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On recursively differentiable k-quasigroups

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Abstract. Recursive differentiability of linear k-quasigroups ($k \geq 2$) is studied in the present work. A k-quasigroup is recursively r-differentiable (r is a natural number) if its recursive derivatives of order up to r are quasigroup operations. We give necessary and sufficient conditions of recursive 1-differentiability (respectively, r-differentiability) of the k-group (Q, B), where $B(x_1, ..., x_k) = x_1 \cdot x_2 \cdot ... \cdot x_k, \forall x_1, x_2, ..., x_k \in Q$, and $(Q, \cdot)$ is a finite binary group (respectively, a finite abelian binary group). The second result is a generalization of a known criterion of recursive r-differentiability of finite binary abelian groups [4]. Also we consider a method of construction of recursively r-differentiable finite binary quasigroups of high order r. The maximum known values of the parameter r for binary quasigroups of order up to 200 are presented.

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The notions "recursive derivative" and "recursively differentiable quasigroup" were introduced in [1], where the authors considered recursive MDS-codes (Maximum Distance Separable codes). The recursive derivative of order $t \geq 0$ of a k-ary groupoid (Q, A) is denoted by $A^{(t)}$ and is defined as follows:

$A^{(0)} = A,$

$A^{(t)}(x_1^k) = A(x_{t+1}, ..., x_k, A^{(0)}(x_1^k), ..., A^{(t-1)}(x_1^k))$ if $1 \leq t < k$;

$A^{(t)}(x_1^k) = A(A^{(t-k)}(x_1^k), ..., A^{(t-1)}(x_1^k))$ if $t \geq k, \forall x_1, ..., x_k \in Q,$

where we denoted the sequence $x_1, x_2, ..., x_k$ by x_1^k. A k-ary quasigroup (Q, A) is called *recursively r-differentiable* if the recursive derivatives $A^{(0)}, A^{(1)}, ..., A^{(r)}$ are quasigroup operations ($r \geq 0$).

The length n of the codewords in a k-recursive code

$$C(n, A) = \{(x_1, ..., x_k, A^{(0)}(x_1^k), ..., A^{(n-k-1)}(x_1^k)) | x_1, ..., x_k \in Q\}$$

given on an alphabet Q of q elements, where $A : Q^k \to Q$ is the defining k-ary quasigroup operation, satisfies the condition $n \leq r + k + 1$, where r is the maximum order of recursive differentiability of (Q, A). On the other hand, $C(n, A)$ is an MDS-code if and only if $d = n - k + 1$, where d is the minimum Hamming distance of this code. At present it is an open problem to determine all triplets (n, d, q) of natural numbers such that there exists an MDS-code C of lenght n, on an alphabet of q elements, with $|C| = q^k$ and with the minimum Hamming distance d, for each

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$k \geq 2$. This general problem implies, in particular, the problem of determining the maximum order of recursive differentiability of finite k-ary quasigroups ($k \geq 2$).

Let $(Q, *)$ be a binary quasigroup. Denoting by $\overset{t}{*}$ the recursive derivative of order t of the operation $*$, we have:

$$x \overset{0}{*} y = x * y,$$
$$x \overset{1}{*} y = y * (x * y),$$
$$x \overset{t}{*} y = (x \overset{t-2}{*} y) * (x \overset{t-1}{*} y),\ \forall t \geq 2 \text{ and } \forall x, y \in Q.$$

It is known that there exist recursively 1-differentiable binary finite quasigroups of any order, except 1, 2, 6, and possibly 14, 18, 26 and 42 [1]. Some estimations of the maximum (known) order r of recursive differentiability of finite n-quasigroups ($n \geq 2$) are given in [1–4]. General properties of recursively differentiable binary quasigroups are studied in [4, 6, 7].

The recursive differentiability of k-ary quasigroups is closely connected to the orthogonality of the recursive derivatives [1,4,6]. It is shown in [1] that a k-quasigroup defines an MDS-code of length n if and only if its first $n-k-1$ recursive derivatives are strongly orthogonal. Hence the defining k-quasigroup operation of a recursive MDS-code of length n is recursively $(n-k-1)$-differentiable. On the other hand, it is known that a system of binary quasigroups is strongly orthogonal if and only if it is (simply) orthogonal [5]. Another "special property" of binary quasigroups is given in [1]: the recursive derivatives of order up to r of a finite binary quasigroup $(Q, *)$ are quasigroup operations if and only if $(Q, *)$ defines a recursive MDS-code of length $r+3$. So, a finite binary quasigroup $(Q, *)$ is recursively r-differentiable if and only if its recursive derivatives of order up to r are mutually orthogonal. The last statement implies the fact that there do not exist recursively 1-differentiable quasigroups of orders 2 and 6 and that $r \leq q-2$, where $q = |Q|$ and r is the order of the recursive differentiability of the quasigroup Q. Recall that there do not exist orthogonal latin squares of order 2 or 6, and the number of mutually orthogonal latin squares on a set of q elements does not exceed $q-1$ [5]. The mentioned above results imply the following lemma.

Lemma 1. *The maximum order r of recursive differentiability of a finite binary quasigroup of order q satisfies the inequality $r \leq q-2$.*

It is shown in [1] that there exist recursively $(q-2)$-differentiable finite binary quasigroups of every primary order $q \geq 3$. However, it is an open problem to find the maximum order r of recursive differentiability of finite k-ary quasigroups of order q, for $k \geq 2$ and non-primary q.

Recursive differentiability of linear n-ary quasigroups ($n \geq 2$) is studied in the present work. In particular, we give necessary and sufficient conditions of recursive 1-differentiability (respectively, r-differentiability) of an n-group (Q, B), where $B(x_1^n) = x_1 \cdot x_2 \cdot ... \cdot x_n, \forall x_1, x_2, ..., x_n \in Q$, and $(Q, \cdot)$ is a finite binary group (respectively, a finite abelian binary group). The second result is a generalization of a known criterion for finite binary abelian groups, given in [4]. Also we consider a method of construction of recursively differentiable finite binary quasigroups of high (in particular, maximum) order r. The maximum known values of the order r of recursive differentiability of finite binary quasigroups of order up to 200, are qiven in Table 1.

Lemma 2. *Let $n \geq 2$ and let (Q_i, A_i) be a recursively r_i-differentiable n-quasigroup, $i = 1, \ldots, m$. Then the direct product $(Q_1 \times ... \times Q_m, B)$,*

$$B((x_{11}^{1m}), ..., (x_{n1}^{nm})) = (A_1(x_{11}^{n1}), ..., A_m(x_{1m}^{nm})), \tag{1}$$

$\forall (x_{11}^{1m}), ..., (x_{n1}^{nm}) \in Q_1 \times ... \times Q_m$, is a recursively r-differentiable n-quasigroup, where $r = min\{r_1, ..., r_m\}$.

Proof. Remind that an n-ary groupoid (Q, B) is an n-ary quasigroup if each element u_i in the equality $B(u_1, ..., u_n) = u_{n+1}$ is uniquely determined by the remaining n elements. Hence, we get from (1) that $(Q_1 \times ... \times Q_m, B)$ is an n-quasigroup. To find the recursive derivatives of B we'll consider the following two cases:

(i) $1 \leq t < n$

$$B^{(t)}((x_{11}^{1m}), ..., (x_{n1}^{nm})) =$$
$$= B((x_{t+1,1}^{t+1,m}), ..., (x_{n1}^{nm}), B^{(0)}((x_{11}^{1m}), ..., (x_{n1}^{nm})), ..., B^{(t-1)}((x_{11}^{1m}), ..., (x_{n1}^{nm}))) =$$
$$B((x_{t+1,1}^{t+1,m}), ..., (x_{n1}^{nm}), (A_1^{(0)}(x_{11}^{n1}), ..., A_m^{(0)}(x_{1m}^{nm})), ..., (A_1^{(t-1)}(x_{11}^{n1}), ..., A_m^{(t-1)}(x_{1m}^{nm}))) =$$
$$= (A_1^{(t)}(x_{11}^{n1}), ..., A_m^{(t)}(x_{1m}^{nm}));$$

(ii) $t \geq n$

$$B^{(t)}((x_{11}^{1m}), ..., (x_{n1}^{nm})) = B(B^{(t-n)}((x_{11}^{1m}), ..., (x_{n1}^{nm})), ..., B^{(t-1)}((x_{11}^{1m}), ..., (x_{n1}^{nm}))) =$$
$$= B((A_1^{(t-n)}(x_{11}^{n1}), ..., A_m^{(t-n)}(x_{1m}^{nm})), ..., (A^{(t-1)}(x_{11}^{n1}), ..., A_m^{(t-1)}(x_{1m}^{nm}))) =$$
$$= (A_1(A_1^{(t-n)}(x_{11}^{n1}), ..., A_1^{(t-1)}(x_{11}^{n1})), ..., A_m(A_m^{(t-n)}(x_{1m}^{nm}), ..., A_m^{(t-1)}(x_{1m}^{nm}))) =$$
$$= (A_1^{(t)}(x_{11}^{n1}), ..., A_m^{(t)}(x_{1m}^{nm})).$$

Hence, $B^{(t)}((x_{11}^{1m}), ..., (x_{n1}^{nm})) = (A_1^{(t)}(x_{11}^{n1}), ..., A_m^{(t)}(x_{1m}^{nm}))$, for every $t \geq 1$ and every $(x_{11}^{1m}), ..., (x_{n1}^{nm}) \in Q_1 \times ... \times Q_m$. As each of the operations $A_1, ..., A_m$ is recursively r-differentiable, where $r = min\{r_1, ..., r_m\}$, we get that B is recursively r-differentiable. □

Proposition 1. *Let $(Q, \cdot)$ be a finite binary group and $n \geq 2$. The n-ary group (Q, B), where $B(x_1^n) = x_1 \cdot x_2 \cdot ... \cdot x_n, \forall x_1, x_2, ..., x_n \in Q$, is recursively 1-differentiable if and only if the mapping $x \rightarrow x^2$ is a bijection in $(Q, \cdot)$.*

Proof. The n-group (Q, B) is recursively 1-differentiable if and only if the recursive derivative $B^{(1)}$ is a quasigroup operation, i.e. if and only if in the equality

$$B^{(1)}(x_1, ..., x_n) = b, \tag{2}$$

every n elements uniquely determine the remaining $(n+1)$-th one. Taking $x_j = a_j \in Q$ in (2), for every $j = 2, \ldots, n$, we get the equation $x_1 \cdot a_2 \cdot ... \cdot a_n = b$, which has a unique solution in Q. For $i \in \{2, ..., n\}$, taking $x_j = a_j \in Q, \forall j \neq i$, $j \in \{1, ..., n\}$, we have:

$$B^{(1)}(a_1, ..., a_{i-1}, x_i, a_{i+1}, ..., a_n) = b \Leftrightarrow$$
$$\Leftrightarrow B(a_2, ..., a_{i-1}, x_i, a_{i+1}, ..., a_n, B(a_1, ..., a_{i-1}, x_i, a_{i+1}, ..., a_n)) = b \Leftrightarrow$$
$$\Leftrightarrow a_2 \cdot ... \cdot a_{i-1} \cdot x_i \cdot a_{i+1} \cdot ... \cdot a_n \cdot a_1 \cdot ... \cdot a_{i-1} \cdot x_i \cdot a_{i+1} \cdot ... \cdot a_n = b \Leftrightarrow$$
$$\Leftrightarrow (a_1 \cdot ... \cdot a_{i-1} \cdot x_i \cdot a_{i+1} \cdot ... \cdot a_n)^2 = a_1 \cdot b.$$

Hence, denoting $a_1 \cdot ... \cdot a_{i-1} \cdot x_i \cdot a_{i+1} \cdot ... \cdot a_n$ by y, we get that the n-group (Q, B) is recursively 1-differentiable if and only if, for each $b \in Q$, the equation $y^2 = b$ has a unique solution. □

Corollary 1. *There exist finite recursively 1-differentiable n-quasigroups of any odd order $q \geq 3$, for every $n \geq 2$.*

Proof. This statement follows from the fact that the mapping $x \to x^2$ is a bijection in every finite binary group of odd order $q \geq 3$. □

Theorem 1. *Let $(Q, \cdot)$ be a finite binary abelian group and let $n \geq 2, r \geq 1$ be two natural numbers. The n-ary group (Q, B), where $B(x_1^n) = x_1 \cdot x_2 \cdot ... \cdot x_n$, for every $x_1, x_2, ..., x_n \in Q$, is recursively r-differentiable if and only if the mappings $x \to x^{s_i^k}$ are bijections in the group $(Q, \cdot)$, $\forall i = 1, \ldots, n$ and $\forall k = 1, \ldots, r$, where the sequences $(s_i^k)_{k \geq 0}$ are defined as follows:*

1. $k = 0$

$$s_1^0 = ... = s_n^0 = 1;$$

2. $1 \leq k < n$

$$s_t^k = s_t^0 + ... + s_t^{k-1}, \ \ \forall t = 1, \ldots, k;$$
$$s_t^k = 1 + s_t^0 + ... + s_t^{k+1}, \ \ \forall t = k+1, \ldots, n;$$

3. $k \geq n$

$$s_t^k = s_t^{k-n} + ... + s_t^{k-1}, \ \forall t = 1, \ldots, n.$$

Proof. As $(Q, \cdot)$ is an abelian group and $B^{(0)}(x_1^n) = x_1 \cdot ... \cdot x_n$, the recursive derivatives $B^{(1)}$ and $B^{(2)}$ as follows:

$$B^{(1)}(x_1^n) = B(x_2, ..., x_n, B^{(0)}(x_1^n)) = x_2 \cdot ... \cdot x_n \cdot x_1 \cdot ... \cdot x_n = x_1 \cdot x_2^2 \cdot ... \cdot x_n^2;$$
$$B^{(2)}(x_1^n) = B(x_3, ..., x_n, B^{(0)}(x_1^n), B^{(1)}(x_1^n)) = x_3 \cdot ... \cdot x_n \cdot x_1 \cdot ... \cdot x_n \cdot x_1 \cdot x_2^2 \cdot ... \cdot x_n^2 =$$
$$= x_1^2 \cdot x_2^3 \cdot x_3^4 \cdot ... \cdot x_n^4.$$

Let denote $B^{(k)}(x_1^n) = x_1^{s_1^k} \cdot x_2^{s_2^k} \cdot ... \cdot x_n^{s_n^k}$, for every $k \geq 0$. To find the sequences $(s_i^k)_{k \geq 0}$, where $i = 1, \ldots, n$, we will consider the following two cases:

1. $0 \leq k < n$

$$B^{(k)}(x_1^n) = B(x_{k+1}, ..., x_n, B^{(0)}(x_1^n), ..., B^{(k-1)}(x_1^n)) =$$

$$= x_{k+1} \cdot ... \cdot x_n \cdot x_1^{s_1^0} \cdot ... \cdot x_n^{s_n^0} \cdot ... \cdot x_1^{s_1^{k-1}} \cdot ... \cdot x_n^{s_n^{k-1}} =$$

$$= x_1^{s_1^0+...+s_1^{k-1}} \cdot ... \cdot x_k^{s_k^0+...+s_k^{k-1}} \cdot x_{k+1}^{1+s_{k+1}^0+...+s_{k+1}^{k-1}} \cdot ... \cdot x_n^{1+s_n^0+...+s_n^{k-1}};$$

2. $k \geq n$

$$B^{(k)}(x_1^n) = B(B^{(k-n)}(x_1^n), ..., B^{(k-1)}(x_1^n)) = B^{(k-n)}(x_1^n) \cdot ... \cdot B^{(k-1)}(x_1^n) =$$

$$= x_1^{s_1^{k-n}+...+s_1^{k-1}} \cdot ... \cdot x_n^{s_n^{k-n}+...+s_n^{k-1}}.$$

The recursive derivatives $B^{(k)}$, where $k = 1, 2, \ldots, r$, are quasigroup operations if and only if the mappings $x \to x^{s_i^k}$ are bijections in the group $(Q, \cdot)$, $\forall i = 1, \ldots, n$ and $\forall k = 1, \ldots, r$.

□

Corollary 2. [4] *A finite binary abelian group $(Q, \cdot)$ is recursively r-differentiable $(r \geq 1)$ if and only if the mappings $x \to x^{s_i^k}$ are bijections, $\forall i = 1, 2$ and $\forall k = 1, ..., r$, where the sequences $(s_1^k)_{k\geq 0}$ and $(s_2^k)_{k\geq 0}$ are defined as follows:*

$$s_1^0 = s_2^0 = 1;\ s_1^1 = 1, s_2^1 = 2;\ s_i^k = s_i^{k-2} + s_i^{k-1}, \forall k \geq 2, \forall i = 1, 2.$$

Note that $(s_1^k)_{k\geq 0}$ and $(s_2^k)_{k\geq 0}$ are Fibonacci sequences.

We will give bellow an algorithm of construction of binary linear (over $\mathbb{Z}_n$) quasigroups, which are recursively differentiable of high order.

Lemma 3. [7] *If $(Q, *)$ is a binary quasigroup then, for every $x, y \in Q$ and $\forall s \geq 1$,*

$$x \overset{s}{*} y = y \overset{s-1}{*} (x * y). \tag{3}$$

Lemma 4. *Let $a \in \mathbb{Z}\backslash\{0\}$ and $x * y = ax + y, \forall x, y \in \mathbb{Z}$. The following statements hold:*

1. *There exist $u_s,\ v_s \in \mathbb{Z}$ such that $x \overset{s}{*} y = u_s x + v_s y,\ \forall x, y \in \mathbb{Z}, \forall s \geq 1$;*
2. *If $n \geq 2$ is a natural number, $k \in \{1, ..., n-1\}$ and $a = n-k$, then there exists $b_{s+2} \in \mathbb{Z}$ such that $v_{s+2} = nb_{s-2} + (-kc_s + c_{s+1})$, for $\forall s \geq 1$, where c_s and c_{s+1} are the rests from dividing v_s and v_{s+1} by n, respectively.*

Proof. 1. In this case $x \overset{1}{*} y = y * (x * y) = ax + (a+1)y, \forall x, y \in \mathbb{Z}$. Denoting $x \overset{s-1}{*} y = u_{s-1}x + v_{s-1}y$ and using the mathematical induction and (3), we get $x \overset{s}{*} y = u_{s-1}y + v_{s-1}(ax + y) = av_{s-1}x + (u_{s-1} + v_{s-1})y$.

2. As $x \overset{s+2}{*} y = (x \overset{s}{*} y) * (x \overset{s+1}{*} y) = (au_s + u_{s+1})x + (av_s + v_{s+1})y$, the following equalities hold:

$$\begin{aligned} v_{s+2} = av_s + v_{s+1} = (n-k)(nb_s + c_s) + (nb_{s+1} + c_{s+1}) = \\ = n(nb_s + c_s - kb_s + b_{s+1}) + (-kc_s + c_{s+1}), \end{aligned}$$

where c_s and c_{s+1} are the rests from dividing v_s and v_{s+1} by n, respectively.

□

Now, let consider the operation $x * y = \overline{a}x + y$ on the ring $\mathbb{Z}_n$ of integers modulo n, where $(a, n) = 1$. Then $(\mathbb{Z}_n, *)$ is a quasigroup and, according to the previous lemma, there exist $\overline{u_s}, \overline{v_s} \in \mathbb{Z}_n$ such that $x \overset{s}{*} y = \overline{u_s}x + \overline{v_s}y, \forall s \geq 0$.

Theorem 2. *Let $n \geq 2, a = n - k, k \in \{1, ..., n-1\}, (a, n) = 1$ and $x * y = \overline{a}x + y, \forall x, y \in \mathbb{Z}_n$. If, for some $s \geq 1$, the recursive derivatives $(\mathbb{Z}_n, \overset{s}{*})$ and $(\mathbb{Z}_n, \overset{s+1}{*})$, where $x \overset{i}{*} y = \overline{u_i}x + \overline{v_i}y,\ i = s, s+1$, are quasigroups, then $(\mathbb{Z}_n, \overset{s+2}{*})$ is a quasigroup if and only if $(-kc_s + c_{s+1}, n) = 1$, where c_s and c_{s+1} are the rests from dividing v_s and v_{s+1} by n, respectively.*

Proof. We have: $x \overset{s+2}{*} y = \overline{u_{s+2}}x + \overline{v_{s+2}}y = \overline{av_{s+1}}x + (\overline{u_{s+1}} + \overline{v_{s+1}})y$, so $\overline{v_{s+2}} = \overline{u_{s+1}} + \overline{v_{s+1}} = \overline{-kc_s + c_{s+1}}$, where c_s and c_{s+1} are the rests from dividing v_s and v_{s+1} by n, respectively. If $(\mathbb{Z}_n, \overset{s}{*})$ and $(\mathbb{Z}_n, \overset{s+1}{*})$ are quasigroups, then $(av_{s+1}, n) = 1$, hence $(\mathbb{Z}_n, \overset{s+2}{*})$ is a quasigroup if and only if $(-kc_s + c_{s+1}, n) = 1$. □

Using Theorem 2, we get, for example, that the quasigroups $(\mathbb{Z}_7, *), x*y = 4x+y$, and $(\mathbb{Z}_{11}, *), x*y = 3x+y$, are recursively 5- and 9-differentiable, respectively. Recall that the order r of recursive differentiability of a binary quasigroup, defined on a set of q elements, satisfies the inequality $r \leq q-2$. The following corollary gives all values of the element a such that the quasigroup $(\mathbb{Z}_p, *)$, where $x * y = \overline{a}x + y, \forall x, y \in \mathbb{Z}_p$, is recursively differentiable of maximum order, for each odd prime p, up to 19.

Corollary 3. *Let $(\mathbb{Z}_n, *)$, where $x * y = \overline{a}x + y, \forall x, y \in \mathbb{Z}_n$, be a quasigroup. The following statements hold:*

1. *$(\mathbb{Z}_3, *)$ is recursively 1-differentiable if and only if $a = 1$;*
2. *$(\mathbb{Z}_5, *)$ is recursively 3-differentiable if and only if $a = 3$;*
3. *$(\mathbb{Z}_7, *)$ is recursively 5-differentiable if and only if $a = 1$ or 4;*
4. *$(\mathbb{Z}_{11}, *)$ is recursively 9-differentiable if and only if $a = 3$ or 4;*
5. *$(\mathbb{Z}_{13}, *)$ is recursively 11-differentiable if and only if $a = 5, 8$ or 11;*
6. *$(\mathbb{Z}_{17}, *)$ is recursively 15-differentiable if and only if $a = 7$ or 10;*
7. *$(\mathbb{Z}_{19}, *)$ is recursively 17-differentiable if and only if $a = 1, 5$ or 7.*

The known estimations $r_0 \leq r$ of the order r of recursive differentiability of binary finite quasigroups of order $q \leq 200$ are given in the following Table 1. In the cell with coordinates (m, k) we give the known value of the parameter r for quasigroups of order $m + k$. Remark that the cell $(0, 0)$ contains the known value of r for the quasigroups of order 200. An analogous table containing the maximum known length of recursive MDS-codes, defined by quasigroups of order up to 100, is given in [2] and we use it in the first ten lines of Table 1.

	0	1	2	3	4	5	6	7	8	9
0(200)	$r \geq 2$	0	0	1	2	3	0	5	6	7
10	1	9	1	11	?	1	14	15	?	17
20	2	2	1	21	2	23	?	25	2	27
30	1	29	30	1	1	3	1	35	1	2
40	1	39	?	41	2	1	1	45	1	47
50	4	1	2	51	3	3	5	4	4	57
60	2	59	3	5	62	4	3	65	3	3
70	4	69	6	71	3	3	3	5	4	77
80	5	79	3	81	4	4	4	3	6	87
90	3	5	4	3	4	4	4	95	4	7
100	2	99	1	101	6	1	1	105	1	107
110	1	1	5	111	1	3	2	1	1	5
120	1	119	1	1	2	123	1	125	126	1
130	1	129	1	5	1	1	6	135	1	137
140	2	1	1	9	1	3	1	1	2	147
150	1	149	6	1	1	3	1	155	1	1
160	3	5	1	161	2	1	1	165	1	167
170	1	1	2	171	1	3	5	1	1	177
180	2	179	1	1	6	3	1	9	2	1
190	1	189	1	191	1	1	2	195	1	197

Table 1. Estimations of the parameter r
(order of recursive differentiability) in the case of binary quasigroups

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