BULETINUL ACADEMIEI DE ŞTIINŢE
A REPUBLICII MOLDOVA. MATEMATICA
Number 3(100), 2022, Pages 15–21
ISSN 1024–7696, E-ISSN 2587–4322

A fixed point theorem for p-contraction mappings in partially ordered metric spaces and application to ordinary differential equations

Ahmed Chaouki Aouine

Abstract. In this paper, we prove a fixed point theorem for p-contraction mappings in partially ordered metric spaces. As an application, we investigate the possibility of optimally controlling the solution of the ordinary differential equations.

Mathematics subject classification: 47H10, 54H25.
Keywords and phrases: Fixed point, p-contraction type maps, partially ordered metric spaces, ordinary differential equation.

1 Introduction and Preliminaries

The applications of fixed point theorems are very important in diverse disciplines of mathematics, statistics, engineering and economics in dealing with problems arising in: approximation theory, potential theory, game theory, mathematical economics, theory of differential equations, theory of integral equations, etc.

In this paper, we prove a fixed point theorem for p-contraction mappings in partially ordered metric spaces and we apply this theorem to ordinary differential equation. For this aim we need the following definitions.
First of all, we define the fixed point of mapping A.

Definition 1. [2] Let $A, S : X \to X$ be two mappings. A point $u \in X$ is said to be

i) a fixed point of A if $Au = u$,

ii) a coincidence point of A and S if $Au = Su$. The point $z = Au = Su$ is called a point of coincidence of A and S.

iii) a common fixed point of A and S if $Au = Su = u$.

iv) A and S are weakly compatible iff they commute at their coincidence point.

Also, we must mention the famous Banach contraction.

Definition 2. [4] A mapping $T : X \longrightarrow X$ is said to be a Banach contraction mapping if it satisfies the following inequality:

$$d(T(x), T(y)) \leq \lambda d(x, y),$$

for all $x, y \in X$, where $0 < \lambda < 1$. It is well known that a Banach contraction mapping T on a complete metric space X has a unique fixed point.
Let X be a topological space and $Y \subset X$ be equipped with relativized topology.

DOI: https://doi.org/10.56415/basm.y2022.i3.p15

Definition 3. A mapping $T : Y \subset X \longrightarrow X$ is said to be a weak topological contraction if Y is T-invariant and T is continuous and closed such that for each non-empty closed subset A of Y with $T(A) = A$, A is a singleton set. Further, if the diameter $\delta(T^n(Y)) \to 0$ as $n \to \infty$ then the mapping T is said to be a strong topological contraction.

Remark 1. If X is a bounded metric space (i.e., $\delta(X)$, the diameter of X, is finite) and T is a Banach contraction, then clearly T is a weak topological contraction. In 2008, H. K. Pathak and N. Shahzad introduced in the following definition the notion of p-contraction which is more general than the Banach contraction principle.

Definition 4. [8] Let (X, d) be a metric space. A mapping $T : Y \subset X \longrightarrow X$ is said to be a metric p-contraction (or simply p-contraction) mapping if Y is T-invariant and it satisfies the following inequality:

$$d(T(x), T^2(x)) \leq p(x)d(x, T(x)) \tag{1}$$

for all x in Y, where $p : Y \longrightarrow [0, 1]$ is a function such that $p(x) < 1$ for all $x \in Y$ and $\sup_{x\in Y} p(Tx) = \alpha < 1$. Further, if $\cap_{n=0}^{\infty} T^n(Y)$ is a singleton set, where $T^n(Y) = T(T^{n-1}(Y))$ for each $n \in \mathbb{N}$ and $T^0(Y) = Y$, then T is said to be a strong p-contraction.

Remark 2. 1) If $p(x) \leq 1$ for all $x \in Y$, then the p-contraction mapping is said to be a fundamental contraction which is also known as a Banach operator.

2) If $Y = X$ and $y = T(x)$, then a Banach contraction mapping is a fundamental contraction.

3) If $p(x) \leq 1$ for all $x \in Y$ and $\sup_{x\in Y} p(Tx) = 1$, then the p-contraction mapping is said to be fundamentally p-non-expansive. In particular when $p(x) = 1$ for all $x \in Y$, then the fundamentally p-non-expansive mapping is said to be fundamentally non-expansive.

4) If $\sup_{x\in Y} p(x) < 1$ (*or* ≤ 1), then $\sup_{x\in Y} p(Tx) < 1$ (or ≤ 1) since $T(Y) \subset Y$.

Remark 3. The concept of p-contraction is more general than the Banach contraction principle, see [8], example 2.1.

Remark 4. A p-contraction mapping is not continuous in general, see [8], example 2.2.

In 1976, Caristi [3] proved the following theorem.

Theorem 1. *Let (X, d) be complete and $\varphi : X \longrightarrow \mathbb{R}$ a lower semi-continuous function with a finite lower bound. Let $T : X \longrightarrow X$ be any (not necessarily continuous) function such that*

$$d(y, Ty)(y) \leq \varphi(y) - \varphi(Ty)$$

for each $y \in X$. Then T has a fixed point.

Remark 5. In general, a p-contraction does not satisfy Caristi's condition but every fundamental contraction does.

Definition 5. [6] A metric space (X, d) is said to be T-orbitally complete if T is a self-mapping of X and if any Cauchy subsequence $\{T^{n_i}x\}$ in orbit $O(x, T)$, $x \in X$, converges in X.

Definition 6. [6] An operator $T : X \longrightarrow X$ on X is said to be orbitally continuous if $T^{n_i}x \longrightarrow u$, then $T(T^{n_i}x) \longrightarrow Tu$ as $i \longrightarrow \infty$.

Definition 7. [6] An operator $T : X \longrightarrow X$ on X is said to be weakly orbitally continuous if $T^{n_i}x \longrightarrow u$, then $d(T^{n_i}x, T(T^{n_i}x)) \longrightarrow d(u, Tu)$ as $i \longrightarrow \infty$.

Remark 6. It is clear that a complete metric space is orbitally complete with respect to any self-mapping of X and that a continuous mapping is always orbitally continuous and an orbitally continuous mapping is always weakly orbitally continuous, but the converse implications are not true in general, see [8], example 2.3.

The aim of this paper is to prove a fixed point result for self-mapping which satisfies p-contraction condition in partially ordered metric spaces. For this purpose we need the following definitions.

Definition 8. [1] Let X be a nonempty set. Then $(X, d, \preceq)$ is called an ordered metric space iff:
(i) (X, d) is a metric space; (ii) $(X, \preceq)$ is partially ordered set.

Definition 9. [6] Let $(X, \preceq)$ be a partially ordered set. $x, y \in X$ are called comparable if $x \preceq y$ or $y \preceq x$ holds.

2 Main Results

Theorem 2. *Let $(X, \leq)$ be a partially ordered set. Suppose that there exists a metric d in X such that (X, d) is T-orbitally complete, T is a non-decreasing mapping such that there exists $x_0 \in X$ with $x_0 \leq T(x_0)$ and T is a strongly fundamental contraction mapping with $T(x) \leq x$. Assume that either T is orbitally continuous or X is such that*

$$\text{if a sequence } x_n \to x \text{ in } X \text{ is non-decreasing, then } x_n \leq x. \tag{2}$$

Then T has a fixed point. Moreover, if

$$\text{for each } x \in X, \text{ there exists } z \in X \text{ which is comparable to } x \text{ and } T(x), \tag{3}$$

therefore, the fixed point is unique.

Proof. First, we show that T has a fixed point. Let x_0 be an arbitrary point of X. We construct an iterative sequence $\{x_n\}$ defined by $x_{n+1} = T(x_n) = T^n(x_0)$. Since $x_0 \leq T(x_0)$ and T is a nondecreasing function, we have by induction

$$x_0 \leq T(x_0) \leq T^2(x_0) \leq T^3(x_0) \leq ... \leq T^n(x_0) \leq T^{n+1}(x_0) \leq ....$$

As $x_n \leq x_{n+1}$ for each $n \in \mathbb{N}$, applying (1) we get

$$d(x_1, x_2) \leq p(x_0)d(x_0, x_1),$$

$$\begin{array}{rcl} d(x_2,x_3) & \leq & p(x_1)d(x_1,x_2) \\ & \leq & p(x_0)p(x_1)d(x_0,x_1). \end{array}$$

By induction we obtain

$$d(x_n,x_{n+1}) \leq \prod_{i=1}^{n} p_i d(x_0,x_1), \tag{4}$$

where $p_i = p(x_{i-1}) = p(T^{i-1}(x_0))$, $i \in \mathbb{N}$. Since $\max\{p(x_0), \sup p(Tx)\} \leq \lambda < 1$, using (4) we have

$$d(x_n,x_{n+1}) \leq \lambda^n d(x_0,x_1), \forall n \in \mathbb{N}.$$

For $m > n$, $m,n \in \mathbb{N}$, we get

$$\begin{array}{rcl} d(x_n,x_m) & \leq & d(x_n,x_{n+1}) + d(x_{n+1},x_{n+2}) + \cdots + d(x_{m-1},x_m) \\ & \leq & [\lambda^n + \lambda^{n+1} + \cdots + \lambda^{m-1}]d(x_0,x_1) \\ & = & \dfrac{\lambda^n(1-\lambda^{m-n})}{1-\lambda} d(x_0,x_1) \\ & < & \dfrac{\lambda^n}{1-\lambda} d(x_0,x_1). \end{array}$$

Therefore, the sequence $\{x_n\}$ is a Cauchy sequence in X. Since X is T-orbitally complete, it follows that there exists a Cauchy subsequence $\{T^{n_i}(x_0)\}$ of $\{x_n\}$ in the orbit $O(x,Tx)$, $x \in X$, which converges to a point $z \in X$. Suppose that T is orbitally continuous. Then

$$\begin{array}{rcl} z & = & \lim\limits_{n\to\infty} x_{n_i} = \lim\limits_{n\to\infty} T^{n_i}(x_0) \\ & = & \lim\limits_{n\to\infty} T^{n_i+1}(x_0) = \lim\limits_{n\to\infty} T(T^{n_i}(x_0)) \\ & = & T(z), \end{array}$$

which shows that z is a fixed point of T. Hence $T(z) = z$. If case (2) holds, then

$$\begin{array}{rcl} d(T(z),z) & \leq & d(T(z),T(x_n)) + d(T(x_n),z) \\ & \leq & p(x_n)d(z,x_n) + d(x_{n+1},z) \\ & \leq & d(z,x_n) + d(x_{n+1},z). \end{array}$$

Since $d(z,x_n) \to 0$ then we obtain $T(z) = z$. To prove the uniqueness of the fixed point, let w be another fixed point of T. From (3) there exists $x \in X$ which is comparable to w and z. Monotonicity implies that $T^n(x)$ is comparable to $T^n(w) = w$ and $T^n(z) = z$ for $n = 1,2,....$ Then

$$\begin{array}{rcl} d(z,T^n(x)) & = & d(T^n(z),T^n(x)) \\ & \leq & p(T^{n-1}(z))d(T^{n-1}(z),T^{n-1}(x)). \end{array} \tag{5}$$

Therefore

$$d(z,T^n(x)) \leq d(z,T^{n-1}(x)).$$

Consequently, the sequence $\{\gamma_n\}$ defined by $\gamma_n = d(z, T^n(x))$ is nonnegative and nonincreasing and so $\lim_{n\to\infty} d(z, T^n(x)) = \gamma \geq 0$. Now, we show that $\gamma = 0$. On the contrary, assume that $\gamma > 0$. By passing to the limit in (5), we get

$$\gamma \leq \sup_{x\in X} p(Tx)\gamma < \gamma,$$

which is a contradiction and so $\gamma = 0$. Simillarly, it can be proved that $\lim_{n\to\infty} d(w, T^n(x)) = 0$. Finally,

$$d(z, w) \leq d(z, T^n(x)) + d(T^n(x), w),$$

and taking the limit as $n \to \infty$, we obtain $z = w$. □

3 Application to ordinary differential equations

Inspired by the papers of Pathak and Shahzad [8] and Aouine and Aliouche [3], we investigate the possibility of optimally controlling the solution of the ordinary differential equation (6) via dynamic programming.

Let A be a compact subset of $\mathbb{R}^m$ and for each given $a \in A$, $F_a : \mathbb{R}^n \to \mathbb{R}^n$ be a strong p-contraction mapping such that $F_a(x) = f(x, a)$, $\forall x \in \mathbb{R}^n$, where $f : \mathbb{R}^n \times A \to \mathbb{R}^n$ is a given bounded function which satisfies the following generalized contractive condition:

$$|f(x, a) - f(y, a)| \leq q(|x - y|)|x - y|,\ x, y \in \mathbb{R}^n, a \in A, \tag{*}$$

where $q : \mathbb{R}_+ \to [0, 1]$ is a function with $\sup_{t\geq 0} q(t) \leq \lambda < 1$. We will now study the possibility of optimally controlling the solution $x(\cdot)$ of the ordinary differential equation

$$\begin{cases} \dot{x}(s) = f(x(s), \alpha(s)) \quad (t < s < T) \\ x(t) = x_0. \end{cases} \tag{6}$$

Here $T > 0$ is a fixed terminal time, and $x \in \mathbb{R}^n$ is a given initial point, taken on by our solution $x(\cdot)$ at the starting time $t \geq 0$. At later times $t < s < T$, $x(\cdot)$ evolves according to the ODE (6). The function $\alpha(\cdot)$ appearing in (6) is a control, that is, some appropriate scheme for adjusting parameters from the set A as time evolves, there by affecting the dynamics of the system modelled by (6). Let us write

$$A = \{\alpha : [0, T] \to A | \alpha(\cdot) \text{ is measurable}\}, \tag{7}$$

to denote the set of admissible controls. Then since

$$|f(x, a)| \leq C,\ |f(x, a) - f(y, a)| \leq q(|x - y|)|x - y|,\ x, y \in \mathbb{R}^n, a \in A, \tag{8}$$

where q is defined in $(*)$, we have

$$|F_a(x) - F_a(y)| \quad \leq \quad q(|x - y|)|x - y| \tag{9}$$

$$\leq \quad p(x)|x-y|,$$

where

$$p(x) = \sup_{y\in\mathbb{R}^n} q(|x-y|),$$

for all x in $\mathbb{R}^n$, where $p : \mathbb{R}^n \to [0,1]$ is a function such that $\sup_{x\in\mathbb{R}^n} p(x) = \lambda < 1$. We see that for each control $\alpha(\cdot) \in A$. We have proved that the ODE (6) has a unique generalized contractive continuous solution $x(\cdot) = x^{\alpha(\cdot)}(\cdot)$, existing on the time interval $[t,T]$ and solving the ODE for a.e. time $t < s < T$. We call $x(\cdot)$ the response of the system to the control $\alpha^*(\cdot)$ and $x(s)$ the state of the system at time s.

Our aim is to find control $\alpha^*(\cdot)$ which optimally steers the system. In order to define what "optimal" means however, we must first introduce a cost criterion. Given $x \in \mathbb{R}^n$ and $0 \leq t \leq T$, let us define for each admissible control $\alpha(\cdot) \in A$ the corresponding cost

$$C_{x,t}[\alpha(\cdot)] := \int_t^T h(x(s),\alpha(s))ds + g(x(T)), \tag{10}$$

where $x(\cdot) = x^{\alpha(\cdot)}(\cdot)$ solves the ODE (6) and $h : \mathbb{R}^n \times\ A \to \mathbb{R}$, $g : \mathbb{R}^n \to \mathbb{R}$ are given functions. We call h the running cost per unit time and g the terminal cost and will henceforth assume

$$\begin{cases} |H_a(x)|, |g(x)| \leq C, \\ |H_a(x) - H_a(y)| \leq p(x)|x-y|,\ |g(x)-g(y)| \leq p(x)|x-y|, \\ x, y \in \mathbb{R}^n, a \in A, \end{cases} \tag{11}$$

for some constant C, where $p : \mathbb{R}^n \to [0,1]$ is a function such that $\sup_{x\in\mathbb{R}^n} p(x) = \lambda < 1$ and for each given $a \in A$, $H_a : \mathbb{R}^n \to \mathbb{R}^n$ is a strongly fundamental contraction mapping such that $H_a(x) = h(x,a)$ for all $x \in \mathbb{R}^n$.

Given now $x \in \mathbb{R}^n$ and $0 \leq t \leq T$, we would like to find if a control $\alpha^*(\cdot)$ is possible which minimizes the cost functional (10) among all admissible controls. To investigate the above problem we shall apply the method of dynamic programming. We now turn our attention to the value function $u(x,t)$ defined by

$$u(x,t) := \inf_{\alpha(\cdot)\in A} C_{x,t}[\alpha(\cdot)](x \in \mathbb{R}^n, 0 \leq t \leq T). \tag{12}$$

The plan is this: having defined $u(x,t)$ as the least cost given we start at the position x at time t, we want to study u as a function of x and t. We are therefore embedding our given control problem (6) and (10) into the larger class of all such problems, as x and t vary. This idea then can be used to show that u solves a certain Hamilton–Jacobi type PDE, and finally to show conversely that a solution of this PDE helps us to synthesize an optimal feedback control. Let us fix $x \in \mathbb{R}^n$, $0 \leq t \leq T$. Following the technique of Evans [7], p. 553, we can obtain the optimality conditions in the form given below: For each $\xi > 0$ so small that $t + \xi \leq T$,

$$u(x,t) := \inf_{\alpha(\cdot)\in A} \left\{ \int_t^{t+\xi} h(x(s),\alpha(s))ds + u(x(t+\xi), t+\xi) \right\}, \tag{13}$$

where $x(\cdot) = x^{\alpha(\cdot)}$, solves the ODE (6) for the control $\alpha(\cdot)$.

Acknowledgement. The author would like to thank the referee for his valuable comments and suggestions.

References

[1] M. Abbas, N. Talat, S. Radenović *Common fixed points of four maps in partially ordered metric spaces*, Appl. Math. Lett **24** (2011), No. 9, 1520-1526.

[2] A. C. Aouine, A. Aliouche *Coincidence and common fixed points theorem with an application in dynamic programming*, Bul. Acad. Ştiinţe Repub. Mold. Mat., **3(94)** (2020), 12–28.

[3] A. C. Aouine, A. Aliouche *Fixed point theorems of Kannan type with an application to control theory*, Applied Mathematics E-Notes **21**, (2021), 238–249.

[4] S. Banach *Sur les opérations dans les ensembles abstraits et leur application aux équations intégrales*, Fund. Math. **3**, (1922), 133–181.

[5] J. Caristi *Fixed point theorems for mappings satisfying the inwardness conditions*, Trans. Amer. Math. Soc. **215**, (1976), 241–251.

[6] Lj. Ćirić, M. Abbas, R. Saadati, N. Hussain *Common fixed points of almost generalized contractive mappings in ordered metric spaces*, Appl. Math. Comput. **217**, (2011), 5784–5789.

[7] L. C. Evans *Partial Differential Equations*, American Mathematical Society **12** (1998).

[8] H. K. Pathak, N. Shahzad *Fixed points for generalized contractions and applications to control theory.* Nonlinear Analysis **68**, (2008), 2181–2193.

A. C. Aouine
Mohamed-Cherif Messaadia University Souk Ahras, 41000,
Algeria.
E-mail: *a.aouine@univ-soukahras.dz*,
chawki81@gmail.com

Received April 15, 2022