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A self-similar solution for the two-dimensional Broadwell system via the Bateman equation

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Abstract. A self-similar solution of the Broadwell system is found. Here the solution is sought using a reduction that transforms the given system into a system of differential equations. Further, the solution is constructed using the Painlevé series. Here the system already passes the Painlevé test and it is possible to find the solution if the equations in resonance satisfy the solution of the two-dimensional Bateman equation. Exact solution of the Bateman equation is established, allowing to find new explicit solution for the original system. In the process of calculations, we use the Wolfram Mathematica program. The proof of these results is carried out at a rigorous mathematical level.

Mathematics subject classification: 35L40, 35Q20, 35C06.
Keywords and phrases: Broadwell system, self-similar solution, Painlevé test, Bateman equation.

1 Introduction

We consider the well-known two-dimensional Broadwell model [7,10,11,15,17,20]

$$\begin{aligned}
&\partial_t u + \partial_x u = \frac{1}{\varepsilon}(wz - uv),\\
&\partial_t v - \partial_x v = \frac{1}{\varepsilon}(wz - uv), \quad x, y \in \mathbb{R},\ t > 0,\\
&\partial_t w + \partial_y w = \frac{1}{\varepsilon}(uv - wz),\\
&\partial_t z - \partial_y z = \frac{1}{\varepsilon}(uv - wz).
\end{aligned} \tag{1}$$

Here $u(x, y, t), v(x, y, t), w(x, y, t), z(x, y, t)$ are the densities of particle groups, ε is the Knudsen parameter. It is required to find a self-similar solution of the system (1). As is known, most of the equations of mathematical physics describe various physical processes, for example, the Burgers equation, the Korteweg-de Vries equation, the Allen-Kahn equation, etc. One of such equations is the discrete kinetic Boltzmann equation [22] (see p.1). We consider the so-called Broadwell model [7, 11, 20], which is a consequence in the discrete case when the collision integral on the right side of the Boltzmann equation is replaced by a finite sum. From here, the given system of equations is directly obtained. The physical interpretation of the

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system can be found in [7,20]. Works [1,5,11] are devoted to finding exact solutions of kinetic systems by means of the Bateman equation [3,6,13,14]. These systems are non-integrable (kinetic systems Carleman [1], Godunov-Sultangazin [9,11] (one-dimensional model of Broadwell), McKean [5,12], two-dimensional model of Broadwell). As a result, the Painlevé test fails. Here, in resonance, the author obtained the Bateman equation and, knowing its implicit solution, constructed a solution for the original system. Stationary solutions of systems were found in [16,18]. In the works [7,9,19] it is proved that the solution of systems tends to a positive equilibrium state exponentially fast. Also recently in [8,12,18,20], solutions were found that can take both positive and negative values. Nevertheless, ones produce interesting results. In our work, a self-similar solution of the system is presented.

2 Bateman equation

The two-dimensional Bateman equation is an equation of the form [4,6,11]

$$\frac{\partial^2\varphi}{\partial\eta^2}\Big(\frac{\partial\varphi}{\partial\xi}\Big)^2-2\frac{\partial\varphi}{\partial\eta}\frac{\partial\varphi}{\partial\xi}\frac{\partial^2\varphi}{\partial\xi\partial\eta}+\Big(\frac{\partial\varphi}{\partial\eta}\Big)^2\frac{\partial^2\varphi}{\partial\xi^2}=0. \tag{2}$$

This equation has an implicit solution

$$\xi f(\varphi)+\eta g(\varphi)=c, \tag{3}$$

where f, g are arbitrary smooth functions, $c\in\mathbb{R}$. The proof is carried out by direct computation

$$\begin{aligned}\frac{\partial\varphi}{\partial\xi}&=-\frac{f(\varphi)}{\xi f'(\varphi)+\eta g'(\varphi)},\\ \frac{\partial^2\varphi}{\partial\xi^2}&=-\frac{f(\varphi)\Big(-2\xi f'(\varphi)^2-2\eta f'(\varphi)g'(\varphi)+f(\varphi)(\xi f''(\varphi)+\eta g''(\varphi))\Big)}{(\xi f'(\varphi)+\eta g'(\varphi))^3},\\ \frac{\partial\varphi}{\partial\eta}&=-\frac{g(\varphi)}{\xi f'(\varphi)+\eta g'(\varphi)},\\ \frac{\partial^2\varphi}{\partial\eta^2}&=-\frac{g(\varphi)\Big(-2\xi f'(\varphi)g'(\varphi)-2\eta g'(\varphi)^2+g(\varphi)(\xi f''(\varphi)+\eta g''(\varphi))\Big)}{(\xi f'(\varphi)+\eta g'(\varphi))^3},\\ \frac{\partial^2\varphi}{\partial\xi\partial\eta}&=\frac{fg'(\xi f'+\eta g')+g\Big(\xi f'(\varphi)^2+\eta f'(\varphi)g'(\varphi)-f(\xi f''+\eta g'')\Big)}{(\xi f'(\varphi)+\eta g'(\varphi))^3}.\end{aligned} \tag{4}$$

Substituting (4) into (2), we are convinced of the equality.

3 A self-similar solution for the Broadwell system

We look for a self-similar solution in the form (see [21], S.3.3., p. 708)

$$u(x,y,t)=x^{\alpha}U(\xi,\eta), v(x,y,t)=x^{\beta}V(\xi,\eta),$$

$$w(x,y,t) = x^{\gamma}W(\xi,\eta), z(x,y,t) = x^{s}Z(\xi,\eta),$$

where $\xi = tx^A, \eta = yx^B$. System is scale invariant under

$$t = C^k\bar{t}, x = C\overline{x}, y = C^l\overline{y}, u = C^m\overline{u}, v = C^n\overline{v}, w = C^p\overline{w}, z = C^q\overline{z}, C > 0. \qquad (5)$$

The scaling transformation (5) converts system (1) into

$$\begin{aligned}
C^{m-k}\frac{\partial\bar{u}}{\partial\bar{t}} + C^{m-1}\frac{\partial\bar{u}}{\partial\bar{x}} &= \frac{1}{\varepsilon}(C^{p+q}\bar{w}\bar{z} - C^{m+n}\bar{u}\bar{v}),\\
C^{n-k}\frac{\partial\bar{v}}{\partial\bar{t}} - C^{n-1}\frac{\partial\bar{v}}{\partial\bar{x}} &= \frac{1}{\varepsilon}(C^{p+q}\bar{w}\bar{z} - C^{m+n}\bar{u}\bar{v}),\\
C^{p-k}\frac{\partial\bar{w}}{\partial\bar{t}} + C^{p-l}\frac{\partial\bar{w}}{\partial\bar{y}} &= \frac{1}{\varepsilon}(C^{m+n}\bar{u}\bar{v} - C^{p+q}\bar{w}\bar{z}),\\
C^{q-k}\frac{\partial\bar{z}}{\partial\bar{t}} - C^{q-l}\frac{\partial\bar{z}}{\partial\bar{y}} &= \frac{1}{\varepsilon}(C^{m+n}\bar{u}\bar{v} - C^{p+q}\bar{w}\bar{z}),
\end{aligned}$$

Equating the powers of C yields the following system of linear algebraic equations for the constants m, k, p, q, l and n:

$$m - 1 - m + k = 0, p + q - m + k = 0, m + n - m + k = 0,$$

$$n - 1 - n + k = 0, p + q - n + k = 0, m + n - n + k = 0,$$

$$p - l - p + k = 0, m + n - p + k = 0, p + q - p + k = 0,$$

$$q - l - q + k = 0, m + n - q + k = 0, p + q - q + k = 0.$$

This system has a unique solution

$$k = l = 1, n = p = m = q = -1.$$

In this case according to the formulas from (see [21], S.3.3., p. 708)

$$\alpha = \beta = \gamma = s = -1, A = B = -1.$$

Then we have

$$u(x,y,t) = \frac{1}{x}U(\xi,\eta), v(x,y,t) = \frac{1}{x}V(\xi,\eta), \qquad (6)$$

$$w(x,y,t) = \frac{1}{x}W(\xi,\eta), z(x,y,t) = \frac{1}{x}Z(\xi,\eta), \qquad (7)$$

where $\xi = t/x, \eta = y/x$. Substituting expressions (6)-(7) into (1), we obtain for the first equation

$$\frac{1}{x}U'_{\xi}\frac{1}{x} - \frac{U}{x^2} + \frac{1}{x}(U'_{\xi}(-\frac{t}{x^2}) + U'_{\eta}(-\frac{y}{x^2})) = \frac{1}{\varepsilon}(\frac{1}{x^2}WZ - \frac{1}{x^2}UV). \qquad (8)$$

The rest of the equations are obtained similarly. Hence, we have system

$$\begin{aligned} U'_\xi(1-\xi) - U'_\eta \eta &= U + \frac{1}{\varepsilon}(WZ - UV), \\ V'_\xi(1+\xi) + V'_\eta \eta &= -V + \frac{1}{\varepsilon}(WZ - UV), \\ W'_\xi + W'_\eta &= \frac{1}{\varepsilon}(UV - WZ), \\ Z'_\xi - Z'_\eta &= \frac{1}{\varepsilon}(UV - WZ). \end{aligned} \tag{9}$$

We apply the Painlevé expansion [2]:

$$\begin{aligned} U(\xi,\eta) &= \varphi^{-p}\sum_{j=0}^{\infty} U_j(\xi,\eta)\varphi^j, V(\xi,\eta) = \varphi^{-\beta}\sum_{j=0}^{\infty} V_j(\xi,\eta)\varphi^j, \\ W(\xi,\eta) &= \varphi^{-\gamma}\sum_{j=0}^{\infty} W_j(\xi,\eta)\varphi^j, Z(\xi,\eta) = \varphi^{-q}\sum_{j=0}^{\infty} Z_j(\xi,\eta)\varphi^j, \end{aligned} \tag{10}$$

where $\varphi = \varphi(\xi,\eta)$ is an analytic function in a neighborhood of the manifold $\varphi(\xi,\eta) = 0$. Firstly, we find the dominant terms

$$U = U_0\varphi^{-p}, V = V_0\varphi^{-\beta}, W = W_0\varphi^{-\gamma}, Z = Z_0\varphi^{-q}, \tag{11}$$

where p, β, γ, q are positive integers. Substituting the leading terms of (11) into our original system, we have

$$\begin{aligned} &(1-\xi)(U'_{0\xi}\varphi^{-p} - p\varphi^{-p-1}\varphi'_\xi U_0) - \eta(U'_{0\eta}\varphi^{-p} - p\varphi^{-p-1}\varphi'_\eta U_0) = \\ &\qquad = U_0\varphi^{-p} + \frac{1}{\varepsilon}(W_0 Z_0 \varphi^{-\gamma-q} - U_0 V_0 \varphi^{-p-\beta}), \\ &(1+\xi)(V'_{0\xi}\varphi^{-\beta} - \beta\varphi^{-\beta-1}\varphi'_\xi V_0) + \eta(V'_{0\eta}\varphi^{-p} - p\varphi^{-p-1}\varphi'_\eta V_0) = \\ &\qquad = -V_0\varphi^{-\beta} + \frac{1}{\varepsilon}(W_0 Z_0 \varphi^{-\gamma-q} - U_0 V_0 \varphi^{-p-\beta}), \\ &W'_{0\xi}\varphi^{-\gamma} - \gamma\varphi^{-\gamma-1}\varphi'_\xi W_0 + W'_{0\eta}\varphi^{-\gamma} - \gamma\varphi^{-\gamma-1}\varphi'_\eta W_0 = \\ &\qquad = \frac{1}{\varepsilon}(U_0 V_0 \varphi^{-p-\beta} - W_0 Z_0 \varphi^{-\gamma-q}), \\ &Z'_{0\xi}\varphi^{-q} - q\varphi^{-q-1}\varphi'_\xi Z_0 - Z'_{0\eta}\varphi^{-q} + q\varphi^{-q-1}\varphi'_\eta Z_0 = \\ &\qquad = \frac{1}{\varepsilon}(U_0 V_0 \varphi^{-p-\beta} - W_0 Z_0 \varphi^{-\gamma-q}). \end{aligned} \tag{12}$$

Multiplying the first equation of (12) by φ^{p+1} and taking into account that $\varphi(\xi,\eta)=0$, we have $p=\beta=\gamma=q=1$. From here

$$\begin{aligned}
-(1-\xi)\varphi'_\xi U_0+\varphi'_\eta U_0\eta &= \frac{1}{\varepsilon}(W_0Z_0-U_0V_0),\\
-(1+\xi)\varphi'_\xi V_0-\varphi'_\eta V_0\eta &= \frac{1}{\varepsilon}(W_0Z_0-U_0V_0),\\
-\varphi'_\xi W_0-\varphi'_\eta W_0 &= \frac{1}{\varepsilon}(U_0V_0-W_0Z_0),\\
-\varphi'_\xi Z_0+\varphi'_\eta Z_0 &= \frac{1}{\varepsilon}(U_0V_0-W_0Z_0).
\end{aligned} \tag{13}$$

Solving the system (13), we obtain solution

$$\begin{aligned}
U_0(\xi,\eta) &= -\frac{\varepsilon(\eta\varphi_\eta+(\xi+1)\varphi_\xi)(\varphi_\eta^2-\varphi_\xi^2)}{(\eta^2-1)\varphi_\eta^2+2\xi\eta\varphi_\xi\varphi_\eta+\xi^2\varphi_\xi^2},\\
V_0(\xi,\eta) &= \frac{\varepsilon(\eta\varphi_\eta+(\xi-1)\varphi_\xi)(\varphi_\eta^2-\varphi_\xi^2)}{(\eta^2-1)\varphi_\eta^2+2\xi\eta\varphi_\xi\varphi_\eta+\xi^2\varphi_\xi^2},\\
W_0(\xi,\eta) &= -\frac{\varepsilon(\varphi_\eta-\varphi_\xi)(\eta\varphi_\eta+(\xi-1)\varphi_\xi)(\eta\varphi_\eta+(\xi+1)\varphi_\xi)}{(\eta^2-1)\varphi_\eta^2+2\xi\eta\varphi_\xi\varphi_\eta+\xi^2\varphi_\xi^2},\\
Z_0(\xi,\eta) &= \frac{\varepsilon(\varphi_\eta+\varphi_\xi)(\eta\varphi_\eta+(\xi-1)\varphi_\xi)(\eta\varphi_\eta+(\xi+1)\varphi_\xi)}{(\eta^2-1)\varphi_\eta^2+2\xi\eta\varphi_\xi\varphi_\eta+\xi^2\varphi_\xi^2}.
\end{aligned} \tag{14}$$

The truncated Painlevé expansion has the form

$$\begin{aligned}
U=U_0\varphi^{-1}+U_1, V=V_0\varphi^{-1}+V_1,\\
W=W_0\varphi^{-1}+W_1, Z=Z_0\varphi^{-1}+Z_1,
\end{aligned} \tag{15}$$

where U_0, V_0, W_0, Z_0 are defined by (14) and U_1, V_1, W_1, Z_1 are arbitrary functions.

Substituting (15) into (9), we have

$$\begin{aligned}
&\varphi^{-1}\Big(U'_{0\xi}(1-\xi)-\eta U'_{0\eta}-U_0-\frac{1}{\varepsilon}(W_0Z_1+W_1Z_0-U_0V_1-U_1V_0)\Big)+\\
&\quad+\varphi^{-2}\Big(-U_0\varphi'_\xi(1-\xi)+\eta\varphi'_\eta U_0-\frac{1}{\varepsilon}(W_0Z_0-U_0V_0)\Big)+\\
&\quad+\varphi^{0}\Big(U'_{1\xi}(1-\xi)-\eta U'_{1\eta}-U_1-\frac{1}{\varepsilon}(W_1Z_1-U_1V_1)\Big)=0,
\end{aligned}$$

$$\begin{aligned}
&\varphi^{-1}\Big(V'_{0\xi}(1+\xi)+\eta V'_{0\eta}+V_0-\frac{1}{\varepsilon}(W_0Z_1+W_1Z_0-U_0V_1-U_1V_0)\Big)+\\
&\quad+\varphi^{-2}\Big(-V_0\varphi'_\xi(1+\xi)-\eta\varphi'_\eta V_0-\frac{1}{\varepsilon}(W_0Z_0-U_0V_0)\Big)+\\
&\quad+\varphi^{0}\Big(V'_{1\xi}(1+\xi)+\eta V'_{1\eta}+V_1-\frac{1}{\varepsilon}(W_1Z_1-U_1V_1)\Big)=0,
\end{aligned}$$

$$
\begin{gathered}
\varphi^{-1}\Big(W'_{0\xi}+W'_{0\eta}-\frac{1}{\varepsilon}(U_0V_1+U_1V_0-W_0Z_1-W_1Z_0)\Big)+\\
+\varphi^{-2}\Big(-W_0\varphi'_\xi-\varphi'_\eta W_0-\frac{1}{\varepsilon}(U_0V_0-W_0Z_0)\Big)+\\
+\varphi^{0}\Big(W'_{1\xi}+W'_{1\eta}-\frac{1}{\varepsilon}(U_1V_1-W_1Z_1)\Big)=0,
\end{gathered}
$$

$$
\begin{gathered}
\varphi^{-1}\Big(Z'_{0\xi}-Z'_{0\eta}-\frac{1}{\varepsilon}(U_0V_1+U_1V_0-W_0Z_1-W_1Z_0)\Big)+\\
+\varphi^{-2}\Big(-Z_0\varphi'_\xi+\varphi'_\eta Z_0-\frac{1}{\varepsilon}(U_0V_0-W_1Z_1)\Big)+\\
+\varphi^{0}\Big(Z'_{1\xi}-Z'_{1\eta}-\frac{1}{\varepsilon}(U_1V_1-W_1Z_1)\Big)=0.
\end{gathered}
$$

The coefficients at φ^{-2} give the well-known expressions defined by (14). Assuming that $U_1=V_1=W_1=Z_1=0$, the coefficients at φ^0 are satisfied. It remains to consider at φ^{-1}. Equating each coefficient of φ^{-1} to zero, we have

$$
\begin{aligned}
U'_{0\xi}(1-\xi)-\eta U'_{0\eta}-U_0&=\frac{1}{\varepsilon}(W_0Z_1+W_1Z_0-U_0V_1-U_1V_0),\\
V'_{0\xi}(1+\xi)+\eta V'_{0\eta}+V_0&=\frac{1}{\varepsilon}(W_0Z_1+W_1Z_0-U_0V_1-U_1V_0),\\
W'_{0\xi}+W'_{0\eta}&=\frac{1}{\varepsilon}(U_0V_1+U_1V_0-W_0Z_1-W_1Z_0),\\
Z'_{0\xi}-Z'_{0\eta}&=\frac{1}{\varepsilon}(U_0V_1+U_1V_0-W_0Z_1-W_1Z_0).
\end{aligned}
\tag{16}
$$

We rewrite the system (16) as

$$U'_{0\xi}(1-\xi)-\eta U'_{0\eta}-U_0=V'_{0\xi}(1+\xi)+\eta V'_{0\eta}+V_0, \tag{17}$$
$$U'_{0\xi}(1-\xi)-\eta U'_{0\eta}-U_0=-W'_{0\xi}-W'_{0\eta}, \tag{18}$$
$$U'_{0\xi}(1-\xi)-\eta U'_{0\eta}-U_0=-Z'_{0\xi}+Z'_{0\eta}, \tag{19}$$
$$U'_{0\xi}(1-\xi)-\eta U'_{0\eta}-U_0=0. \tag{20}$$

Substituting (14) into (17), we have

$$\frac{4\eta\varepsilon(\eta\varphi_\eta^2+(-1+\eta^2+\xi^2)\varphi_\eta\varphi_\xi+\eta\xi\varphi_\xi^2)(\varphi_{\eta\eta}\varphi_\xi^2+\varphi_\eta(-2\varphi_\xi\varphi_{\xi\eta}+\varphi_\eta\varphi_{\xi\xi}))}{((-1+\eta^2)\varphi_\eta^2+2\eta\xi\varphi_\eta\varphi_\xi+\xi^2\varphi_\xi^2)^2}=0,$$

which contains one of the equations – the Bateman equation. Similarly, the equations (18), (19) also yield the given equation. Finally, substituting (14) into (20), we have condition using the Wolfram Mathematica

$$
\begin{aligned}
&-2\eta\varphi_\eta^5+\varphi_\eta^4\Big(\eta^2(-1+\eta^2)\varphi_{\eta\eta}-2\xi\varphi_\xi-2\eta\xi\varphi_{\xi\eta}+\varphi_{\xi\xi}-\eta^2\varphi_{\xi\xi}+\\
&+2\eta^2\xi\varphi_{\xi\xi}-\xi^2\varphi_{\xi\xi}-\eta^2\xi^2\varphi_{\xi\xi}\Big)+\xi\varphi_\xi^4\Big(\eta^2(2+\xi)\varphi_{\eta\eta}-\\
&-2\varphi_\xi-2\eta\varphi_{\xi\eta}+\xi\varphi_{\xi\xi}-\xi^3\varphi_{\xi\xi}\Big)-2\eta\varphi_\eta^3\varphi_\xi\Big(-2\eta^2\xi\varphi_{\eta\eta}-2\varphi_\xi-\eta\varphi_{\xi\eta}+\\
&+\eta^3\varphi_{\xi\eta}+2\eta\xi\varphi_{\xi\eta}-\eta\xi^2\varphi_{\xi\eta}+\varphi_{\xi\xi}-\eta^2\varphi_{\xi\xi}-\xi\varphi_{\xi\xi}+\eta^2\xi\varphi_{\xi\xi}-\\
&-\xi^2\varphi_{\xi\xi}+\xi^3\varphi_{\xi\xi}\Big)+\varphi_\eta^2\varphi_\xi^2\Big(\eta^2(-1+\eta^2+2\xi+5\xi^2)\varphi_{\eta\eta}+4\xi\varphi_\xi+4\eta\varphi_{\xi\eta}-\\
&-4\eta^3\varphi_{\xi\eta}-4\eta^3\xi\varphi_{\xi\eta}-4\eta\xi^2\varphi_{\xi\eta}+4\eta\xi^3\varphi_{\xi\eta}-3\varphi_{\xi\xi}+3\eta^2\varphi_{\xi\xi}+\\
&+2\eta^2\xi\varphi_{\xi\xi}+4\xi^2\varphi_{\xi\xi}-5\eta^2\xi^2\varphi_{\xi\xi}-\xi^4\varphi_{\xi\xi}\Big)+\\
&2\varphi_\eta\varphi_\xi^3\Big(\eta(1+\xi)(-1+\eta^2+\xi^2)\varphi_{\eta\eta}-\eta\varphi_\xi-\\
&-(1+\xi)\Big((\eta^2-(-1+\xi)^2)(1+\xi)\varphi_{\xi\eta}+2\eta(-1+\xi)\xi\varphi_{\xi\xi}\Big)\Big)=0.
\end{aligned}
\tag{21}
$$

Equating coefficients to zero at the same degrees, we obtain

$$\xi:-4f^3g^2(f')^2+2fg^4(f')^2+2f^4gf'g'+f^4g^2f''-f^2g^4f''=0,$$

$$\eta:-4f^3g^2f'g'+2fg^4f'g'+2f^4g(g')^2+f^4g^2g''-f^2g^4g''=0,$$

$$\xi^2:0,$$

$$
\begin{aligned}
\xi^2\eta:\ &8f^4g(f')^2+4f^5f'g'-8f^3g^2f'g'-4f^4g(g')^2-\\
&-2f^5gf''+2fg^5f''-f^6g''+f^2g^4g''=0,
\end{aligned}
$$

$$\xi^2\eta^2:0,\eta^2:0,$$

$$
\begin{aligned}
\eta^2\xi:\ &4f^3g^2(f')^2+8f^4gf'g'-4f^2g^3f'g'-8f^3g^2(g')^2-\\
&-f^4g^2f''+g^6f''-2f^5gg''+2fg^5g''=0,
\end{aligned}
$$

$$\xi^3:4f^5(f')^2-4f^4gf'g'-f^6f''+f^2g^4f''=0,$$

$$\xi^3\eta:0,$$

$$\xi^3\eta^2:-12f^3g^2(f')^2-8f^4gf'g'+12f^2g^3f'g+8f^3g^2(g')^2+$$

$$+ 6f^4g^2f'' - 6f^2g^4f'' + 4f^5gg'' - 4f^3g^3g'' = 0,$$

$$\xi^3\eta^3 : 0,$$

$$\eta^3 : 4f^3g^2f'g' - 4f^2g^3(g')^2 - f^4g^2g'' + g^6g'' = 0,$$

$$\eta^3\xi : 0,$$

$$\eta^3\xi^2 : -8f^2g^3(f')^2 - 12f^3g^2f'g' + 8fg^4f'g' + 12f^2g^3(g')^2 + \\ + 4f^3g^3f'' - 4fg^5f'' + 6f^4g^2g'' - 6f^2g^4g'' = 0,$$

$$\xi^4 : 0,$$

$$\xi^4\eta : -8f^4g(f')^2 - 2f^5f'g' + 8f^3g^2f'g' + 2f^4g(g')^2 + \\ + 4f^5gf'' - 4f^3g^3f'' + f^6g'' - f^4g^2g'' = 0,$$

$$\xi^4\eta^2 : 0, \xi^4\eta^3 : 0, \xi^4\eta^4 : 0, \eta^4 : 0,$$

$$\eta^4\xi : -2fg^4(f')^2 - 8f^2g^3f'g' + 2g^5f'g' + 8fg^4(g')^2 + \\ + f^2g^4f'' - g^6f'' + 4f^3g^3g'' - 4fg^5g'' = 0,$$

$$\eta^4\xi^2 : 0; \eta^4\xi^3 : 0; \eta^4\xi^4 : 0,$$

$$\xi^5 : -2f^5(f')^2 + 2f^4gf'g' + f^6f'' - f^4g^2f'' = 0,$$

$$\xi^5\eta : 0,$$

$$\eta^5 : -2fg^4f'g' + 2g^5(g')^2 + f^2g^4g'' - g^6g'' = 0.$$

This system of equations is satisfied for $g(\varphi) = \pm f(\varphi)$. Taking this equality into (3), we have for $g(\varphi) = f(\varphi)$

$$\varphi = F\Big(\frac{c}{\xi + \eta}\Big), \tag{22}$$

where F is an arbitrary invertible function. And finally, to get the final solution of our system, we substitute (22) in (15) and take into account the formula (6).

We can formulate a proposition.

Proposition. *A self-similar solution of (1) is*

$$u(x,y,t)=\frac{1}{x}\frac{U_0}{\varphi}, v(x,y,t)=\frac{1}{x}\frac{V_0}{\varphi}, w(x,y,t)=\frac{1}{x}\frac{W_0}{\varphi}, z(x,y,t)=\frac{1}{x}\frac{Z_0}{\varphi},$$

where U_0, V_0, W_0, Z_0 are defined by (14) and φ satisfies the two-dimensional Bateman equation (2) and (21). Solution for φ is

$$\varphi(\xi,\eta)=F\Big(\frac{c}{\xi\pm\eta}\Big), c\in\mathbb{R}.$$

Here F is an arbitrary invertible function.

Solutions of system (1) are for $g(\varphi)=f(\varphi)$

$$\begin{gathered} u(x,y,t)=0, v(x,y,t)=0,\\ w(x,y,t)=0, z(x,y,t)=-\frac{1}{x}\frac{2c\varepsilon F'(\frac{c}{\xi+\eta})}{(\xi+\eta)^2F(\frac{c}{\xi+\eta})} \end{gathered} \tag{23}$$

and for $g(\varphi)=-f(\varphi)$

$$\begin{gathered} u(x,y,t)=0, v(x,y,t)=0,\\ w(x,y,t)=-\frac{1}{x}\frac{2c\varepsilon F'(\frac{c}{\xi-\eta})}{(\xi-\eta)^2F(\frac{c}{\xi-\eta})}, z(x,y,t)=0, \end{gathered} \tag{24}$$

where $\xi=t/x, \eta=y/x$ are self-similar variables.

Comment. We give an example of a solution that is not described in the work [11]:

$$\begin{aligned} u(x,y,t)&=-\frac{14}{15}-\frac{16}{5(-\frac{71}{60}+\frac{7}{60}\sqrt{191}\tan(\frac{7}{120}\sqrt{191}(1+t+2x+3y)))},\\ v(x,y,t)&=3+\frac{48}{5(-\frac{71}{60}+\frac{7}{60}\sqrt{191}\tan(\frac{7}{120}\sqrt{191}(1+t+2x+3y)))},\\ w(x,y,t)&=2+\frac{12}{5(-\frac{71}{60}+\frac{7}{60}\sqrt{191}\tan(\frac{7}{120}\sqrt{191}(1+t+2x+3y)))},\\ z(x,y,t)&=1-\frac{24}{5(-\frac{71}{60}+\frac{7}{60}\sqrt{191}\tan(\frac{7}{120}\sqrt{191}(1+t+2x+3y)))}. \end{aligned}$$

This solution is taken from [20].

4 Conclusion

We investigated the two-dimensional Broadwell system. We found the self-similar solutions for this system using the Bateman equation.

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