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Optimal control of jump-diffusion processes with random parameters

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Abstract. Let $X(t)$ be a controlled jump-diffusion process starting at $x \in [a, b]$ and whose infinitesimal parameters vary according to a continuous-time Markov chain. The aim is to minimize the expected value of a cost function with quadratic control costs until $X(t)$ leaves the interval (a, b), and a termination cost that depends on the final value of $X(t)$. Exact and explicit solutions are obtained for important processes.

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1 Introduction

Let $\{Y(t), t \geq 0\}$ be a continuous-time Markov chain with state space $E = \{1, 2, \ldots, k\}$. We consider the two-dimensional process $\{(X(t), Y(t)), t \geq 0\}$, where $X(t)$ is defined by the stochastic differential equation

$$\mathrm{d}X(t) = \mu[X(t), Y(t)]\,\mathrm{d}t + b_0\,u[X(t), Y(t)]\,\mathrm{d}t + \sigma[X(t), Y(t)]\,\mathrm{d}B(t) + \epsilon\,\mathrm{d}N(t), \quad (1)$$

in which b_0 and ϵ are positive constants, $\{B(t), t \geq 0\}$ is a standard Brownian motion and $\{N(t), t \geq 0\}$ is a time-homogeneous Poisson process with rate $\lambda > 0$. That is, $\{X(t), t \geq 0\}$ is a controlled jump-diffusion process with random infinitesimal mean $\mu(\cdot, \cdot)$ and variance $\sigma^2(\cdot, \cdot)$. We assume that $\{B(t), t \geq 0\}$ and $\{N(t), t \geq 0\}$ are independent processes.

Jump-diffusion processes are very important in mathematical finance, where they are used as models for the evolution of stock or commodity prices. Moreover, because of frequent regime changes, the fact that the parameters $\mu(\cdot, \cdot)$ and $\sigma^2(\cdot, \cdot)$ are random is more realistic.

In this paper, we are looking for the control $u^*(x, i)$ that minimizes the expected value of the cost function

$$J(x, i) := \int_0^{T(x,i)} \left\{ \frac{1}{2} q_{0,i}\, u^2[X(t), i] + \theta_i \right\} \mathrm{d}t + K_i[X(T(x, i))], \quad (2)$$

where $q_{0,i} > 0$, $\theta_i \in \mathbb{R}$ and $T(x, i)$ is the first-passage time

$$T(x, i) = \inf\{t \geq 0 : X(t) \notin (a, b) \mid X(0) = x \in [a, b], Y(0) = i\}, \quad (3)$$

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for $i = 1, 2, \dots, k$. If the constant θ_i is positive (respectively, negative), then the optimizer wants the process to leave the continuation region as soon (respectively, late) as possible. Furthermore, we assume that the termination cost is of the form

$$K_i[X(T(x,i))] = a_i X^2(T(x,i)) + b_i X(T(x,i)) + k_i, \tag{4}$$

where a_i, b_i and k_i are constants, for $i = 1, 2, \dots, k$. Depending on the values of these constants (and the other parameters in the problem), the aim can be to try to leave the interval (a, b) through a rather than b, or vice versa.

The problem set up above is an extension of the so-called *LQG homing* problem studied by Whittle [7] for n-dimensional diffusion processes. He showed that it is sometimes possible to obtain the exact optimal control by computing a mathematical expectation for the corresponding uncontrolled process. Lefebvre [3] extended Whittle's results to the case of one-dimensional jump-diffusion processes with deterministic infinitesimal parameters. The optimal control then becomes approximate, rather than exact. At any rate, even if one is able to reduce the stochastic optimal control problem to a purely probabilistic problem, computing the mathematical expectation needed to obtain the optimal control is usually very difficult, especially in two or more dimensions.

For applications of LQG homing problems, see in particular Ionescu *et al.* [1] and [2], as well as Lefebvre [4] and [5]. See also Makasu [6] for the solution to a two-dimensional problem.

In the next section, we will give the system of non-linear differential-difference equations that we must solve to determine the optimal controls $u^*(x, i)$, for $i = 1, 2, \dots, k$. In Section 3, exact solutions to particular problems for important processes will be found explicitly.

2 Dynamic programming

We define the value function

$$F(x,i) = \inf_{u[X(t),i],\ 0 \le t \le T(x,i)} E[J(x,i)], \tag{5}$$

for $i = 1, 2, \dots, k$. To obtain the dynamic programming equation satisfied by the function $F(x, i)$, we will use the following results: first, by definition, a Poisson process starts at $N(0) = 0$, and the number $N(t)$ of events in the interval $(0, t]$ has a Poisson distribution with parameter λt, which implies that

$$P[N(\Delta t) = 0] = e^{-\lambda \Delta t} = 1 - \lambda \Delta t + o(\Delta t) \tag{6}$$

and

$$P[N(\Delta t) = 1] = \lambda \Delta t e^{-\lambda \Delta t} = \lambda \Delta t + o(\Delta t). \tag{7}$$

Next, we assume that $B(0) = 0$; then, as is well known, we can write that

$$E[B(\Delta t)] = 0 \tag{8}$$

and

$$E[B^2(\Delta t)] = V[B(\Delta t)] = \Delta t. \tag{9}$$

Finally, in the case of the continuous-time Markov chain $\{Y(t), t \geq 0\}$, when it enters state i, it remains there for a random time τ_i having an exponential distribution with parameter denoted by ν_i. Then, it will move to state $j \neq i$ with probability $p_{i,j}$, with $\sum_{j \neq i} p_{i,j} = 1$. Therefore, when $Y(0) = i$,

$$P[Y(\Delta t) = i] = P[\tau_i > \Delta t] = e^{-\nu_i \Delta t} = 1 - \nu_i \Delta t + o(\Delta t) \tag{10}$$

and

$$P[Y(\Delta t) = j] = \left(1 - e^{-\nu_i \Delta t}\right) p_{i,j} = \nu_i p_{i,j} \Delta t + o(\Delta t), \tag{11}$$

for $j \neq i$.

Using the standard arguments, we obtain the following dynamic programming equation (DPE):

$$\begin{aligned} 0 \quad = \quad \inf_{u(x,i)} \Bigg\{ & \frac{1}{2} q_{0,i} u^2(x,i) + \theta_i + [\mu(x,i) + b_0 u(x,i)] F'(x,i) \\ & + \frac{1}{2}\sigma^2(x,i) F''(x,i) + \sum_{j \neq i} \nu_i p_{i,j} \left[F(x,j) - F(x,i)\right] \\ & + \lambda \left[F(x+\epsilon, i) - F(x,i)\right] \Bigg\}. \end{aligned} \tag{12}$$

From Eq. (12), we deduce at once that, in terms of $F(x,i)$, the optimal control $u^*(x,i)$ is

$$u^*(x,i) = -\frac{b_0}{q_{0,i}} F'(x,i). \tag{13}$$

We can now state the following proposition, obtained by substituting the above expression into the DPE.

Proposition 2.1. *The value functions $F(x,i)$, $i = 1, \ldots, k$, satisfy the system of non-linear second-order differential-difference equations*

$$\begin{aligned} 0 \quad = \quad & \theta_i + \mu(x,i) F'(x,i) - \frac{1}{2}\frac{b_0^2}{q_{0,i}} \left[F'(x,i)\right]^2 \\ & + \frac{1}{2}\sigma^2(x,i) F''(x,i) + \sum_{j \neq i} \nu_i p_{ij} \left[F(x,j) - F(x,i)\right] \\ & + \lambda \left[F(x+\epsilon, i) - F(x,i)\right]. \end{aligned} \tag{14}$$

The system is valid for $a < x < b$. Moreover, because the jump size is a positive constant ϵ, the boundary conditions are

$$F(a,i) = K_i(a) \quad \textit{and} \quad F(x,i) = K_i(x) \quad \textit{if } b \leq x < b + \epsilon. \tag{15}$$

In the next section, we will find exact solutions to the above system in important particular cases.

3 Particular cases

For the sake of simplicity, we assume that the state space of the Markov chain $\{Y(t), t \geq 0\}$ is the set $\{1, 2\}$; that is, $k = 2$ in Proposition 2.1. Then, we have that $p_{i,j} = 1$.

First, making use of Taylor's formula, we can write that

$$F(x+\epsilon, i) = F(x,i) + \epsilon F'(x,i) + \frac{1}{2}\epsilon^2 F''(x,i) + o(\epsilon^2), \tag{16}$$

which implies that the system (14) can be rewritten as follows:

$$\begin{aligned} 0 \quad \approx \quad & \theta_i + \mu(x,i) F'(x,i) - c_i^2 \left[F'(x,i)\right]^2 + \frac{1}{2}\sigma^2(x,i) F''(x,i) \\ & + \nu_i \left[F(x,j) - F(x,i)\right] + \lambda \left[\epsilon F'(x,i) + \frac{1}{2}\epsilon^2 F''(x,i)\right] \end{aligned} \tag{17}$$

for $i = 1, 2$, where

$$c_i^2 := \frac{1}{2} \frac{b_0^2}{q_{0,i}}. \tag{18}$$

If ϵ is small, then the solution to the above system should be a good approximation to the exact solution that we are looking for. Furthermore, if $F(x,i)$ is a polynomial of degree 1 or 2, then the solution to (17) is actually the *exact* solution to our problem.

Case I. Assume that the interval $[a,b]$ is $[0,1]$, and that $\mu(x,i) \equiv \mu_i \in \mathbb{R}$, for $i = 1, 2$. If $\sigma^2(x,i) \equiv \sigma_i^2$, then the continuous part of the process $\{X(t), t \geq 0\}$ is a Wiener process with random infinitesimal parameters. The Wiener process is surely among the most important diffusion processes.

Suppose that

$$K_i[X(T(x,i))] = b_i X(T(x,i)) + k_i, \tag{19}$$

where $b_i \neq 0$, for $i = 1, 2$. So, we take $a_i = 0$ in Eq. (4). The boundary conditions are therefore

$$F(0,i) = k_i \quad \text{and} \quad F(x,i) = b_i x + k_i \quad \text{if } 1 \leq x < 1 + \epsilon. \tag{20}$$

Hence, in general, the optimizer should try to make the controlled process leave the interval $(0,1)$ through the origin, so that we expect $u^*(x,i)$ to be negative. However the sign of the optimal control also depends, in particular, on the value of θ_i. If $\theta_i > 0$ is large and x is close to 1, it might be better to leave the interval through $x \geq 1$ and accept the larger termination cost.

Now, let us try a value function $F(x,i)$ of the same form as $K_i(x)$. Substituting this expression into the system (17), we find that

$$0 \quad = \quad \theta_1 + \mu_1 b_1 - c_1^2 b_1^2 + \nu_1 \left[(b_2 - b_1)x + (k_2 - k_1)\right] + \lambda \epsilon b_1, \tag{21}$$

$$0 \quad = \quad \theta_2 + \mu_2 b_2 - c_2^2 b_2^2 - \nu_2 \left[(b_2 - b_1)x + (k_2 - k_1)\right] + \lambda \epsilon b_2. \tag{22}$$

We deduce from the above equations that a necessary condition for the solution to be valid is that we must have $b_1 = b_2$, so that the constants $b_1 \neq 0$, k_1 and k_2 must be such that

$$0 = \theta_1 + \mu_1 b_1 - c_1^2 b_1^2 + \nu_1 (k_2 - k_1) + \lambda \epsilon b_1, \tag{23}$$
$$0 = \theta_2 + \mu_2 b_1 - c_2^2 b_1^2 - \nu_2 (k_2 - k_1) + \lambda \epsilon b_1. \tag{24}$$

Let us choose the following parameters:

$$\mu_1 = -1,\ \mu_2 = 0,\ \lambda = \epsilon = \theta_i = \nu_i = b_0 = q_{0,i} = 1, \text{ for } i = 1, 2.$$

Then, one can check that the system (23), (24) is satisfied if $b_1 = 2$, $k_1 = 0$ and $k_2 = 1$. Thus, we have that

$$F(x, 1) = 2x \quad \text{and} \quad F(x, 2) = 2x + 1 \quad \text{if } 0 < x < 1. \tag{25}$$

Furthermore, the functions $F(x, i)$ satisfy the boundary conditions in (20) with $b_1 = b_2 = 2$, for $i = 1, 2$.

From Eq. (13), we obtain that the optimal control is given by

$$u^*(x, 1) = u^*(x, 2) \equiv -2. \tag{26}$$

For other choices of the parameters $q_{0,1}$ and $q_{0,2}$, $u^*(x, 1)$ and $u^*(x, 2)$ could be different, but they are always constant in this first example.

Remarks. (i) If instead of $\mu_2 = 0$ above, we rather have $\mu_2 = -2$, then we find that the system is satisfied if $b_1 = -2$ (together with $k_1 = 0$, $k_2 = 1$). Therefore, we have a second explicit solution to the problem considered. Moreover, notice that the optimal control $u^*(x, i)$ would then be positive.

(ii) Since the solution to our problem does not depend on $\sigma_1^2(x, i)$ and $\sigma_2^2(x, i)$, it is valid, in particular, in the case of a Wiener process with random parameters and Poissonian jumps, as mentioned above.

(iii) We can easily find other particular solutions. For instance, if $\mu_1 = 0$, $\mu_2 = 1/3$, $k_1 = 0$ and $k_2 = 1/2$, then $b_1 = 3$, etc.

Case II. Assume again that the continuation region is the interval $(0, 1)$. This time, we take $\mu(x, i) = -\gamma_i x$, for $i = 1, 2$. If the constant γ_i is positive and if $\sigma^2(x, i) \equiv \sigma_i^2$, $\{X(t), t \geq 0\}$ is then an Ornstein-Uhlenbeck process with random parameters and Poissonian jumps. The Ornstein-Uhlenbeck process is also among the most important diffusion processes for applications.

We choose the termination cost function in Eq. (19), and we try a solution $F(x, i) = K_i(x)$ of the system (17). We obtain the following system:

$$0 = \theta_1 - \gamma_1 b_1 x - c_1^2 b_1^2 + \nu_1 \left[(b_2 - b_1)x + (k_2 - k_1)\right] + \lambda \epsilon b_1, \tag{27}$$
$$0 = \theta_2 - \gamma_2 b_2 x - c_2^2 b_2^2 - \nu_2 \left[(b_2 - b_1)x + (k_2 - k_1)\right] + \lambda \epsilon b_2. \tag{28}$$

Therefore, we must have (for the terms in x)

$$0 = -\gamma_1 b_1 + \nu_1 (b_2 - b_1), \tag{29}$$

$$0 = -\gamma_2 b_2 - \nu_2 (b_2 - b_1) \tag{30}$$

and (for the constant terms)

$$\begin{aligned} 0 &= \theta_1 - c_1^2 b_1^2 + \nu_1 (k_2 - k_1) + \lambda \epsilon b_1, &(31) \\ 0 &= \theta_2 - c_2^2 b_2^2 - \nu_2 (k_2 - k_1) + \lambda \epsilon b_2. &(32) \end{aligned}$$

Let us choose the parameters $\gamma_1 = 1$, $\gamma_2 = -1/2$, $\lambda = \epsilon = \nu_1 = \nu_2 = b_0 = 1$, $\theta_1 = -1/2$, $\theta_2 = -1$, $q_{0,1} = 1$ and $q_{0,2} = 2$. We find that a solution of the above systems (that also satisfies the appropriate boundary conditions) is

$$F(x,1) = x + k_1 \quad \text{and} \quad F(x,2) = 2x + k_2 \quad \text{if } 0 < x < 1, \tag{33}$$

for any choice of the constants k_1 and k_2 such that

$$k_2 - k_1 = 0. \tag{34}$$

Furthermore, the optimal control is

$$u^*(x,1) = u^*(x,2) \equiv -1. \tag{35}$$

Case III. Finally, we take $\mu(x,i) = \mu_i x$, where $\mu_i \in \mathbb{R}$, and $\sigma^2(x,i) = \sigma_i^2 x^2$, for $i = 1, 2$. Therefore, the continuous part of the process $\{X(t), t \geq 0\}$ is a geometric Brownian motion, which is widely used in financial mathematics. Because this diffusion process is always positive (if it starts at $X(0) > 0$), we assume that $a > 0$ in the interval $[a, b]$. We choose the interval $[1, 2]$ and the termination cost function in (4), with $a_1 = a_2$ and $b_1 = b_2$. The boundary conditions are

$$F(1,i) = a_1 + b_1 + k_i \quad \text{and} \quad F(x,i) = a_1 x^2 + b_1 x + k_i \quad \text{if } 2 \leq x < 2 + \epsilon. \tag{36}$$

Proceeding as in the previous cases, we try a solution of the same form as the function $K_i(\cdot)$. We then obtain the system

$$\begin{aligned} 0 &= \theta_1 + \mu_1 x (2a_1 x + b_1) - c_1^2 (2a_1 x + b_1)^2 + \sigma_1^2 x^2 a_1 \\ &\quad + \nu_1 (k_2 - k_1) + \lambda \left[\epsilon (2a_1 x + b_1) + \epsilon^2 a_1\right], &(37) \\ 0 &= \theta_2 + \mu_2 x (2a_1 x + b_1) - c_2^2 (2a_1 x + b_1)^2 + \sigma_2^2 x^2 a_1 \\ &\quad - \nu_2 (k_2 - k_1) + \lambda \left[\epsilon (2a_1 x + b_1) + \epsilon^2 a_1\right]. &(38) \end{aligned}$$

For the sake of simplicity, let us choose $\lambda = \epsilon = 1$. We then deduce that we must have (for the terms in x^2)

$$\begin{aligned} 0 &= 2\mu_1 a_1 - 4c_1^2 a_1^2 + \sigma_1^2 a_1, &(39) \\ 0 &= 2\mu_2 a_1 - 4c_2^2 a_1^2 + \sigma_2^2 a_1, &(40) \end{aligned}$$

(for the terms in x)

$$0 = \mu_1 b_1 - 4c_1^2 a_1 b_1 + 2a_1, \tag{41}$$

$$0 = \mu_2 b_1 - 4c_2^2 a_1 b_1 + 2a_1 \tag{42}$$

and (for the constant terms)

$$0 = \theta_1 - c_1^2 b_1^2 + \nu_1(k_2 - k_1) + b_1 + a_1, \tag{43}$$

$$0 = \theta_2 - c_2^2 b_1^2 - \nu_2(k_2 - k_1) + b_1 + a_1. \tag{44}$$

We can check that the function

$$F(x,i) = x^2 + x + k_i \quad \text{for } 1 < x < 2 \tag{45}$$

is a solution to our problem if

$$\mu_1 = 0,\ \mu_2 = -1,\ \sigma_1^2 = 2,\ \sigma_2^2 = 3,\ b_0 = \nu_1 = \nu_2 = q_{0,1} = 1$$

and $q_{0,2} = 2$, together with

$$\theta_1 = -\frac{3}{2} - (k_2 - k_1) \quad \text{and} \quad \theta_2 = -\frac{7}{4} + (k_2 - k_1). \tag{46}$$

It follows that the optimal controls are affine functions of x:

$$u^*(x,1) = -(2x+1) \quad \text{and} \quad u^*(x,2) = -\frac{1}{2}(2x+1). \tag{47}$$

4 Conclusion

In this paper, we considered a difficult problem, namely an optimal control problem for jump-diffusion processes with random parameters, when in addition the final time is a first-passage time random variable. The aim was to obtain exact and explicit solutions to such problems.

In Section 3, we were able to solve three particular problems for very important diffusion processes. Wiener processes, Ornstein-Uhlenbeck processes and geometric Brownian motions appear in numerous applications.

For the discrete part of the jump-diffusion processes, we assumed that jumps occurred according to a time-homogeneous Poisson process and that the jump size was a positive constant. This enabled us, making use of Taylor's formula, to transform a system of differential-difference equations into an approximate system of differential equations. However, this approximate system becomes an exact one in the case when the value function is a polynomial of degree equal to 1 or 2.

It would be interesting to try to generalize the results obtained in this paper to the case of a random jump size. We could also assume that there can be both positive and negative jumps that are generated by two independent Poisson processes.

Finally, as mentioned above, the aim was to obtain analytical solutions to the problem set up in Section 1. When the state space E of the Markov chain contains many values, it should at least be possible to use numerical methods to determine the value functions and the optimal controls.

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