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Some integrals for groups of bounded linear operators on finite-dimensional non-Archimedean Banach spaces

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Abstract. In this paper, we extend the Volkenborn integral and Shnirelman integral for groups of bounded linear operators on finite-dimensional non-Archimedean Banach spaces over $\mathbb{Q}_p$ and $\mathbb{C}_p$ respectively. When the ground field is a complete non-Archimedean valued field, which is also algebraically closed, we give some functional calculus for groups of infinitesimal generator A such that A is a nilpotent operator on finite-dimensional non-Archimedean Banach spaces.

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1 Introduction and Preliminaries

Throughout this paper, $\mathbb{K}$ is a non-Archimedean non trivially complete valued field with valuation $|\cdot|$, X is a non-Archimedean Banach space over $\mathbb{K}$, $\mathbb{Q}_p$ is the field of p-adic numbers ($p \geq 2$ being a prime) equipped with p-adic valuation $|\cdot|_p$ and $\mathbb{Z}_p$ denotes the ring of p-adic integers (the ring of p-adic integers $\mathbb{Z}_p$ is the unit ball of $\mathbb{Q}_p$). We denote the completion of algebraic closure of $\mathbb{Q}_p$ under the p-adic valuation $|\cdot|_p$ by $\mathbb{C}_p$ and $B(X)$ denotes the set of all bounded linear operators on X.
The study of Archimedean C_0-semigroup or C_0-group of bounded linear operators was first attempted by Yosida and Hille [8]. From [8], Corollary 2.5, if A is the infinitesimal generator of a C_0-semigroup then it is closed and $\overline{D(A)} = X$. By [8], (b) of Theorem 2.4:

$$\text{For } x \in X,\ t \in \mathbb{R}^+,\ \int_0^t T(s)ds \in D(A),$$

and

$$\text{for } x \in X,\ T(t)x - T(s)x = \int_s^t T(u)Axdu = \int_s^t AT(u)xdu.$$

This is thanks to the Haar measure on the topological group $(\mathbb{R}, +)$.
In the non-Archimedean analysis, there is no Haar measure on a subset of $\mathbb{Q}_p$ into $\mathbb{Q}_p$, see Theorem 5. When $\mathbb{K} = \mathbb{C}_p$, it is useful to use the Shnirelman integral defined

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as: let $f(z)$ be a $\mathbb{C}_p$-valued function defined for all $z \in \mathbb{C}_p$ such that $|z-a|_p = r$ where $a \in \mathbb{C}_p$ and $r > 0$ with $r \in |\mathbb{C}_p|_p$. Let $\Gamma \in \mathbb{C}_p$ such that $|\Gamma|_p = r$. Then the Shnirelman integral of f is defined as the following limit, if it exists,

$$\int_{a,\Gamma} f(z)dz = \lim_{n\to\infty}{}' \frac{1}{n} \sum_{\zeta^n=1} f(a+\zeta\Gamma), \tag{1}$$

where $\lim'$ indicates that the limit is taken over n such that $gcd(n,p) = 1$. For more details, we refer to [2], [4] and [9]. But there is a different results in non-Archimedean analysis, by [2], Theorem 1, we have:

$$\int_{a,\Gamma} e^z dz = e^a,$$

and

$$\int_{a,\Gamma} (z-a)e^z dz = 0.$$

Recently, Diagana [3] introduced the notion of C_0-groups of bounded linear operators on a free non-Archimedean Banach space, for more details we refer to [3] and [5]. In [5], A. El Amrani, A. Blali, J. Ettayb and M. Babahmed introduced the notions of C-groups and cosine families of bounded linear operators on non-Archimedean Banach space. Let $r > 0$, $\Omega_r = \{t \in \mathbb{K} : |t| < r\}$ [5]. We have the following definition.

Definition 1. [5] Let $r > 0$ be a real number. A one-parameter family $(T(t))_{t\in\Omega_r}$ of bounded linear operators from X into X is a group of bounded linear operators on X if

(i) $T(0) = I$, where I is the unit operator of X.

(ii) For all $t, s \in \Omega_r$, $T(t+s) = T(t)T(s)$.

The group $(T(t))_{t\in\Omega_r}$ will be called of class C_0 or strongly continuous if the following condition holds:

- For each $x \in X$, $\lim\limits_{t\to 0} \|T(t)x - x\| = 0$.

A group of bounded linear operators $(T(t))_{t\in\Omega_r}$ is uniformly continuous if and only if $\lim\limits_{t\to 0} \|T(t) - I\| = 0$.
The linear operator A defined by

$$D(A) = \{x \in X : \lim_{t\to 0} \frac{T(t)x - x}{t} \text{ exists}\},$$

and

$$Ax = \lim_{t\to 0} \frac{T(t)x - x}{t}, \text{ for each } x \in D(A),$$

is called the infinitesimal generator of the group $(T(t))_{t\in\Omega_r}$.

In this paper, we extend to Volkenborn integral and Shnirelman integral for studying the C_0-groups of bounded linear operators on some non-Archimedean Banach spaces and we show some results about it. Now, we assume that $\mathbb{K}=\mathbb{C}_p$. We have the following definition.

Definition 2. [4] Let $f(z)$ be a $\mathbb{C}_p$-valued function defined for all $z\in\mathbb{C}_p$ such that $|z-a|_p=r$ where $a\in\mathbb{C}_p$ and $r>0$ with $r\in|\mathbb{C}_p|_p$. Let $\Gamma\in\mathbb{C}_p$ such that $|\Gamma|_p=r$. Then the Shnirelman integral of f is defined as the following limit, if it exists,

$$\int_{a,\Gamma} f(z)dz = \lim_{n\to\infty}{}' \frac{1}{n}\sum_{\zeta^n=1} f(a+\zeta\Gamma),$$

where $\lim'$ indicates that the limit is taken over n such that $gcd(n,p)=1$.

Theorem 1. [1] *Let* $f(z)=\sum_{n\in\mathbb{N}} a_n f_n(z)$ *where the series on the right converges uniformly to* $f(z)$ *for all points* $z\in\mathbb{C}_p$ *such that* $|z-a|_p=|\gamma|_p$. *Suppose that for all* $n\in\mathbb{N}$, $\int_{a,\gamma} f_n(z)dz$ *exists.*
Then $\int_{a,\gamma} f(z)dz$ *exists and* $\int_{a,\gamma} f(z)dz=\sum_{n\in\mathbb{N}} a_n\int_{a,\gamma} f_n(z)dz$.

Lemma 1. [1] *Let* p *be any integer such that* $0<|p|<n$. *Then*

$$\sum_{i=1}^{n}\xi_i^{(n)p}=0.$$

Now, let $f(z)=a_0+a_1z+a_2z^2+\cdots$ be a power series converging for all $z\in\mathbb{C}_p$ such that $|z|_p<R\ \Big(R>0\Big)$, we have the following:

Theorem 2. [1] *If* $|a|_p<R$ *and* $|\gamma|_p<R$, *then*

$$\int_{a,\gamma} f(z)dz=f(a).$$

Corollary 1. [1] *With the same hypothesis as in Theorem 2, we have:*

$$\int_{a,\gamma}(z-a)f(z)dz=0.$$

Theorem 3. [1] *Let* $f(z)=\sum_{k=0}^{\infty}a_kz^k$ *be a power series converging for all* $z\in\mathbb{C}_p$ *such that* $|z|_p<R$ $(R>0)$. *Suppose that* $x,r\in\mathbb{C}_p$ *such that* $|x|_p,|r|_p<R$. *Then,*

$$\int_{0,r}\frac{zf(z)}{z-x}dz=\begin{cases} f(x) & if\ |x|_p<|r|_p,\\ 0 & if\ |x|_p>|r|_p.\end{cases}$$

Theorem 4. [1] *With the same hypothesis as in Theorem 3, we have:*

$$\int_{0,r} \frac{zf(z)}{(z-x)^{n+1}}dz = \frac{f^n(x)}{n!} \text{ for } |x|_p < |r|_p.$$

In the next, we assume that $\mathbb{K} = \mathbb{Q}_p$. There is no Newton-Leibniz formula in the p-adic analysis. There is no $\mathbb{Q}_p$-valued Lebesgue measure $\int_{\mathbb{Q}_p} f(t)dt$ is not well defined as usual.

Theorem 5. *[7] Additive, translation invariant and bounded $\mathbb{Q}_p$-valued measure on clopens of $\mathbb{Z}_p$ is the zero measure.*

We denote by $C(\mathbb{Z}_p, \mathbb{Q}_p)$ the space of all functions defined and continuous from $\mathbb{Z}_p$ into $\mathbb{Q}_p$.

Theorem 6. *[7] Let $f \in C(\mathbb{Z}_p, \mathbb{Q}_p)$. The function defined on $\mathbb{N}$ by*

$$F(0) = 0, \; F(n) = f(0) + f(1) + \cdots + f(n-1)$$

is uniformly continuous. The extended function is denoted by $Sf(x)$ (called indenite sum of f). If f is strictly differentiable, so is Sf.

We denote by $C_s^1(\mathbb{Z}_p, \mathbb{Q}_p)$ the space of all functions defined and strictly differentiable in $\mathbb{Z}_p$ taking values in $\mathbb{Q}_p$. For more details, we refer to [7].

Definition 3. [7] The Volkenborn integral of $f \in C_s^1(\mathbb{Z}_p, \mathbb{Q}_p)$ is defined by

$$\int_{\mathbb{Z}_p} f(t)dt = \lim_{n\to\infty} p^{-n} \sum_{j=0}^{p^n-1} f(j) = \lim_{n\to\infty} \frac{Sf(p^n) - Sf(0)}{p^n} = (Sf)'(0).$$

Lemma 2. *[7] For all $t \in \Omega^*_{p^{\frac{-1}{p-1}}}$,*

$$\int_{\mathbb{Z}_p} e^{tu}du = \frac{t}{e^t - 1}.$$

2 Integral for C_0-groups on finite-dimentional Banach space over $\mathbb{C}_p$

In this section, let $\mathbb{K} = \mathbb{C}_p$ and let Ω_r be the open ball of $\mathbb{K}$ centered at 0 with radius $r > 0$. We always assume that r is suitably chosen such that $t \in \Omega_r \mapsto T(t)$ is well-defined, we have the following definition.

Definition 4. Let $r > 0$ be a real number. A one-parameter family $(T(t))_{t\in\Omega_r}$ of bounded linear operators from $\mathbb{C}_p^n$ into $\mathbb{C}_p^n$ is said to be analytic group of bounded linear operators on $\mathbb{C}_p^n$ if

(i) $T(0) = I$, where I is the unit operator of $\mathbb{C}_p^n$.

(ii) For all $t, s \in \Omega_r$, $T(t+s) = T(t)T(s)$.

(iii) For all $x \in X$, $t \to T(t)x$ is analytic on Ω_r.

We extend the following definition.

Definition 5. Let $(T(t))_{t\in\Omega_r}$ be analytic group of bounded linear operators on $\mathbb{C}_p^n$. The group $(T(t))_{t\in\Omega_r}$ is said to be integrable in the sense of Schnirelman if for all $a \in \Omega_r$ and $\gamma \in \Omega_r\backslash\{0\}$, the sequence $(S_n)_n \subset B(\mathbb{C}_p^n)$ defined by

$$S_n = \sum_{\zeta^n=1} T(a+\zeta\gamma),$$

converges strongly as $n \to \infty$ $\Big($the limit is taken over n such that $gcd(n,p)=1\Big)$ to a bounded linear operator. More precisely

$$\int_{a,\gamma} T(t)dt = \lim_{n\to\infty}{}' \frac{1}{n} \sum_{\zeta^n=1} T(a+\zeta\gamma),$$

where $\lim'$ indicates that the limit is taken over n such that $gcd(n,p)=1$.

Lemma 3. *Let $(T(t))_{t\in\Omega_r}$ be analytic group on $\mathbb{C}_p^n$ such that $\int_{a,\gamma} T(t)dt$ exists and $\sup_{t\in\Omega_r}\|T(t)\|\leq M$ where $a\in\Omega_r$ and $\gamma\in\Omega_r\backslash\{0\}$. Then*

(i) For all $x \in \mathbb{C}_p^n$, $\|\int_{a,\gamma} T(t)xdt\| \leq M\|x\|$.

(ii) For all $a \in \Omega_r$, $x \in \mathbb{C}_p^n$, $\int_{a,\gamma} T(t)xdt = T(a)\int_{0,\gamma} T(t)xdt$.

Proof. Let $(T(t))_{t\in\Omega_r}$ be analytic group on $\mathbb{C}_p^n$ such that $\int_{a,\gamma} T(t)dt$ exists, then

(i) It suffices to apply Definition 5.

(ii) By Definition 5, for all $x \in \mathbb{C}_p^n$ and for each $a \in \Omega_r$, then

$$\begin{aligned}\int_{a,\gamma} T(t)xdt &= \lim_{n\to\infty}{}' \frac{1}{n} \sum_{\zeta^n=1} T(a+\zeta\gamma)x \\ &= T(a)\lim_{n\to\infty}{}' \frac{1}{n} \sum_{\zeta^n=1} T(\zeta\gamma)x \\ &= T(a)\int_{0,\gamma} T(t)xdt.\end{aligned}$$

□

Definition 6. [6] Let $A \in B(\mathbb{C}_p^n)$. A is said to be nilpotent of index d, if there is an integer number $d \leq n$ such that $A^n = 0_{\mathbb{C}_p^n}$ and $A^{n-1} \neq 0_{\mathbb{C}_p^n}$ $\Big($where $0_{\mathbb{C}_p^n}$ denotes the null operator from $\mathbb{C}_p^n$ into $\mathbb{C}_p^n\Big)$.

Example 1. Let $A \in B(\mathbb{C}_p^4)$ be defined by

$$\begin{pmatrix} 0 & a & b & c \\ 0 & 0 & a & b \\ 0 & 0 & 0 & a \\ 0 & 0 & 0 & 0 \end{pmatrix} \text{ where } a,\ b,\ c \in \mathbb{C}_p.$$

It is easy to see that A is nilpotent of index 4.

Proposition 1. *Let A be a nilpotent operator of index n on $\mathbb{C}_p^n$ such that $\|A\| < p^{\frac{-1}{p-1}}$. Then $e^{tA} = \displaystyle\sum_{k=0}^{n-1} \frac{t^k A^k}{k!}$.*

Proof. Since A is nilpotent operator of index n on $\mathbb{C}_p^n$. Then,

$$\begin{aligned} e^{tA} &= \sum_{k \in \mathbb{N}} \frac{t^k A^k}{k!} \\ &= \sum_{k=0}^{n-1} \frac{t^k A^k}{k!}. \end{aligned}$$

□

Theorem 7. *Let e^{tA} be a C_0-group of infinitesimal generator A on $\mathbb{C}_p^n$ such that A is nilpotent operator of index n on $\mathbb{C}_p^n$. Then for all $x \in \mathbb{C}_p^n$, $\displaystyle\int_{a,\gamma} e^{tA} x dt = e^{aA} x$.*

Proof. Let $e^{tA} = \displaystyle\sum_{k=0}^{n-1} \frac{t^k A^k}{k!}$. Using Proposition 1 and Theorem 2, we have for all $x \in \mathbb{C}_p^n$,

$$\begin{aligned} \int_{a,\gamma} e^{tA} x dt &= \sum_{k=0}^{n-1} \frac{A^k}{k!} \int_{a,\gamma} t^k x dt \\ &= \sum_{k=0}^{n-1} \frac{a^k A^k}{k!} x = e^{aA} x. \end{aligned}$$

□

Corollary 2. *Under the hypothesis of Theorem 7, for all* $x \in \mathbb{C}_p^n$,

$$\int_{a,\gamma} (t-a)e^{tA}xdt = 0.$$

Remark 1. Let $A \in B(\mathbb{C}_p^n)$ be a nilpotent operator, then e^{tA} is integrable in the sense of Shnirelman.

Set for all $\lambda \in \rho(A),\ R(\lambda, A) = (\lambda I - A)^{-1}$ where $\rho(A)$ is the resolvent set of the linear operator A defined on X, we have the following:

Proposition 2. *Let* $A \in B(\mathbb{C}_p^n)$. *If* A *is a nilpotent operator of index* n, *then for all* $\lambda \in \mathbb{C}_p^*$, $R(\lambda, A)$ *exists. Furthermore, for each* $\lambda \in \mathbb{C}_p^*$, *we have*

$$R(\lambda, A) = \sum_{k=0}^{n-1} \frac{A^k}{\lambda^{k+1}}.$$

Proof. Let $\lambda \in \mathbb{C}_p^*$, then

$$\begin{aligned}(\lambda I - A)\Big(\sum_{k=0}^{n-1} \frac{A^k}{\lambda^{k+1}}\Big) &= \sum_{k=0}^{n-1} \frac{A^k}{\lambda^k} - \sum_{k=0}^{n-1} \frac{A^{k+1}}{\lambda^{k+1}} \\ &= I.\end{aligned}$$

On the other hand,

$$\begin{aligned}\Big(\sum_{k=0}^{n-1} \frac{A^k}{\lambda^{k+1}}\Big)(\lambda I - A) &= \sum_{k=0}^{n-1} \frac{A^k}{\lambda^k} - \sum_{k=0}^{n-1} \frac{A^{k+1}}{\lambda^{k+1}} \\ &= I.\end{aligned}$$

Consequently, for all $\lambda \in \mathbb{C}_p^*,\ R(\lambda, A) = \sum_{k=0}^{n-1} \frac{A^k}{\lambda^{k+1}}$. □

Proposition 3. *Let* A *be a nilpotent operator of index* n *on* $\mathbb{C}_p^n$ *and* $r = \frac{-1}{p-1}$. *Then*

$$\text{for all } t \in \Omega_r,\ e^{tA} = \int_{0,\gamma} \lambda e^{\lambda t} R(\lambda, A) d\lambda, \text{ where } \gamma \in \Omega_r \backslash \{0\}.$$

Proof. By Proposition 2, for all $\lambda \in \Omega_{\frac{-1}{p-1}} \backslash \{0\}$, $R(\lambda, A)$ has a polynomial function form on $\mathbb{C}_p^n$, hence it is analytic on $\Omega_{\frac{-1}{p-1}} \backslash \{0\}$. Using Theorem 4, we obtain

$$\begin{aligned}\int_{0,\gamma} \lambda e^{\lambda t} R(\lambda, A) &= \int_{0,\gamma} \sum_{k=0}^{n-1} \lambda^{-k} e^{t\lambda} A^k d\lambda \\ &= \sum_{k=0}^{n-1} A^k \int_{0,\gamma} \lambda^{-k} e^{t\lambda} d\lambda\end{aligned}$$

$$
\begin{aligned}
&= \sum_{k=0}^{n-1} A^k \frac{\left(e^{t\lambda}\right)^{(k)}(0)}{k!} \\
&= \sum_{k=0}^{n-1} A^k \frac{t^k}{k!} = e^{tA}.
\end{aligned}
$$

□

We have the following proposition.

Proposition 4. *Let A and B be nilpotent operators on $\mathbb{C}_p^n$ and let e^{tA} and e^{tB} be two C_0-groups of infinitesimal generators A and B respectively. If $R(\lambda, A)$ and $R(\lambda, B)$ commute, then e^{tA} and e^{tB} commute.*

Proof. By Proposition 3, we have

$$e^{tA} = \int_{0,\gamma} \lambda e^{\lambda t}(\lambda I - A)^{-1} d\lambda \text{ and } e^{tB} = \int_{0,\gamma} \lambda e^{\lambda t}(\lambda I - B)^{-1} d\lambda.$$

Asumme that $R(\lambda, A)$ and $R(\lambda, B)$ commute, then

$$
\begin{aligned}
e^{tA}e^{tB} &= \int_{0,\gamma} \lambda e^{\lambda t}(\lambda I - A)^{-1} d\lambda \int_{0,\gamma} \lambda e^{\lambda t}(\lambda I - B)^{-1} d\lambda \\
&= \int_{0,\gamma}\int_{0,\gamma} \lambda e^{\lambda t}(\lambda I - A)^{-1} \lambda e^{\lambda t}(\lambda I - B)^{-1} d\lambda d\lambda \\
&= \int_{0,\gamma}\int_{0,\gamma} \lambda e^{\lambda t}(\lambda I - B)^{-1} \lambda e^{\lambda t}(\lambda I - A)^{-1} d\lambda d\lambda \\
&= e^{tB}e^{tA}.
\end{aligned}
$$

□

We have the following:

Proposition 5. *Let A and $(A_k)_{k\in\mathbb{N}}$ be nilpotent operators on $\mathbb{C}_p^n$. If, $R(\lambda, A_k) \to R(\lambda, A)$ as $k \to \infty$, then e^{tA_k} converges to e^{tA} as $k \to \infty$.*

Proof. From Proposition 3, we have

$$\text{for all } t \in \Omega_r,\ e^{tA} = \int_{0,\gamma} \lambda e^{\lambda t} R(\lambda, A) d\lambda,$$

where $\gamma \in \Omega_r \backslash \{0\}$ and $r = \frac{-1}{p-1}$ and

$$\text{for all } t \in \Omega_r,\ k \in \mathbb{N},\ \ e^{tA_k} = \int_{0,\gamma} \lambda e^{\lambda t} R(\lambda, A_k) d\lambda.$$

Moreover,

$$e^{tA_k} - e^{tA} = \int_{0,\gamma} \lambda e^{t\lambda}[R(\lambda, A_k) - R(\lambda, A)]d\lambda$$

is well-defined. Set

$$M = \max_{|\lambda|_p = |\gamma|_p} |\lambda e^{t\lambda}|_p < \infty.$$

Since $R(\lambda, A_k) \to R(\lambda, A)$ as $k \to \infty$, it follows that for each $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that for all $k \geq N$, $\|R(\lambda, A_k) - R(\lambda, A)\| \leq \frac{\varepsilon}{M}$. Consequently

$$\begin{aligned} \|e^{tA_k} - e^{tA}\| &\leq \left\| \int_{0,\gamma} \lambda e^{t\lambda}[R(\lambda, A_k) - R(\lambda, A)]d\lambda \right\| \\ &\leq \max_{|\lambda|_p = |\gamma|_p} |\lambda e^{t\lambda}|_p \|R(\lambda, A_k) - R(\lambda, A)\| \\ &\leq M \cdot \frac{\varepsilon}{M} \\ &= \varepsilon, \end{aligned}$$

whenever $k \geq N$, then e^{tA_k} converges to e^{tA} as $k \to \infty$. □

3 Integral of groups of linear operators on $\mathbb{Q}_p^n$

From now on we assume that $\mathbb{K} = \mathbb{Q}_p$, we extend the Volkenborn integral to some non-Archimedean Banach spaces.

Definition 7. Let $f \in C_s^1(\mathbb{Z}_p, \mathbb{Q}_p^n)$. The sequence $(S_m)_m \subset B(\mathbb{Q}_p^n)$ defined by

$$S_m = p^{-m} \sum_{j=0}^{p^m - 1} f(j),$$

converges strongly as $m \to \infty$ to a bounded linear operator. More precisely

$$\int_{\mathbb{Z}_p} f(t)dt = \lim_{m \to \infty} p^{-m} \sum_{j=0}^{p^m - 1} f(j).$$

Set $B_r(\mathbb{Q}_p^n) = \{A \in B(\mathbb{Q}_p^n) : 0 < \|A\| < r\}$ where $r = p^{\frac{-1}{p-1}}$.

Proposition 6. *Let $A \in B_r(\mathbb{Q}_p^n)$ be invertible diagonal operator, then $(e^{tA})_{t \in \mathbb{Z}_p}$ is C^1 function and $(e^A - I)^{-1} \in B(\mathbb{Q}_p^n)$.*

Proof. Let $A \in B_r(\mathbb{Q}_p^n)$ be invertible diagonal operator, then

$$\text{for all } i \in \{1, \cdots, n\},\ Ae_i = a_i e_i,$$

where $a_i \in \mathbb{Q}_p^*$ such that $|a_i|_p < r$ and $(e_i)_{1 \leq i \leq n}$ is the canonical basis of $\mathbb{Q}_p^n$. Hence, for all $t \in \Omega_r$, e^{tA} exists and is given by

$$\text{for all } i \in \{1, \cdots, n\},\ e^{tA}e_i = e^{ta_i}e_i.$$

Hence e^{tA} is C^∞ that is C^1. Moreover,

$$\text{for all } i \in \{1, \cdots, n\},\ \Big(e^A - I\Big)e_i = \Big(e^{a_i} - 1\Big)e_i.$$

We have for all $i \in \{1, \cdots, n\},\ 1 - e^{a_i} \neq 0$. Consequently, $\det(e^A - I) \neq 0$, then $e^A - I$ is invertible. Moreover,

$$\text{for all } i \in \{1, \cdots, n\},\ \Big(e^A - I\Big)^{-1} e_i = \Big(\tfrac{1}{e^{a_i}-1}\Big)e_i.$$

Hence $\|(e^A - I)^{-1}\| = \sup_{1\le i\le n} \Big|\dfrac{1}{e^{a_i} - 1}\Big|_p = \dfrac{1}{\inf\limits_{1\le i\le n} |e^{a_i} - 1|_p} < \infty$. Consequently, $(e^A - I)^{-1} \in B(\mathbb{Q}_p^n)$. □

Proposition 7. *Let $A \in B_r(\mathbb{Q}_p^n)$ be invertible diagonal operator such that $\int_{\mathbb{Z}_p} e^{tA}dt$ exists. Then for all $x \in \mathbb{Q}_p^n$, $(e^A - I)\int_{\mathbb{Z}_p} e^{tA}xdt = Ax$.*

Proof. Let $A \in B_r(\mathbb{Q}_p^n)$ be invertible diagonal operator. By Proposition 6, the C_0-group $(e^{tA})_{t\in\mathbb{Z}_p}$ is locally analytic function and $(e^A - I)^{-1} \in B(\mathbb{Q}_p^n)$. Let $x \in \mathbb{Q}_p^n$, set $S_m x = p^{-m}\sum\limits_{j=0}^{p^m-1} e^{jA}x$. Hence for all $x \in \mathbb{Q}_p^n$, we have

$$\begin{aligned}(e^A - I)S_m x &= S_m(e^A - I)x \\ &= \frac{e^{p^m A}x - x}{p^m}.\end{aligned}$$

By assumption, for all $x \in \mathbb{Q}_p^n$, we have

$$\int_{\mathbb{Z}_p} e^{tA}xdt = \lim_{m\to\infty} S_m x.$$

Then, for all $x \in \mathbb{Q}_p^n$, we have

$$\begin{aligned}(e^A - I)\int_{\mathbb{Z}_p} e^{tA}xdt &= (e^A - I)\lim_{m\to\infty} S_m x \\ &= \lim_{m\to\infty} \frac{e^{p^m A}x - x}{p^m} \\ &= Ax.\end{aligned}$$

□

Example 2. Let $r = \frac{-1}{p-1}$ and let $A \in B(\mathbb{Q}_p^2)$ defined by

$$\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \text{ where } a,\ b \in \Omega_r^*.$$

Then, for all $t \in \mathbb{Z}_p$, we have

$$e^{tA} = \begin{pmatrix} e^{at} & 0 \\ 0 & e^{bt} \end{pmatrix}.$$

Hence,

$$\int_{\mathbb{Z}_p} e^{tA} dt = \begin{pmatrix} \int_{\mathbb{Z}_p} e^{at} dt & 0 \\ 0 & \int_{\mathbb{Z}_p} e^{bt} dt \end{pmatrix}.$$

Thus, for all $x = \begin{pmatrix} u \\ v \end{pmatrix} \in \mathbb{Q}_p^2$, we have

$$\begin{aligned} \int_{\mathbb{Z}_p} e^{tA} x dt &= \begin{pmatrix} \frac{a}{e^a-1} & 0 \\ 0 & \frac{b}{e^b-1} \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} \\ &= \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \cdot \begin{pmatrix} \frac{1}{e^a-1} & 0 \\ 0 & \frac{1}{e^b-1} \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} \\ &= (e^A - I)^{-1} A x. \end{aligned}$$

Definition 8. Let $A \in B(\mathbb{Q}_p^n)$. A is said to be scalar multiple of identity operator on $\mathbb{Q}_p^n$, if $A = aI$ for some $a \in \mathbb{Q}_p$ and I is the identity operator on $\mathbb{Q}_p^n$.

Example 3. Let A be invertible scalar multiple of identity operator on $\mathbb{Q}_p^n$ such that $A = aI$, where $a \in \Omega_r^*$ with $r = \frac{-1}{p-1}$. Hence for all $t \in \mathbb{Z}_p$, $T(t) = e^{ta} I$, then for all $x \in \mathbb{Q}_p^n$ and $a \in \Omega_r^*$, we have

$$\int_{\mathbb{Z}_p} T(tu) x du = \frac{a}{e^a - 1} x = (T(1) - I)^{-1} A x. \tag{2}$$

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