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Nuclear Identification of Some New Loop Identities of Length Five

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Abstract. In this work, we discovered a dozen of new loop identities we called identities of 'second Bol-Moufang type'. This was achieved by using a generalized and modified nuclear identification model originally introduced by Drápal and Jedlička. Among these twelve identities, eight of them were found to be distinct (from well known loop identities), among which two pairs axiomatize the weak inverse property power associative conjugacy closed (WIP PACC) loop. The four other new loop identities individually characterize the Moufang identities in loops. Thus, now we have eight loop identities that characterize Moufang loops. We also discovered two (equivalent) identities that describe two varieties of Buchsteiner loops. In all, only the extra identities which the Drápal and Jedlička nuclear identification model tracked down could not be tracked down by our own nuclear identification model. The dozen laws $\{Q_i\}_{i=1}^{12}$ induced by our nuclear identification form four cycles in the following sequential format: $\left(Q_{4i-j}\right)_{i=1}^{3}$, $j = 0, 1, 2, 3$, and also form six pairs of dual identities. With the help of twisted nuclear identification, we discovered six identities of lengths five that describe the abelian group variety and commutative Moufang loop variety (in each case). The second dozen identities $\{Q_i^*\}_{i=1}^{12}$ induced by our twisted nuclear identification were also found to form six pairs of dual identities. Some examples of loops of smallest order that obey non-Moufang laws (which do not necessarily imply the other) among the dozen laws $\{Q_i\}_{i=1}^{12}$ were found.

Mathematics subject classification: 20N02, 20N05.
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1 Introduction

The first classification of the varieties of loops of Bol-Moufang was done by Fenyves in [11, 12] and concluded by Phillips and Vojtěchovský in [34, 35]. Jaíyéọlá et al. [25–27] and Ilojide et al. [14] used the identities therein to classify varieties of quasi neutrosophic triplet loops (called Fenyves BCI-Algebras) and also to study their isotopy and holomorphy. We shall refer to the identities described by the Bol-Moufang type of loops as 'first Bol-Moufang type' while we shall introduce what we call 'second Bol-Moufang type' of loops.

An identity of length four is said to be of Bol-Moufang type (first Bol-Moufang type) if:

1. It has 3 distinct variables with one of them appearing twice on both sides.

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2. The variables appear in the same order on both sides.

Coté et al. [7] and Akhtar et al. [2] classified loops of generalized Bol-Moufang type.

An identity of length four is said to be of generalized Bol-Moufang type if:

1. It has 3 distinct variables with one of them appearing twice on both sides.

2. The variables do not necessarily appear in the same order on both sides.

One of such loops of generalized Bol-Moufang type, namely Frute loops were studied by Jaíyéọlá et al. [24, 28].

An identity of length five will be said to be of second Bol-Moufang type if:

1. It has 3 distinct variables with one of them appearing 3 times.

2. The variables appear in the same order on both sides.

Two of such loops of second Bol-Moufang type are described by the identities

$$(xy \cdot x) \cdot xz = x((yx \cdot x)z), \qquad \text{(LWPC)}$$
$$zx \cdot (x \cdot yx) = (z(x \cdot xy))x \qquad \text{(RWPC)}$$

which Phillips [32] showed axiomatize the variety of loops that are weak inverse property power associative conjugacy closed (WIP PACC) loops. George et al. [13] studied loops that obey LWPC (RWPC) identity and were able to link them up with some loop identities that are not of Bol-Moufang type.

Theorem 1.1. *(George et al. [13])*

Let Q be a loop.

1. *Q is an LWPC-loop if and only if Q is an LCC-loop and $\underbrace{(xy \cdot x)x = x(yx \cdot x)}_{P_\lambda(x,y)}$.*

2. *Q is a RWPC-loop if and only if Q is an RCC-loop and $\underbrace{x(x \cdot yx) = (x \cdot xy)x}_{P_\rho(x,y)}$.*

3. *A CC-loop Q is a power associative WIP-loop if and only if Q fulfills the laws $P_\lambda(x, y)$ and $P_\rho(x, y)$.*

Drápal and Jedlička [9] investigated interactions between loop nuclei and loop identities. With the aid of nuclear identification, they considered some varieties of loops of first Bol-Moufang type and non-Bol-Moufang type in which not all the nuclei necessarily coincide. Drapal and Kinyon [10] recently used nuclear identification to obtain the identities of Osborn loops.

2 Preliminaries

A quasigroup $(Q,\cdot)$ consists of a non-empty set Q with a binary operation $\cdot$ on Q such that given $a,b\in Q$, the equations $ax=b$ and $ya=b$ have unique solutions $x,y\in Q$ respectively. We shall sometimes refer to $(Q,\cdot)$ as simply Q.

For any $x\in Q$, define the right translation map $R(x)$ and left translation map $L(x)$ of x in $(Q,\cdot)$ by $yR(x)=y\cdot x=yx$ and $yL(x)=x\cdot y=xy$, respectively. It is clear that $(Q,\cdot)$ is a quasigroup if and only if the left and right translation maps are bijections. Since the translation maps are bijections, then the inverse maps $R^{-1}(x)$ and $L^{-1}(x)$ exist and are thus defined by $yR^{-1}(x)=y/x$ and $yL^{-1}(x)=x\backslash y$.

A loop $(Q,\cdot)$ is a quasigroup with an identity element, 1, such that $1x=x1=x$, for all $x\in Q$. The right and left inverse maps $\rho : x\mapsto x^\rho$ and $\lambda : x\mapsto x^\lambda$ are unary operations that take an element x in a loop to its right and left inverses x^ρ and x^λ respectively, such that $x\cdot x^\rho=1=x^\lambda\cdot x$. A loop in which $x^\rho=x^\lambda$ for all elements x is said to have 2-sided inverse. See [5,6,17,31] for a general overview on quasigroups and loops.

A loop is a weak inverse property loop if it satisfies any one of the following identities:

$$x(yx)^\rho=y^\rho \quad \text{or} \quad (xy)^\lambda x=y^\lambda. \tag{1}$$

A loop $(Q,\cdot)$ is called a left Bol (right Bol) loop if for all $x,y,z\in Q$ it satisfies

$$(x\cdot yx)z=x(y\cdot xz), \tag{LB}$$
$$(yx\cdot z)x=y(xz\cdot x). \tag{RB}$$

A loop $(Q,\cdot)$ is called a Moufang loop if for all $x,y,z\in Q$ any of the following identities is satisfied

$$(xz\cdot x)y=x(z\cdot xy), \tag{LM}$$
$$(yx\cdot z)x=y(x\cdot zx), \tag{RM}$$
$$xy\cdot zx=(x\cdot yz)x, \tag{MM2}$$
$$xy\cdot zx=x(yz\cdot x). \tag{MM1}$$

A loop is said to be conjugacy closed (CC-loop) if it satisfies the two identities:

$$(xy)/x\ \cdot xz=x(yz), \tag{LCC}$$
$$zx\ \cdot x\backslash(yx)=(zy)x. \tag{RCC}$$

A loop $(Q,\cdot)$ is called a left central loop (LC-loop) if it satisfies the following identity for all $x,y,z\in Q$:

$$(x\cdot xy)z=x(x\cdot yz). \tag{2}$$

A loop $(Q,\cdot)$ is called a right central loop (RC-loop) if for all $x,y,z\in Q$ it satisfies the identity

$$y(zx\cdot x)=(yz\cdot x)x. \tag{3}$$

$(Q,\cdot)$ is called a central loop (C-loop) if it satisfies the identity

$$(yx\cdot x)z = y(x\cdot xz). \tag{4}$$

LC-loops, RC-loops and C-loops are among the varieties of loops of first Bol-Moufang type. Phillips and Vojtěchovský [35, 36], Kinyon et al. [37], Ramamurthi and Solarin [38], Jaiyéọlá [15,16], Adéníran and Jaiyéọlá [1], Jaiyéọlá and Adéníran [19–22] and Solarin [40], Beg [3, 4] have studied them. Fenyves [12] gave three equivalent identities that define each of LC-loops and RC-loops, and only one identity that defines C-loops. But, Phillips and Vojtěchovský [35] gave four equivalent identities that define each of LC-loops and RC-loops. Three of the four identities given by Phillips and Vojtěchovský are the same as the three already given by Fenyves. Jaíyéọlá [18] introduced and studied the generalized forms of LC-loop, RC-loop and C-loops. Jaiyéọlá and Adéníran [23] characterized Osborn-Buchsteiner loops with a new identity that is obeyed by LC-loop.

A loop $(Q,\cdot)$ is called a Buchsteiner loop, if for all $x, y, z \in Q$

$$(BUCH) \qquad x\backslash(xy\cdot z) = (y\cdot zx)/x. \tag{5}$$

A loop is power associative if subloops generated by every single element are groups.

The left alternative property (LAP) of a loop is defined as $xx\cdot y = x\cdot xy$, the right alternative property (RAP) is given by $y\cdot xx = yx\cdot x$. A loop is an alternative loop if it is left and right alternative. Flexible loops satisfy $x\cdot yx = xy\cdot x$. A loop Q is said to have the 3-power associative (3-PA) property if $xx\cdot x = x\cdot xx$.

A loop Q satisfies the left inverse property (LIP) if $x^\lambda\cdot xy = y$ and the right inverse property (RIP) if $xy\cdot y^\rho = x$. An inverse property loop is a loop that satisfies both the (LIP) and the (RIP).

The left nucleus N_λ, the middle nucleus N_μ and the right nucleus N_ρ of a loop Q are defined by

$$\begin{aligned} N_\lambda(Q) &= \{a\in Q:\ a\cdot xy = ax\cdot y\ \ \forall\ x,\ y\in Q\},\\ N_\mu(Q) &= \{a\in Q:\ xa\cdot y = x\cdot ay\ \ \forall\ x,\ y\in Q\},\\ N_\rho(Q) &= \{a\in Q:\ xy\cdot a = x\cdot ya\ \ \forall\ x,\ y\in Q\}.\end{aligned}$$

The intersection

$$N(Q) = N_\rho(Q)\cap N_\lambda(Q)\cap N_\mu(Q)$$

is called the nucleus of Q.

A triple of bijections $(U,\ V,\ W)$ is called an autotopism of a loop Q provided that

$$xU\cdot yV = (xy)W \tag{6}$$

for all $x,\ y\in Q$. The set of such triples forms a group $Atp(Q)$ called the autotopism group of Q.

It is easy to see that

$$a \in N_\lambda(Q) \Leftrightarrow (L(a),\ I,\ L(a)) \in Atp(Q), \tag{7}$$

$$a \in N_\mu(Q) \Leftrightarrow (R^{-1}(a),\ L(a),\ I) \in Atp(Q), \tag{8}$$

$$a \in N_\rho(Q) \Leftrightarrow (I,\ R(a),\ R(a)) \in Atp(Q). \tag{9}$$

Denote the autotopisms of (7), (8) and (9) by $\alpha_\lambda(x), \alpha_\mu(x)$ and $\alpha_\rho(x)$ respectively.

By generalizing and modifying the nuclear identification model in [9], we say a loop identity is nuclear identifiable if it can be expressed using autotopisms $\alpha_a^i(x)\alpha_b^j(x)\alpha_c^k(x)\alpha_d^l(x)$, where $i,\ j,\ k,\ l \in \{-1,\ 1\}$ and $a,\ b,\ c,\ d \in \{\lambda,\ \rho,\ \mu\}$.

We now state some existing results which we shall be using in this work.

Lemma 2.1. *[33]*

Let Q be a loop. The following are equivalent for any $x \in Q$:

1. *Q is a WIPL.*
2. *$R^{-1}(x) = \rho L(x)\lambda$.*
3. *$L^{-1}(x) = \lambda R(x)\rho$.*

Theorem 2.2. *[33]*

Let Q be a WIPL. If $(U,\ V,\ W) \in Atp(Q)$:

1. *$(V,\ \lambda W\rho,\ \lambda U\rho) \in Atp(Q)$.*
2. *$(\rho W\lambda,\ U,\ \rho V\lambda) \in Atp(Q)$.*

In this work, we shall consider some loops of second Bol-Moufang type. We shall investigate loops of length five with two coinciding nuclei relative to weak inverse property.

3 Main Results

3.1 Nuclear Identification

Definition 3.1. *Let Q be a loop which obeys an identity $Id = Id_\alpha$ where Id is equivalently expressible by the autotopism α. Let $\alpha_\lambda(x) = (L(x),\ I,\ L(x))$, $\alpha_\rho(x) = (I,\ R(x),\ R(x))$ and $\alpha_\mu(x) = (R^{-1}(x),\ L(x),\ I)$. Then the identity $Id = Id_\alpha$ is said to be nuclear identifiable in Q if α can be expressed as $\alpha_\eta^\epsilon(x)\alpha_\xi^\omega(x)\alpha_\chi^\kappa(x)\alpha_\zeta^\psi(x)$, where $\epsilon,\ \omega,\ \kappa,\ \psi \in \{-1,\ 1\}$ and $\eta,\ \xi,\ \chi,\ \zeta \in \{\lambda,\ \rho,\ \mu\}$.*

We shall code such identity $Id = Id_\alpha$ as $(\eta,\ \xi,\ \chi,\ \zeta\ ;\epsilon,\ \omega,\ \kappa,\ \psi)$ and replace 1 and -1 by $+$ and $-$ in concrete instances. Using Definition 3.1, a dozen identities of second Bol-Moufang type which are directly or indirectly affiliated with the loop identities induced by nuclear identification in [9] (except the extra identities) are presented (cf. Table 4).

Lemma 3.2. *Let Q be a loop satisfying an identity $(\eta,\ \xi,\ \chi,\ \zeta;\ \epsilon,\ \omega,\ \kappa,\ \psi)$ such that $\zeta = \eta = \xi \neq \chi$. Then $N_\eta = N_\chi$.*

Proof. Let Q be a loop satisfying an identity $(\eta,\ \xi,\ \chi,\ \zeta;\ \epsilon,\ \omega,\ \kappa,\ \psi)$ which is equivalently expressible by the autotopism α. Then $\alpha = \alpha_\eta^\epsilon(x)\alpha_\xi^\omega(x)\alpha_\chi^\kappa(x)\alpha_\zeta^\psi(x)$, where $\epsilon,\ \omega,\ \kappa,\ \psi \in \{-1,\ 1\}$ and $\eta,\ \xi,\ \chi,\ \zeta \in \{\lambda,\ \rho,\ \mu\}$. With the hypothesis $\zeta = \eta = \xi \neq \chi$, $N_\eta = N_\chi$. □

Definition 3.3. *A triple $\alpha \in SYM(Q)^3$ of a loop Q is said to be an I-shift of an $(U,\ V,\ W) \in Atp(Q)$ if $\alpha = (\rho W\lambda,\ U,\ \rho V\lambda)$.*

Theorem 3.4. *The dozen loop identities $\{Q_i\}_{i=1}^{12}$ of Table 4 induced by the nuclear identification of Definition 3.1 form the following cycles in a WIPL:*
$(Q_1, Q_5, Q_9),\ (Q_2, Q_6, Q_{10}),\ (Q_3, Q_7, Q_{11}),\ (Q_4, Q_8, Q_{12})$ in which each of the identities is the I-shift of the preceding.

Proof. Based on Definition 3.3, for any loop Q, the I-shift of any $(U,\ V,\ W) \in Atp(Q)$ will give a triple $(\rho W\lambda,\ U,\ \rho V\lambda) \in SYM(Q)^3$ which is not necessarily in $Atp(Q)$. For $(\rho W\lambda,\ U,\ \rho V\lambda) \in Atp(Q)$, Q must be a WIPL going by Lemma 2.1 and Theorem 2.2. Note that $\rho = \lambda$ in all the loops identified by nuclear identification in Definition 3.1.

The autotopic equivalence of Q_1 is $(R^{-2}(x)L^{-1}(x)R(x), L(x), L^{-1}(x))$ and its I-shift is

$$\big(\rho L^{-1}(x)\lambda,\ R^{-2}(x)L^{-1}(x)R(x), \rho L(x)\lambda\big) = \big(R(x), R^{-2}(x)L^{-1}(x)R(x), R^{-1}(x)\big)$$

which is the autotopic equivalence of Q_5. Furthermore, the I-shift of the autotopism of Q_5 is $\big(L(x), R(x), L^2(x)R(x)L^{-1}(x)\big)$ and this characterizes Q_9. The I-shift of the autotopism of Q_9 gives the autotopism of Q_1.

Similarly, the autotopism of Q_2 is $\big(R(x), L^{-2}(x)R^{-1}(x)L(x), R^{-1}(x)\big)$ and the I-shift of this autotopism is

$$\big(\rho R^{-1}(x)\lambda, R(x), \rho L^{-2}(x)R^{-1}(x)L(x)\lambda\big) = \big(L(x), R(x), R^2(x)L(x)R^{-1}(x)\big)$$

which is the autotopism that characterizes Q_6. Computing the I-shift of the autotopism of Q_6 gives $\big(L^{-2}(x)R^{-1}(x)L(x), L(x), L^{-1}(x)\big)$ which is the autotopism for Q_{10}. The I-shift of the autotopism of Q_{10} gives the autotopism for Q_2.

The arguments for the other two cycles are similar. □

Lemma 3.5. *The dozen loop identities $\{Q_i\}_{i=1}^{12}$ of Table 4 induced by the nuclear identification of Definition 3.1 are made up of six pairs of dual identities:*
$\{Q_1, Q_2\}, \{Q_3, Q_4\}, \{Q_5, Q_{10}\}, \{Q_6, Q_9\}, \{Q_7, Q_{12}\}, \{Q_8, Q_{11}\},$

Proof. This follows by checking the identities of $\{Q_i\}_{i=1}^{12}$ in Table 4 for duality. □

Corollary 3.6. *Let Q be a loop. The following are equivalent to each other:*
(a) Q obeys Q_1 and WIP, (b) Q obeys Q_2 and WIP, (c) Q is a Moufang loop.

Proof. This follows from Theorem 3.4. □

3.2 Loops of Second Bol-Moufang Type

Lemma 3.7.

1. *In a loop, each of the following identities* Q_1, Q_3, Q_7, Q_8 *implies* $P_\lambda(x, y)$.
2. *In a loop, each of the following identities* Q_2, Q_4, Q_{11}, Q_{12} *implies* $P_\rho(x, y)$.
3. *A loop in which any of the identittes* Q_5, Q_6, Q_9, Q_{10} *is obeyed is a flexible loop.*
4. *A flexible loop obeys* $P_\lambda(x, y)$ *and* $P_\rho(x, y)$.
5. *Any loop that obeys* $P_\lambda(x, y)$ *or* $P_\rho(x, y)$ *has* 2*-sided inverse.*

Proof. This is easily achieved by using the identities in a loop. □

Theorem 3.8. *Let* Q *be a loop.*

1. Q *is a* Q_1*-loop if and only if* Q *is a left Bol loop and* $P_\lambda(x, y)$ *is satisfied.*
2. Q *is a* Q_2*-loop if and only if* Q *is a right Bol loop and* $P_\rho(x, y)$ *is satisfied.*
3. Q *is a* Q_3*-loop if and only if* Q *is an LCC-loop and* $P_\lambda(x, y)$ *is satisfied.*
4. Q *is a* Q_4*-loop if and only if* Q *is an RCC-loop and* $P_\rho(x, y)$ *is satisfied.*
5. Q *is a* Q_5*-loop if and only if* Q *is a right Moufang loop.*
6. Q *is a* Q_6*-loop if and only if* Q *is an MM1-loop if and only if* Q *is a MM2-loop.*
7. Q *is a* Q_7*-loop if and only if* Q *is an RCC and* $P_\lambda(x, y)$ *is satisfied.*
8. Q *is a* Q_8*-loop if and only if* Q *is a Buchsteiner loop and* $P_\lambda(x, y)$ *is satisfied.*
9. Q *is a* Q_9*-loop if and only if* Q *is an MM1-loop if and only if* Q *is an MM2-loop.*
10. Q *is a* Q_{10}*-loop if and only if* Q *is an LM-loop.*
11. Q *is a* Q_{11}*-loop if and only if* Q *is a Buchsteiner loop and* $P_\rho(x, y)$ *is satisfied.*
12. Q *is a* Q_{12}*-loop if and only if* Q *is an LCC-loop and* $P_\rho(x, y)$ *is satisfied.*
13. Q *is a WIP PACC-loop if and only if* Q *is a* Q_3*-loop and a* Q_4*-loop if and only if* Q *is a* Q_7*-loop and a* Q_{12}*-loop.*
14. Q *is a* Q_8*-loop if and only if* Q *is a* Q_{11}*-loop.*

Proof. 1. Let Q be a Q_1-loop, then by Lemma 3.7(1), $P_\lambda(x, y)$ is satisfied. Note that

$$x(yx \cdot xz) = (x(yx \cdot x))z \Rightarrow$$
$$x(y \cdot xz) = (x \cdot yx)z.$$

Conversely, suppose Q is a left Bol loop and $P_\lambda(x,y)$ is satisfied. Then,

$$\begin{aligned} x(y\cdot xz) &= (x\cdot yx)z \Rightarrow \\ x(yx\cdot xz) &= (x(yx\cdot x))z \Rightarrow \\ x(yx\cdot xz) &= ((xy\cdot x)x)z. \end{aligned}$$

2. This follows from the mirror argument of 1.

3. Let Q be a Q_3-loop, then by Lemma 3.7(1), $P_\lambda(x,y)$. So, identity Q_3 becomes

$$\begin{aligned} (xy\cdot x)\cdot xz &= x((x\backslash((xy\cdot x)x))z) \Rightarrow \\ y\cdot xz &= x((x\backslash(yx))z) \Rightarrow \\ (xy)/x\ \cdot xz &= x(yz). \end{aligned}$$

For the converse, suppose Q is an LCC-loop, then

$$\begin{aligned} y\cdot xz &= x((x\backslash(yx))z) \Rightarrow \\ (xy\cdot x)\cdot xz &= x((x\backslash((xy\cdot x)x))z). \end{aligned}$$

This last identity becomes Q_3 since Q also satisfies $x\backslash((xy\cdot x)x) = yx\cdot x$.

4. This can be proved by mirroring the argument in 3 above.

5. Assume Q is a Q_5-loop, then by Lemma 3.7(3), Q is flexible. Thus,

$$\begin{aligned} (yx\cdot zx)x &= y((xz\cdot x)x) \Rightarrow \\ (yx\cdot zx)x &= y((x\cdot zx)x) \Rightarrow \\ (yx\cdot z)x &= y(xz\cdot x) \\ &= y(x\cdot zx). \end{aligned}$$

Conversely, suppose Q is a right Moufang loop, then

$$\begin{aligned} (yx\cdot z)x &= y(x\cdot zx) \Rightarrow \\ (yx\cdot z)x &= y(xz\cdot x) \Rightarrow \\ (yx\cdot zx)x &= y((x\cdot zx)x) \Rightarrow \\ (yx\cdot zx)x &= y((xz\cdot x)x). \end{aligned}$$

6. Let Q be a Q_6-loop, then by Lemma 3.7 (3), Q is flexible, and by Lemma 3.7, Q_6 satisfies $P_\lambda(x,y)$. Therefore,

$$\begin{aligned} (xy\cdot zx)x &= x((yz\cdot x)x) \Rightarrow \\ (xy\cdot zx)x &= ((x\cdot yz)x)x \Rightarrow \\ (xy\cdot zx) &= (x\cdot yz)x. \end{aligned}$$

Conversely, suppose Q is an MM2-loop, then

$$\begin{aligned}(xy\cdot zx) &= (x\cdot yz)x \Rightarrow\\ (xy\cdot zx)x &= ((x\cdot yz)x)x \Rightarrow\\ (xy\cdot zx)x &= x(yz\cdot x)x = x((yz\cdot x)x).\end{aligned}$$

Again, if Q is a Q_6-loop, then

$$\begin{aligned}(xy\cdot zx)x &= x((yz\cdot x)x) \Rightarrow\\ (xy\cdot zx)x &= (x(yz\cdot x))x \Rightarrow\\ xy\cdot zx &= x(yz\cdot x).\end{aligned}$$

Thus, Q is an MM1-loop. Conversely, suppose Q is a MM1-loop, then

$$\begin{aligned}(xy\cdot zx) &= x(yz\cdot x) \Rightarrow\\ (xy\cdot zx)x &= (x(yz\cdot x))x \Rightarrow\\ (xy\cdot zx)x &= x((yz\cdot x)x).\end{aligned}$$

7. Let Q be a Q_7-loop, then by Lemma 3.7(2), Q satisfies $P_\lambda(x,y)$ or $x\backslash(xy\cdot x)x) = (yx\cdot x)$. Thus,

$$\begin{aligned}(y(xz\cdot x))x &= yx\cdot(zx\cdot x) \Rightarrow\\ (y(xz\cdot x))x &= yx\cdot x\backslash((xz\cdot x)x) \Rightarrow\\ yz\cdot x &= yx\cdot x\backslash zx.\end{aligned}$$

Conversely, let Q be an RCC-loop, then

$$\begin{aligned}yz\cdot x &= yx\cdot x\backslash zx \Rightarrow\\ (y(xz\cdot x))x &= yx\cdot x\backslash((xz\cdot x)x)\end{aligned}$$

and the result follows since Q also satisfies $x\backslash(xy\cdot x)x) = (yx\cdot x)$.

8. Suppose Q is a Q_8-loop, then Q satisfies

$$\begin{aligned}x((y\cdot zx)x) &= ((xy\cdot z)x)x \Rightarrow\\ x(yz\cdot x) &= ((xy\cdot z/x)x)x \Rightarrow\\ ((x(yz\cdot x))/x)/x &= xy\cdot z/x.\end{aligned} \tag{10}$$

By Lemma 3.7(4), Q satisfies $P_\lambda(x,y)$ or equivalently,

$$((x\cdot yx)/x)/x = x(y/x). \tag{11}$$

Use (11) in (10) to get

$$x((yz)/x)) = xy\cdot z/x \Rightarrow$$

$$\begin{aligned} x((y\cdot zx)/x)) &= xy\cdot z \Rightarrow \\ (y\cdot zx)/x &= x\backslash(xy\cdot z). \end{aligned}$$

Conversely, suppose Q is a Buchsteiner loop and $P_\lambda(x,y)$ or equivalently $(x\cdot yx)/x)/x = x(y/x)$, then

$$\begin{aligned} (y\cdot zx)/x &= x\backslash(xy\cdot z) \Rightarrow \\ x((y\cdot zx)/x)) &= xy\cdot z \Rightarrow \\ x((yz)/x)) &= xy\cdot z/x \Rightarrow \\ ((x(yz\cdot x))/x)/x &= xy\cdot z/x \Rightarrow \\ x((y\cdot zx)x) &= ((xy\cdot z)x)x. \end{aligned}$$

9. Suppose Q is a Q_9-loop, then

$$\begin{aligned} x(xy\cdot zx) &= (x(x\cdot yz))x \Rightarrow \\ x(xy\cdot zx) &= x((x\cdot yz)x) \Rightarrow \\ xy\cdot zx &= (x\cdot yz)x. \end{aligned}$$

Thus, Q is a MM2-loop. Conversely, suppose Q is MM2-loop, then

$$\begin{aligned} xy\cdot zx &= (x\cdot yz)x \Rightarrow \\ x(xy\cdot zx) &= x((x\cdot yz)x) \Rightarrow \\ x(xy\cdot zx) &= (x(x\cdot yz))x. \end{aligned}$$

Therefore, Q is an Q_9-loop. Now, suppose Q is Q_9, then by Lemma 3.7(1),

$$\begin{aligned} x(xy\cdot zx) &= (x(x\cdot yz))x \Rightarrow \\ x(xy\cdot zx) &= x(x(yz\cdot x)) \Rightarrow \\ xy\cdot zx &= x(yz\cdot x). \end{aligned}$$

Thus, Q is an MM1-loop. Conversely, suppose Q is MMI-loop, then just reverse the process to get Q_9.

10. The proof is similar to the one in 5.

11. The Q_{11} identity is mirror to Q_8 identity, so a mirror argument will suffice.

12. Suppose Q is Q_{12}-loop, then using Lemma 3.7(2) in the Q_{12} identity, we have

$$\begin{aligned} x((x\cdot yx)z) &= (x((x\cdot yx))/x\cdot xz \Rightarrow \\ x\cdot yz &= (xy)/x\cdot xz. \end{aligned}$$

The converse is easy if we reverse the process and use the fact that Q also satisfies $P_\rho(x,y)$.

13. This follows from 7 and 12 of above and Theorem 1.1.

14. Let Q be a Q_8-loop. Then, Q is a Buchsteiner loop in which $P_\lambda(x,y)$ holds by 8. By Lemma 3.7(5), Q is a Buchsteiner loop with 2-sided inverse, hence, a WIP Buchsteiner loop. Applying Theorem 3.4, Q is a Q_{12}-loop and so $P_\rho(x,y)$ holds. Thus, Q is a Q_{11}-loop by 11. The converse is similar. Therefore, Q is a Q_8-loop if and only if Q is a Q_{11}-loop.

□

Lemma 3.9.

1. *In an LC-loop the identity $P_\rho(x,y)$ is satisfied.*
2. *In an RC-loop the identity $P_\lambda(x,y)$ is satisfied.*
3. *In a C-loop the identities $P_\rho(x,y)$ and $P_\lambda(x,y)$ are satisfied.*
4. *In an extra loop, the identities $P_\rho(x,y)$ and $P_\lambda(x,y)$ are satisfied.*

Proof. 1. Put $z = x$ in (2).

2. Put $y = x$ in (3).
3. A loop is a C-loop if and only if it is an LC-loop and RC-loop.
4. An extra loop is a C-loop.

□

Code	Identity	Label	Equivalent Form(s) (⇔)
$(\mu,\mu,\lambda,\mu;+,+,-,-)$	$x(yx \cdot xz) = ((xy \cdot x)x)z$	Q_1	LB + $P_\lambda(x,y)$
$(\mu,\mu,\rho,\mu;-,-,-,+)$	$(yx \cdot xz)x = y(x(x \cdot zx))$	Q_2	RB + $P_\rho(x,y)$
$(\mu,\mu,\lambda,\mu;+,+,+,-)$	$(xy \cdot x) \cdot xz = x((yx \cdot x)z)$	Q_3	LWPC=LCC + $P_\lambda(x,y)$
$(\mu,\mu,\rho,\mu;-,-,+,+)$	$yx \cdot (x \cdot zx) = (y(x \cdot xz))x$	Q_4	RWPC=RCC + $P_\rho(x,y)$
$(\rho,\rho,\mu,\rho;-,-,-,+)$	$(yx \cdot zx)x = y((xz \cdot x)x)$	Q_5	RM
$(\rho,\rho,\lambda,\rho;+,+,+,-)$	$(xy \cdot zx)x = x((yz \cdot x)x)$	Q_6	MM1 or MM2
$(\rho,\rho,\mu,\rho;-,-,+,+)$	$(y(xz \cdot x))x = yx \cdot (zx \cdot x)$	Q_7	RCC + $P_\lambda(x,y)$
$(\rho,\rho,\lambda,\rho;+,+,-,-)$	$x((y \cdot zx)x) = ((xy \cdot z)x)x$	Q_8	BUCH + $P_\lambda(x,y)$
$(\lambda,\lambda,\rho,\lambda;+,+,+,-)$	$x(xy \cdot zx) = (x(x \cdot yz))x$	Q_9	MM1 or MM2
$(\lambda,\lambda,\mu,\lambda;-,-,+,+)$	$x(xy \cdot xz) = (x(x \cdot yx))z$	Q_{10}	LM
$(\lambda,\lambda,\rho,\lambda;+,+,-,-)$	$(x(xy \cdot z))x = x(x(y \cdot zx))$	Q_{11}	BUCH + $P_\rho(x,y)$
$(\lambda,\lambda,\mu,\lambda;-,-,-,+)$	$x((x \cdot yx)z) = (x \cdot xy) \cdot xz$	Q_{12}	LCC + $P_\rho(x,y)$

Table 1. Summary of new loop identities induced by nuclear identifications and their equivalent forms

Theorem 3.10. *The variety* $(\eta,\ \xi,\ \chi,\ \zeta\ ;\epsilon,\ \omega,\ \kappa,\ \psi)^*$ *consists of all commutative loops in the variety* $(\eta,\ \xi,\ \chi,\ \zeta\ ;\epsilon,\ \omega,\ \kappa,\ \psi)$*, whenever* $\epsilon,\ \omega,\ \kappa,\ \psi \in \{-1, 1\}$ *and* $\eta,\ \xi,\ \chi,\ \zeta \in \{\rho, \lambda, \mu\}$*, such that* $\zeta = \eta = \xi \neq \chi$*.*

Proof. Let Q be a commutative loop.

If $(A, B, C) \in Atp(Q)$, then $(B, A, C) \in Atp(Q)$. Let $(A(x), B(x), C(x))$ be the autotopisms of the loop varieties described by identities $(\eta,\ \xi,\ \chi,\ \zeta\ ;\epsilon,\ \omega,\ \kappa,\ \psi)$ as given in Table 4 and let $(\eta,\ \xi,\ \chi,\ \zeta\ ;\epsilon,\ \omega,\ \kappa,\ \psi)^*$ be the loop varieties determined by $(B(x), A(x), C(x))$ for all $x \in Q$. Table 5 highlights the identities obtained for these varieties (where ABG stands for the variety of abelian groups and CML represents commutative Moufang loop). It can be easily verified that each of these laws describes a variety of commutative loops (abelian grousp and commutative Moufang loops). □

Corollary 3.11. *Let* Q *be a loop.*

1. *The following are equivalent:*
   - *(a)* Q *is a commutative Moufang loop.*
   - *(b)* Q *obeys* Q_1^* *or* Q_2^* *or* Q_5^* *or* Q_6^* *or* Q_9^* *or* Q_{10}^**.*
2. *The following are equivalent:*
   - *(a)* Q *is an abelian group.*
   - *(b)* Q *obeys* Q_3^* *or* Q_4^* *or* Q_7^* *or* Q_8^* *or* Q_{11}^* *or* Q_{12}^**.*

Proof. This follows from Theorem 3.10 and Table 5. □

Lemma 3.12. *The dozen loop identities* $\{Q_i^*\}_{i=1}^{12}$ *of Table 5 induced by twisted nuclear identification are made up of six pairs of dual identities:*
$\{Q_1^*, Q_2^*\}, \{Q_3^*, Q_4^*\}, \{Q_5^*, Q_{10}^*\}, \{Q_6^*, Q_9^*\}, \{Q_7^*, Q_{12}^*\}, \{Q_8^*, Q_{11}^*\}$*.*

Proof. This follows by checking the identities of $\{Q_i^*\}_{i=1}^{12}$ in Table 5 for duality. □

3.3 Examples and Constructions

We shall be using the GAP Package [30] and Library of GAP-LOOPS Package [29] to get some examples of non-Moufang, non-extra loops and non-CC-loops that are of second Bol-Moufang type. In GAP-LOOPS, 'LeftBolLoop(n, m)' returns the mth left Bol loop (LBL) of order $n < 17$ while 'RightBolLoop(n, m)' returns mth right Bol loop (RBL) of order $n < 17$ in the library. Similarly, 'RCCLoop(n, m)' returns the mth right conjugacy closed loop (RCCL) of order $n \leq 27$ while 'LCCLoop(n, m)' returns the mth left conjugacy closed loop (LCCL) of order $n \leq 27$ in the library.

1. Any Moufang loop obeys $Q_1, Q_2, Q_5, Q_6, Q_9, Q_{10}$.
2. Any extra loop obeys any of the identities in the set $\{Q_i\}_{i=1}^{12}$.

3. 'LeftBolLoop(8, i)', $i = 1, 2, \cdots, 6$, is an LBL, which is a non-Moufang loop (i.e. does not obey Q_5, Q_6, Q_9, Q_{10}) and obeys $P_\lambda(x, y)$ (hence a Q_1-loop). It also obeys $P_\rho(x, y)$ but is not an RBL. So, it is not a Q_2-loop. See Proposition 3.2 of [13].

4. LeftBolLoop(8, i), $i = 1, 2, \cdots, 6$, is an LCCL, which is non-Moufang and non-CC loop that obeys $P_\lambda(x, y)$ (hence a Q_3-loop). It also obeys $P_\rho(x, y)$ but is not an RCCL. So, it is not a Q_4-loop. Since it obeys $P_\rho(x, y)$ and it is an LCCL, then, it is a Q_{12}-loop. Though, it obeys $P_\lambda(x, y)$ but it is not an RCCL, hence, not a Q_7-loop. See Proposition 3.2 of [13].

5. According to ([9], Lemma 3.6), a Buchsteiner loop is an LCCL iff it is an RCCL. Assume by contradiction that LeftBolLoop(8, i), $i = 1, 2, \cdots, 6$, is a Buchsteiner loop. Since it is an LCCL, then it should be an RCCL which will be a contradiction. So, LeftBolLoop(8, i), $i = 1, 2, \cdots, 6$, is not a Buchsteiner loop. Hence, LeftBolLoop(8, i), $i = 1, 2, \cdots, 6$, is neither a Q_8-loop nor a Q_{11}-loop.

6. Consider the opposite loop of LeftBolLoop(8, i), $i = 1, 2, \cdots, 6$, i.e. LeftBolLoop(8, i)*= RightBolLoop(8, i), $i = 1, 2, \cdots, 6$. It is an RBL, which is non-Moufang loop (i.e. does not obey Q_5, Q_6, Q_9, Q_{10}) and obeys $P_\rho(x, y)$ (hence a Q_2-loop). It also obeys $P_\lambda(x, y)$ but is not an LBL. So, it is not a Q_1-loop.

7. RightBolLoop(8, i), $i = 1, 2, \cdots, 6$, is an RCCL, which is non-Moufang and non-CC loop that obeys $P_\rho(x, y)$ (hence a Q_4-loop). It also obeys $P_\lambda(x, y)$ but is not an LCCL. So, it is not a Q_3-loop. Since it obeys $P_\lambda(x, y)$ and it is an RCCL, then it is a Q_7-loop. Though, it obeys $P_\rho(x, y)$ but it is not an LCCL, hence, not a Q_{12}-loop. One of such loops was constructed in Example 2.1 of [39].

8. According to ([9], Lemma 3.6), a Buchsteiner loop is an RCCL iff it is an LCCL. Assume by contradiction that RightBolLoop(8, i), $i = 1, 2, \cdots, 6$, is a Buchsteiner loop. Since it is an RCCL, then it should be an LCCL which will be a contradiction. So, RightBolLoop(8, i), $i = 1, 2, \cdots, 6$, is not a Buchsteiner loop. Hence, RightBolLoop(8, i), $i = 1, 2, \cdots, 6$, is neither a Q_8-loop nor a Q_{11}-loop.

9. For $n = 6$:
   (a) When $m = 1$, both $P_\lambda(x, y)$ and $P_\rho(x, y)$ are satisfied by RCCLoop(n, m). Hence, it is a Q_4-loop and Q_7-loop. But it is not a $Q_1, Q_2, Q_3, Q_8, Q_{11}, Q_{12}$-loop.
   (b) When $m = 2, 3$, none of $P_\lambda(x, y)$ and $P_\rho(x, y)$ is satisfied by RCCLoop(n, m). Thus, it does not satisfy any of $\{Q_i\}_{i=1}^{12}$.

10. For $n = 8$:

(a) When $m = 1, 2, 3, 7, 8, 9, 13, 15, 16, 17, 18, 19$, none of $P_\lambda(x, y)$ and $P_\rho(x, y)$ is satisfied by RCCLoop(n, m). Thus, it does not satisfy any of $\{Q_i\}_{i=1}^{12}$

(b) When $m = 4, 5, 6, 10, 11, 12$, both $P_\lambda(x, y)$ and $P_\rho(x, y)$ are satisfied by RCCLoop(n, m). It is also an RBL. Hence, it is a Q_2, Q_4-loop and Q_7-loop. But it is not a $Q_1, Q_3, Q_8, Q_{11}, Q_{12}$-loop.

(c) When $m = 14$, RCCLoop(n, m) satisfies $P_\lambda(x, y)$ but does not satisfy $P_\rho(x, y)$. Hence, it is a Q_7-loop but it is not a $Q_1, Q_2, Q_3, Q_4, Q_8, Q_{11}, Q_{12}$-loop.

11. In Theorem 3.1 and Theorem 3.4 of [13], methods of construction of Q_3 loops were described.

Example 3.13. *Let* $G = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16\}$ *and define* $*$ *on* G *as shown in Table 2.* $(G, *)$ *is a* Q_3*-loop,* Q_4*-loop,* Q_7*-loop,* Q_{12}*-loop,* Q_8*-loop,* Q_{11}*-loop that is power associative, not diassociative, not (Moufang, left Bol, right Bol, LC, RC, C, extra), not (left or right power alternative), right A-loop and left A-loop, not middle A-loop.* $(G, *)$ *is not a* $Q_1, Q_2, Q_5, Q_6, Q_9, Q_{10}$*-loop.*

*	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
1	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
2	2	3	4	1	6	7	8	5	10	11	12	9	14	15	16	13
3	3	4	1	2	7	8	5	6	11	12	9	10	15	16	13	14
4	4	1	2	3	8	5	6	7	12	9	10	11	16	13	14	15
5	5	6	7	8	1	2	3	4	13	14	15	16	11	12	9	10
6	6	7	8	5	2	3	4	1	14	15	16	13	12	9	10	11
7	7	8	5	6	3	4	1	2	15	16	13	14	9	10	11	12
8	8	5	6	7	4	1	2	3	16	13	14	15	10	11	12	9
9	9	10	11	12	16	13	14	15	1	2	3	4	8	5	6	7
10	10	11	12	9	13	14	15	16	2	3	4	1	5	6	7	8
11	11	12	9	10	14	15	16	13	3	4	1	2	6	7	8	5
12	12	9	10	11	15	16	13	14	4	1	2	3	7	8	5	6
13	13	14	15	16	12	9	10	11	7	8	5	6	4	1	2	3
14	14	15	16	13	9	10	11	12	8	5	6	7	1	2	3	4
15	15	16	13	14	10	11	12	9	5	6	7	8	2	3	4	1
16	16	13	14	15	11	12	9	10	6	7	8	5	3	4	1	2

Table 2. A Q_3-loop, Q_4-loop, Q_7-loop, Q_{12}-loop, Q_8-loop, Q_{11}-loop $(G, *)$

Example 3.14. *Let* $G = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16\}$ *and define* $\star$ *on* G *as shown in Table 3.* $(G, \star)$ *is a* Q_3*-loop,* Q_4*-loop,* Q_7*-loop,* Q_{12}*-loop,* Q_8*-loop,* Q_{11}*-loop that is power associative, not diassociative, not (Moufang, left Bol, right Bol, LC, RC, C, extra), not (left or right power alternative), right A-loop and left A-loop, not middle A-loop.* $(G, \star)$ *is not a* $Q_1, Q_2, Q_5, Q_6, Q_9, Q_{10}$*-loop.* $(G, *)$ *and* $(G, \star)$ *are neither isomorphic nor isotopic.*

3.4 Discussion, Conclusion and Future Study

Note that $P_\rho(x,y)$ and $P_\lambda(x,y)$ are satisfied by any dissociative loop (e.g. Moufang or extra loop). In fact, each of the identities in $\{Q_i\}_{i=1}^{12}$ generalizes the extra law in loops, but this is not true of the Moufang law in loops. Among all the loop identities tracked down by the nuclear identification code in (Table 1, [9]), only the Moufang identities got tracked down distinctively as new loop identities by the nuclear identification code introduced in this work (see Q_5, Q_6, Q_9, Q_{10} in Table 1). The importance of $P_\rho(x,y)$ and $P_\lambda(x,y)$ in this current work is the fact that they are associated to equivalent forms of the new loop identities which were not tracked down distinctively by our nuclear identification code (see $Q_1, Q_2, Q_3, Q_4, Q_7, Q_8, Q_{11}, Q_{12}$ in Table 1). Among all the loop identities tracked down by the nuclear identification code in (Table 1, [9]), the left (right) Bol, LCC (RCC) and Buchsteiner identities got tracked down non-distinctively as new loop identities by our nuclear identification code (see $Q_1, Q_2, Q_3, Q_4, Q_7, Q_8, Q_{11}, Q_{12}$ in Table 1). Among the 12 identities tracked down by the nuclear identification code in (Table 1, [9]), only the extra identities are missing in our own work. But our work has been able to discover:

1. eight new loop identities (i.e. $Q_1, Q_2, Q_3, Q_4, Q_7, Q_8, Q_{11}, Q_{12}$) among which the two pairs (Q_3, Q_4) and (Q_7, Q_{12}) axiomatize the weak inverse property power associative conjugacy closed (WIP PACC) loop, while Q_8 and Q_{11} were found to be equivalent.

2. four new loop identities which individually characterize the Moufang identities

$\star$	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
1	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
2	2	1	4	3	6	5	8	7	10	9	12	11	14	13	16	15
3	3	4	1	2	7	8	5	6	11	12	9	10	15	16	13	14
4	4	3	2	1	8	7	6	5	12	11	10	9	16	15	14	13
5	5	6	8	7	1	2	4	3	13	14	16	15	10	9	11	12
6	6	5	7	8	2	1	3	4	14	13	15	16	9	10	12	11
7	7	8	6	5	3	4	2	1	15	16	14	13	12	11	9	10
8	8	7	5	6	4	3	1	2	16	15	13	14	11	12	10	9
9	9	10	12	11	15	16	14	13	1	2	4	3	7	8	6	5
10	10	9	11	12	16	15	13	14	2	1	3	4	8	7	5	6
11	11	12	10	9	13	14	16	15	3	4	2	1	5	6	8	7
12	12	11	9	10	14	13	15	16	4	3	1	2	6	5	7	8
13	13	14	15	16	12	11	10	9	6	5	8	7	4	3	2	1
14	14	13	16	15	11	12	9	10	5	6	7	8	3	4	1	2
15	15	16	13	14	10	9	12	11	8	7	6	5	2	1	4	3
16	16	15	14	13	9	10	11	12	7	8	5	6	1	2	3	4

Table 3. A Q_3-loop, Q_4-loop, Q_7-loop, Q_{12}-loop, Q_8-loop, Q_{11}-loop $(G, \star)$

in loops (i.e. Q_5, Q_6, Q_9, Q_{10}). Thus, we now have eight loop identities that characterize Moufang loop.

Therefore, we have been able to identify a dozen second Bol-Moufang type identities via nuclear identification, among which are first Bol-Moufang type identities (e.g. Moufang) or relations of first Bol-Moufang type identities (RB, LB) or non-Bol-Moufang type identities (LCC, RCC, Buchsteiner).

In [9], loop identities that split into at least two other loop identities (with at least three variables) were nuclear identified (e.g. extra and Moufang). Also, loop identities that do not split into at least two other loop identities (with at least three variables) were also nuclear identified (e.g. left Bol, right Bol, LCC, RCC and Buchsteiner). But, with our own nuclear identification model, loop identities that split into at least two other loop identities (with at least three variables) were nuclear identified (only Moufang) without the company of $P_\rho(x,y)$ or $P_\lambda(x,y)$. While, loop identities that do not split into at least two other loop identities (with at least three variables) were also nuclear identified (e.g. left Bol, right Bol, LCC, RCC and Buchsteiner) with the company of $P_\rho(x,y)$ or $P_\lambda(x,y)$. Thus, $P_\rho(x,y)$ and $P_\lambda(x,y)$ are distinguishing features between our own nuclear identification model and that of [9].

Note that a Q_8-loop and Q_{11}-loop are both Buchsteiner loops with 2-sided inverse. Hence, they are linked to **Buch2SI** in the following chain of varieties of Buchsteiner loops (Csorgo [8]):

$$\mathbf{BuchCS} \subset \mathbf{Buch2SI} \subset \mathbf{BuchWIP} \subset \mathbf{BuchCC}$$

where **BuchCS**, **Buch2SI**, **BuchWIP** and **BuchCC** represent the varieties of Buchsteiner with central square, Buchsteiner with 2-sided inverse, Buchsteiner with WIP and Buchsteiner that is a CC-loop respectively. The identities that describe Q_8-loop and Q_{11}-loop form two varieties of Buchsteiner loops. But we are not sure if the varieties **BuchCS**, **Buch2SI**, **BuchWIP** and **BuchCC** have single identities that describe them.

Just like the dozen laws of (Proposition 1.3, [9]) form four cycles, our dozen laws also form four cycles as well (but in a sequential manner) and also form six pairs of dual identities. Using twisted nuclear identification, the authors in [9] were able to identify six identities of lengths four that describe the abelian group variety and commutative Moufang loop variety (in each case). We also achieved a similar result in this work with the discovery of six identities of length five that describe the abelian group variety and commutative Moufang loop variety (in each case). This second dozen of identities were also found to form six pairs of dual identities.

Code	Autotopism	Identity	Label
$(\mu,\mu,\lambda,\mu;+,+,-,-)$	$(R^{-2}(x)L^{-1}(x)R(x), L(x), L^{-1}(x))$	$x(yx\cdot xz)=((xy\cdot x)x)z$	Q_1
$(\mu,\mu,\rho,\mu;-,-,-,+)$	$(R(x), L^{-2}(x)R^{-1}(x)L(x), R^{-1}(x))$	$(yx\cdot xz)x=y(x(x\cdot zx))$	Q_2
$(\mu,\mu,\lambda,\mu;+,+,+,-)$	$(R^{-2}(x)L(x)R(x), L(x), L(x))$	$(xy\cdot x)\cdot xz=x((yx\cdot x)z)$	Q_3
$(\mu,\mu,\rho,\mu;-,-,+,+)$	$(R(x), L^{-2}(x)R(x)L(x), R(x))$	$yx\cdot(x\cdot zx)=(y(x\cdot xz))x$	Q_4
$(\rho,\rho,\mu,\rho;-,-,-,+)$	$(R(x), R^{-2}(x)L^{-1}(x)R(x), R^{-1}(x))$	$(yx\cdot zx)x=y((xz\cdot x)x)$	Q_5
$(\rho,\rho,\lambda,\rho;+,+,+,-)$	$(L(x), R(x), R^{2}(x)L(x)R^{-1}(x))$	$(xy\cdot zx)x=x((yz\cdot x)x)$	Q_6
$(\rho,\rho,\mu,\rho;-,-,+,+)$	$(R^{-1}(x), R^{-2}(x)L(x)R(x), R^{-1}(x))$	$(y(xz\cdot x))x=yx\cdot(zx\cdot x)$	Q_7
$(\rho,\rho,\lambda,\rho,+,+,-,-)$	$(L^{-1}(x), R(x), R^{2}(x)L^{-1}(x)R^{-1}(x))$	$x((y\cdot zx)x)=((xy\cdot z)x)x$	Q_8
$(\lambda,\lambda,\rho,\lambda;+,+,+,-)$	$(L(x), R(x), L^{2}(x)R(x)L^{-1}(x))$	$x(xy\cdot zx)=(x(x\cdot yz))x$	Q_9
$(\lambda,\lambda,\mu,\lambda;-,-,+,+)$	$(L^{-2}(x)R^{-1}(x)L(x), L(x), L^{-1}(x))$	$x(xy\cdot xz)=(x(x\cdot yx))z$	Q_{10}
$(\lambda,\lambda,\rho,\lambda;+,+,-,-)$	$(L(x), R^{-1}(x), L^{2}(x)R^{-1}(x)L^{-1}(x))$	$(x(xy\cdot z))x=x(x(y\cdot zx))$	Q_{11}
$(\lambda,\lambda,\mu,\lambda;-,-,-,+)$	$(L^{-2}(x)R(x)L(x), L^{-1}(x), L^{-1}(x))$	$x((x\cdot yx)z)=(x\cdot xy)\cdot xz$	Q_{12}

Table 4. Summary of new loop identities induced by nuclear identifications

Code	Autotopism	Identity	Variety	Label
$(\mu,\mu,\lambda,\mu;+,+,-,-)^*$	$(L(x), R^{-2}(x)L^{-1}(x)R(x), L^{-1}(x))$	$x(xy\cdot zx)=y((xz\cdot x)x)$	CML	Q_1^*
$(\mu,\mu,\rho,\mu;-,-,-,+)^*$	$(L^{-2}(x)R^{-1}(x)L(x), R(x), R^{-1}(x))$	$(xy\cdot zx)x=(x(x\cdot yx))z$	CML	Q_2^*
$(\mu,\mu,\lambda,\mu;+,+,+,-)^*$	$(L(x), R^{-2}(x)L(x)R(x), L(x))$	$xy\cdot(xz\cdot x)=x(y(zx\cdot x))$	ABG	Q_3^*
$(\mu,\mu,\rho,\mu;-,-,+,+)^*$	$(L^{-2}(x)R(x)L(x), R(x), R(x))$	$(x\cdot yx)\cdot zx=((x\cdot xy)z)x$	ABG	Q_4^*
$(\rho,\rho,\mu,\rho;-,-,-,+)^*$	$(R^{-2}(x)L^{-1}(x)R(x), R(x), R^{-1}(x))$	$(yx\cdot zx)x=((xy\cdot x)x)z$	CML	Q_5^*
$(\rho,\rho,\lambda,\rho;+,+,+,-)^*$	$(R(x), L(x), R^{2}(x)L(x)R^{-1}(x))$	$(yx\cdot xz)x=x((yz\cdot x)x)$	CML	Q_6^*
$(\rho,\rho,\mu,\rho;-,-,+,+)^*$	$(R^{-2}(x)L(x)R(x), R^{-1}(x), R^{-1}(x))$	$((xy\cdot x)z)x=(yx\cdot x)\cdot zx$	ABG	Q_7^*
$(\rho,\rho,\lambda,\rho,+,+,-,-)^*$	$(R(x), L^{-1}(x), R^{2}(x)L^{-1}(x)R^{-1}(x))$	$x((yx\cdot z)x)=((y\cdot xz)x)x$	ABG	Q_8^*
$(\lambda,\lambda,\rho,\lambda;+,+,+,-)^*$	$(R(x), L(x), L^{2}(x)R(x)L^{-1}(x))$	$x(yx\cdot xz)=(x(x\cdot yz))x$	CML	Q_9^*
$(\lambda,\lambda,\mu,\lambda;-,-,+,+)^*$	$(L(x), L^{-2}(x)R^{-1}(x)L(x), L^{-1}(x))$	$x(xy\cdot xz)=y(x(x\cdot zx))$	CML	Q_{10}^*
$(\lambda,\lambda,\rho,\lambda;+,+,-,-)^*$	$(R^{-1}(x), L(x), L^{2}(x)R^{-1}(x)L^{-1}(x))$	$(x(y\cdot xz))x=x(x(yx\cdot z))$	ABG	Q_{11}^*
$(\lambda,\lambda,\mu,\lambda;-,-,-,+)^*$	$(L^{-1}(x), L^{-2}(x)R(x)L(x), L^{-1}(x))$	$x(y(x\cdot zx))=xy\cdot(x\cdot xz)$	ABG	Q_{12}^*

Table 5. Loop Identities obtained by twisted nuclear identifications

In the conclusion of [9], the authors pointed out the prospect of possibly using another nuclear identification model to track down the LC, RC and C-loop identities which their own nuclear identification model could not track down (except if a restriction in their code is expunged). It is worth mentioning that even though our own nuclear identification model could not track down the LC, RC and C-loop identities, but Lemma 3.9 informs us that LC, RC loop identities imply $P_\rho(x, y), P_\lambda(x, y)$ respectively. Thus, some other nuclear identification models for identities of length five that could track down the LC, RC and C-loop identities might exist.

Future Studies Definitely, the dozen identities discovered in this work are not the only identities of second Bol-Moufang type. There is the need to know if there are some more others that can be nuclear identified like the twelve of this work. Perhaps, the extra law which we could not nuclear-identify could be nuclear-identifiable among the future loop identities of second Bol-Moufang type.

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