,[5,31]):

1) $\varphi(0, v, g) = v$ for all $v \in \mathbb{R}^n$ and $g \in H(f)$;

2) $\varphi(k, \varphi(\tau, v, g), g_\tau) = \varphi(k + \tau, v, g)$ for each $v \in \mathbb{R}^n$, $g \in H(f)$ and $k, \tau \in \mathbb{Z}_+$;

3) $\varphi : \mathbb{Z}_+ \times \mathbb{R}^n \times H(f) \to \mathbb{R}^n$ is continuous.

Denote by $Y := H(f)$ and $(Y, \mathbb{Z}, \sigma)$ a dynamical system of translations on Y, induced by the dynamical system of translations $(C(\mathbb{Z} \times \mathbb{R}^n, \mathbb{R}^n), \mathbb{Z}, \sigma)$. The triple $\langle \mathbb{R}^n, \varphi, (Y, \mathbb{Z}, \sigma)\rangle$ is a cocycle over $(Y, \mathbb{Z}_+, \sigma)$ with the fiber $\mathbb{R}^n$. Hence, the equation (40) generates a cocycle $\langle \mathbb{R}^n, \varphi, (Y, \mathbb{Z}, \sigma)\rangle$ and the non-autonomous dynamical system $\langle (X, \mathbb{Z}_+, \pi), (Y, \mathbb{Z}, \sigma), h\rangle$, where $X := \mathbb{R}^n \times Y$, $\pi := (\varphi, \sigma)$ and $h := pr_2 : X \to Y$.

Definition 49. Recall that the difference equation (40) is called dissipative if for all $t_0 \in \mathbb{R}$ and $u_0 \in \mathbb{R}^n$ there exists a unique solution $x(k; u_0, k_0)$ of the equation (40) passing through the point (u_0, k_0) and if there exists a number $R > 0$ such that $\lim\limits_{k\to+\infty} \sup |x(k; u_0, k_0)| < R$ for all $x_0 \in \mathbb{R}^n$ and $k_0 \in \mathbb{Z}$. In other words, for every solution $x(k; u_0, k_0)$ there is an instant $k_1 = k_0 + l(k_0, u_0)$, such that $|x(k; u_0, k_0)| < R$ for any $k \geq k_1$. If for any $r > 0$ the number $l(k_0, u_0)$ can be chosen independently on k_0 and u_0 with $|u_0| \leq r$, then the equation (40) is called uniformly dissipative.

Lemma 17. *Let $f \in C(\mathbb{Z} \times \mathbb{R}^n, \mathbb{R}^n)$ be regular. If $H(f)$ is compact, then equation (40) is uniformly dissipative if and only if there is a positive number r such that*

$$\limsup_{k\to+\infty} |\varphi(k, u_0, g)| < r \quad (u_0 \in \mathbb{R}^n,\ g \in H(f)) \ .$$

Proof. This statement can be proved using the same arguments as in the proof of Lemma 15 (see [12, ChIII]). □

Remark 16. If $f \in C(\mathbb{Z} \times \mathbb{R}^n, \mathbb{R}^n)$ is regular, $H(f)$ is compact and then equation (40) is uniformly dissipative, then the cocycle φ generated by equation (40) admits a compact global attractor.

Theorem 20. *Suppose that the following assumptions are fulfilled:*

– *$\mu \in \mathbb{R}^m$ is such that the system (30) is monotone;*

– *the matrix* Γ *has rank exactly* $n-1$ *whose kernel is spanned by a strongly positive vector* v*;*

– *the function* $F_\mu \in C(\mathbb{Z}\times X_\mu,\mathbb{R}^n)$ *is recurrent in* $k\in\mathbb{Z}$ *uniformly with respect to* u *on every compact subset from* X_μ*;*

– *the cocycle* ϕ *generated by equation (30) admits a compact global attractor and* $\boldsymbol{I}:=\{I_G|\ G\in H(F_\mu)\}$ *is its Levinson center.*

Then under the condition ($\boldsymbol{D}$) *the following statements hold:*

1. $\alpha(G),\beta(G)\in I_G$ *for any* $G\in H(F_\mu)$ *and, consequently,* $I_G\subseteq[\alpha(G),\beta(G)]$*;*
2. $\phi(k,\alpha(G),G)=\alpha(\sigma(k,G))$ *(respectively,* $\phi(k,\beta(G),G)=\beta(\sigma(k,G))$*) for any* $k\geq 0$ *and* $G\in H(F_\mu)$*;*
3. *the point* $\gamma_*(F_\mu):=(\alpha(F_\mu),F_\mu)\in X=X_\mu\times Y$ *(respectively,* $\gamma^*(F_\mu):=(\beta(F_\mu),F_\mu)\in X$*) is strongly comparable by character of recurrence with the point* F_μ*;*
4. *for any* $h\in H(f)$ *equation (34) has at least two solutions* $\varphi(k,U_0,h)$ *(*$U_0=\mu+\Gamma\alpha(\Phi(h))$*) and* $\varphi(k,V_0,h)$ *(*$V_0=\mu+\Gamma\beta(\Phi(h))$*) defined and bounded on* $\mathbb{Z}$ *which are strongly compatible and belong to Levinson center of (30);*
5. *if the function* $f\in C(\mathbb{Z}\times X_{(\mu,\Gamma)},\mathbb{R}^n)$ *is quasi-periodic (respectively, Bohr almost periodic, almost automorphic, recurrent, pseudo-recurrent and Lagrange stable, uniformly Poisson stable and stable in the sense of Lagrange) in* $k\in\mathbb{Z}$ *uniformly with respect to* u *on every compact subset from* $X_{(\mu,\Gamma)}$*, then* $\varphi(k,u_0,f)$ *and* $\varphi(k,V_0,f)$ *are quasi-periodic (respectively, Bohr almost periodic, almost automorphic, recurrent, pseudo-recurrent and Lagrange stable, uniformly Poisson stable and stable in the sense of Lagrange).*

Proof. Let $(H(F_\mu),\mathbb{Z},\sigma)$ (respectively, $(H(f),\mathbb{Z},\sigma)$) be the shift dynamical system on $H(F_\mu)$ (respectively, on $H(f)$). Denote by $\langle X_\mu,\phi,(H(F_\mu),\mathbb{Z},\sigma)\rangle$ (respectively, $\langle X_{(\mu,\Gamma)},\varphi,(H(f),\mathbb{Z},\sigma)\rangle$, where $X_{(\mu,\Gamma)}=\mu+\Gamma(X_\mu):=\{\mu+\Gamma u|\ u\in X_\mu\}$) the cocycle generated by family of equations (34) (respectively, (33)).

By equality (30) the cocycle ϕ is translation invariant with respect to vector $v\gg 0$. Applying Theorem 15 to nonautonomous dynamical system $\langle X_\mu,\phi,(H(F_\mu),\mathbb{Z},\sigma)\rangle$ we obtain the following statements:

1. $\alpha(G),\beta(G)\in I_G$ for any $G\in H(F_\mu)$ and, consequently, $I_G\subseteq[\alpha(G),\beta(G)]$, where $\alpha(G):=\inf I_G$ (respectively, $\beta(G):=\sup I_G$);
2. $\phi(k,\alpha(G),G)=\alpha(\sigma(k,G))$ (respectively, $\phi(k,\beta(G),G)=\beta(\sigma(k,G))$) for any $k\geq 0$ and $G\in H(F_\mu)$;
3. the point $\gamma_*(F_\mu):=(\alpha(F_\mu),F_\mu)\in X=X_\mu\times Y$ (respectively, $\gamma^*(F_\mu):=(\beta(F_\mu),F_\mu)\in X$) is strongly comparable by character of recurrence with the point F_μ.

Note that the nonautonomous dynamical system $\langle X_{(\mu,\Gamma)}, \varphi, (H(f), \mathbb{Z}, \sigma)\rangle$ is compactly dissipative because $\varphi(k, U, h) = \mu + \Gamma\phi(k, u, \Phi(h))$ for any $h \in H(f)$ ($U = \mu + \Gamma u$) and the cocycle ϕ is so. Let $\mathbf{A} = \{A_h | \ h \in H(f)\}$ be the Levinson center for the compact dissipative cocycle φ generated by equation (33). Denote by $U(h) := \mu + \Gamma\alpha(\Phi(h))$ and $V(h) := \mu + \Gamma\beta(\Phi(h))$. Then by Lemma 16 for any $h \in H(f)$

$$\varphi(k, U(h), h) = \mu + \Gamma\phi(k, \alpha(\Phi(h)), \Phi(h)) \tag{42}$$

(respectively,

$$\varphi(k, V(h), h) = \mu + \Gamma\phi(k, \beta(\Phi(h)), \Phi(h))) \tag{43}$$

is a bounded on $\mathbb{Z}$ solution of equation (34). By Theorem 8 we have $U(h), V(h) \in A_h$, i.e., $U(h)$ and $V(h)$ belongs to the Levinson center of the cocycle φ. Finally, from (42) (respectively, (43)) it follows that $\varphi(k, U(f), f)$ (respectively, $\varphi(k, V(f), f)$) is a strongly compatible solution of equation (33) for $h = f$, because $\phi(k, \alpha(\Phi(h)), \Phi(h))$ (respectively, $\phi(k, \beta(\Phi(h)), \Phi(h)))$ is a strongly compatible solution of equation (34), $\Phi : H(f) \to H(F_\mu)$ is a homeomorphism and $\Phi(f) = F_\mu$. Theorem is proved. $\square$

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10 Conflict of Interest

The author declares that he does not have conflict of interest.

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