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Isostrophy Bryant-Schneider Group-Invariant of Bol Loops

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Abstract. In the recent past, Grecu and Syrbu (in no order of preference) have jointly and individually reported some results on isostrophy invariants of Bol loops. Also, the Bryant-Schneider group of a loop has been found important in the study of the isotopy-isomorphy of some varieties of loops (e.g. Bol loops, Moufang loops, Osborn loops). In this current work, the Bryant-Schneider group of a middle Bol loop was linked with some of the isostrophy-group invariance results of Grecu and Syrbu. In particular, it was shown that some subgroups of the Bryant-Schneider group of a middle Bol loop are equal (or isomorphic) to the automorphism and pseudo-aumorphism groups of its corresponding right (left) Bol loop. Some elements of the Bryant-Schneider group of a middle Bol loop were shown to induce automorphisms and middle pseudo-automorphisms. It was discovered that if a middle Bol loop is of exponent 2, then, its corresponding right (left) Bol loop is a left (right) G-loop.

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1 Introduction

Let Q be a non-empty set. Define a binary operation "$\cdot$" on Q. If $x \cdot y \in Q$ for all $x, y \in Q$, then the pair $(Q, \cdot)$ is called a groupoid or magma. If the equations: $a \cdot x = b$ and $y \cdot a = b$ have unique solutions $x, y \in Q$ for all $a, b \in Q$, then $(Q, \cdot)$ is called a quasigroup. Let $(Q, \cdot)$ be a quasigroup and let there exist a unique element $e \in Q$ called the identity element such that for all $x \in Q$, $x \cdot e = e \cdot x = x$, then $(Q, \cdot)$ is called a loop. We write xy instead of $x \cdot y$ and stipulate that $\cdot$ has lower priority than juxtaposition among factors to be multiplied.

Let $(Q, \cdot)$ be a groupoid and let "a" be a fixed element in Q, then the left and right translations L_a, R_a of $a \in Q$ are respectively defined by $xL_a = a \cdot x$ and $xR_a = x \cdot a$ for all $x \in Q$. It can now be seen that a groupoid $(Q, \cdot)$ is a quasigroup if its left and right translation mappings are permutations. Thence, the inverse mappings L_x^{-1} and R_x^{-1} exist. Thus, for any quasigroup $(Q, \cdot)$, we have two new binary operations: right division (/) and left division (\) and middle translation P_a for any fixed $a \in Q$.

$$x \backslash y = yL_x^{-1} = xP_y \qquad \text{and} \qquad x/y = xR_y^{-1} = yP_x^{-1}$$

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and note that

$$x\backslash y = z \Longleftrightarrow x \cdot z = y \qquad \text{and} \qquad x/y = z \Longleftrightarrow z \cdot y = x.$$

Consequently, $(Q, \backslash)$ and $(Q, /)$ are also quasigroups. The symmetric group $SYM(Q)$ of Q is defined as $SYM(Q) = \{U : Q \to Q \mid U \text{ is a permutation}\}$. For a loop $(Q, \cdot)$, the group generated by its left (right) translations is called the left (right) multiplication group $Mult_{\lambda(\rho)}(Q, \cdot) \leq SYM(Q)$.

$$(x/y)(z\backslash x) = x(zy\backslash x) \tag{1}$$

Middle Bol loops (MBLs) were first studied in the work of Belousov [9], where he gave identity (1) characterizing loops that satisfy the universal anti-automorphic inverse property. After this beautiful characterization by Belousov and the laying of foundations for a classical study of this structure, Gvaramiya [19] proved that a loop $(Q, \circ)$ is middle Bol loop if there exists a right Bol loop $(Q, \cdot)$ such that $x \circ y = (y \cdot xy^{-1})y$ for all $x, y \in Q$. If $(Q, \circ)$ is a middle Bol loop and $(Q, \cdot)$ is the corresponding right Bol loop, then

$$x \circ y = y^{-1}\backslash x \qquad \text{and} \qquad x \cdot y = y//x^{-1} \tag{2}$$

where for every $x, y \in Q$, '$//$' is the left division in $(Q, \circ)$.

Also, if $(Q, \circ)$ is a middle Bol loop and $(Q, \cdot)$ is the corresponding left Bol loop, then

$$x \circ y = x/y^{-1} \qquad \text{and} \qquad x \cdot y = x//y^{-1} \tag{3}$$

where '$//$' is the left division in $(Q, \circ)$. The relations in (2) and (3) and their translational forms shall be of tremendous use in the proofs of results in this current work.

Grecu [16] showed that the right multiplication group of a middle Bol loop coincides with the left multiplication group of the corresponding right Bol loop. After that, middle Bol loops resurfaced in literature in 1994 and 1996 when Syrbu [40, 41] considered them in relation to the universality of the elasticity law. In 2003, Kuznetsov [39], while studying gyrogroups (a special class of Bol loops) established some algebraic properties of middle Bol loop and designed a method of constructing a middle Bol loop from a gyrogroup.

In 2010, Syrbu [42] studied the connections between structure and properties of middle Bol loops and of the corresponding left Bol loops. It was noted that two middle Bol loops are isomorphic if and only if the corresponding left (right) Bol loops are isomorphic, and a general form of the autotopisms of middle Bol loops was deduced. Relations between different sets of elements, such as nucleus, left (right, middle) nuclei, the set of Moufang elements, the center of a middle Bol loop and left Bol loop were established. In 2012, Grecu and Syrbu [17] proved that two middle Bol loops are isotopic if and only if the corresponding right (left) Bol loops are isotopic. In 2012, Drapal and Shcherbacov [13] rediscovered the middle Bol identities in a new way. In 2013, Syrbu and Grecu [44] established a necessary and sufficient condition for the quotient loop of a middle Bol loop and of its corresponding right Bol loop to be isomorphic. In 2014, Grecu and Syrbu [18] established that the commutant (centrum) of a middle Bol loop is an AIP-subloop and gave a necessary and sufficient condition when the commutant is an invariant under the existing isostrophy between middle Bol loop and the corresponding right Bol loop and the same authors presented a study of loops with invariant flexibility law under the isostrophy of loop [43]. Osoba and Oyebo [31] further investigated the multiplication group of middle Bol loop in relation to left Bol loop while Jaiyéọlá [26, 27] studied second Smarandache Bol loops. Second Smarandache nuclei of second Smarandache Bol loops was further studied by Osoba [30] while more results on the algebraic properties of middle Bol loops using its parastrophes was presented by Oyebo and Osoba [34].

For any non-empty set Q, the set of all permutations on Q forms a group $SYM(Q)$ called the symmetric group of Q. Let $(Q,\cdot)$ be a loop and let $A, B, C \in SYM(Q)$. If

$$xA \cdot yB = (x \cdot y)C, \ \forall x, y \in Q$$

then the triple (A, B, C) is called an autotopism and such triples form a group $AUT(Q,\cdot)$ called the autotopism groups of $(Q,\cdot)$. If $A = B = C$, then A is called an automorphism of $(Q,\cdot)$ which forms a group $AUM(Q,\cdot)$ called the automorphism group of $(Q,\cdot)$.

Grecu [16] showed that right multiplication group of a middle Bol loop coincides with the left multiplication group of the corresponding right Bol loop.

Definition 1. Let $(Q,\cdot)$ be a loop.

1. A mapping $\theta \in SYM(Q,\cdot)$ is called a right special map for Q if there exists $f \in Q$ so that $(\theta, \theta L_f^{-1}, \theta) \in AUT(Q,\cdot)$.
2. A mapping $\theta \in SYM(Q,\cdot)$ is called a left special map for Q if there exists $g \in Q$ so that $(\theta R_g^{-1}, \theta, \theta) \in AUT(Q,\cdot)$.
3. A mapping $\theta \in SYM(Q)$ is called a special map for Q if there exist $f, g \in Q$ so that $(\theta R_g^{-1}, \theta L_f^{-1}, \theta) \in AUT(Q,\cdot)$.

From Definition 1, it is clearly seen that

$$(\theta R_g^{-1}, \theta L_f^{-1}, \theta) = (\theta, \theta, \theta)(R_g^{-1}, L_f^{-1}, I),$$

which implies that θ is an isomorphism of $(Q,\cdot)$ onto some f, g-isotope of it.

Theorem 1. *[36] Let the set* $BS(Q,\cdot) = \{\theta \in SYM(Q) : \ \exists\ f, g \in Q \ni (\theta R_g^{-1}, \theta L_f^{-1}, \theta) \in AUT(Q,\cdot)\}$, *then* $BS(Q,\cdot) \leq SYM(Q)$.

Theorem 1 is associated with Theorem 2.

Theorem 2. *(Pflugfelder [35])*

Let $(G,\cdot)$ *and* $(H,\circ)$ *be two isotopic loops. For some* $f, g \in G$, *there exists an* f, g*-principal isotope* $(G,*)$ *of* $(G,\cdot)$ *such that* $(H,\circ) \cong (G,*)$.

In a loop $(Q, \cdot)$, the set of right special maps shall be represented by $BS_\rho(Q, \cdot)$ and will be called the right Bryant-Schneider set of the loop $(Q, \cdot)$. Similarly, the set of left special maps shall be represented by $BS_\lambda(Q, \cdot)$ and called the left Bryant-Schneider set of the loop $(Q, \cdot)$. Also, the set of special maps shall be represented by $BS(Q, \cdot)$ and called the Bryant-Schneider set of the loop $(Q, \cdot)$. Going by Theorem 1, $BS(Q, \cdot)$ forms a group called the Bryant-Schneider group of the loop $(Q, \cdot)$.

Adeniran [1–3] studied the Bryant-Schneider group of conjugacy closed loops. Jaiyéọlá [20] and Jaiyéọlá et al. [21, 22] used the Bryant-Schneider group to study Smarandache loop, Osborn loop and its universality. For more on quasigroups and loops, see Jaiyéọlá [28], Shcherbacov [38] and Pflugfelder [35].

In 2015, Adeniran et al. [6] carried out a study of some isotopic characterisation of generalised Bol loops. In 2017, Jaiyéọlá et al. [23] studied the holomorphic structure of middle Bol loops and showed that the holomorph of a commutative loop is a commutative middle Bol loop if and only if the loop is a middle Bol loop and its automorphism group is abelian. Adeniran et al. [7, 8], Jaiyéọlá and Popoola [29] studied generalised Bol loops.

In 2018, Jaiyéọlá et al. [24], in furtherance to their exploit obtained new algebraic identities of middle Bol loop, where necessary and sufficient conditions for a bivariate mapping of a middle Bol loop to have RIP, LIP, RAP, LAP and flexible property were presented. In 2020, Syrbu and Grecu [43] considered loops with invariant flexibility under the isostrophy. Additional algebraic properties of middle Bol loops were announced by Jaiyéọlá et al. [25] in 2021.

In furtherance to earlier studies, the first two authors in their work [33] unveiled some algebraic characterizations of right and middle Bol loops relative to their cores. Drapal and Syrbu [14] studied middle Bruck loops and total multiplication group.

Definition 2. A groupoid (quasigroup) $(Q, \cdot)$ is said to have

1. left inverse property (LIP) if there exists a mapping $I_\lambda : x \mapsto x^\lambda$ such that $x^\lambda \cdot xy = y$ for all $x, y \in Q$.

2. right inverse property (RIP) if there exists a mapping $I_\rho : x \mapsto x^\rho$ such that $yx \cdot x^\rho = y$ for all $x, y \in Q$.

3. a right alternative property (RAP) if $y \cdot xx = yx \cdot x$ for all $x, y \in Q$.

4. a left alternative property (LAP) if $y \cdot xx = yx \cdot x$ for all $x, y \in Q$.

5. flexibility or elasticity if $xy \cdot x = x \cdot yx$ holds for all $x, y \in Q$.

Note that $I : x \mapsto x^{-1}$ when $I = I_\rho = I_\lambda$.

Definition 3. A loop $(Q, \cdot)$ is said to be

1. an automorphic inverse property loop (AIPL) if $(xy)^{-1} = x^{-1}y^{-1}$ for all $x, y \in Q$.

2. an anti-automorphic inverse property loop (AAIPL) if $(xy)^{-1} = y^{-1}x^{-1}$ for all $x, y \in Q$.

Definition 4. A loop $(Q, \cdot)$ is called a

1. right Bol loop if $(xy \cdot z)y = x(yz \cdot y)$ for all $x, y, z \in Q$.
2. left Bol loop if $(x \cdot yx)z = x(y \cdot xz)$ for all $x, y, z \in Q$.
3. middle Bol loop if $(x/y)(z\backslash x) = (x/(zy))x$ or $(x/y)(z\backslash x) = x((zy)\backslash x)$ for all $x, y, z \in Q$.

Definition 5. Let $(Q, \cdot)$ be a loop.

1. $\phi \in SYM(Q)$ is called a left pseudo-automorphism with companion $a \in Q$ if $(\phi L_a, \phi, \phi L_a) \in AUT(Q, \cdot)$. The set of left pseudo-automorphisms $PS_\lambda(Q, \cdot)$ forms a group called the left pseudo-automorphism group of $(Q, \cdot)$. See [35].
2. $\phi \in SYM(Q)$ is called a right pseudo-automorphism with companion $a \in Q$ if $(\phi, \phi R_a, \phi R_a) \in AUT(Q, \cdot)$. The set of right pseudo-automorphisms $PS_\rho(Q, \cdot)$ forms a group called the left pseudo-automorphism group of $(Q, \cdot)$. See [35].
3. $\phi \in SYM(Q)$ is called a middle pseudo-automorphism with companion $a \in Q$ if $(\phi R_a^{-1}, \phi L_{a^\lambda}^{-1}, \phi) \in AUT(Q, \cdot)$. The set of middle pseudo-automorphisms $PS_\mu(Q, \cdot)$ forms a group called the middle pseudo-automorphism group of $(Q, \cdot)$. See [44].

Definition 6. Let $(Q, \cdot)$ be a loop.

1. The left nucleus of Q is $N_\lambda = \{a \in Q : ax \cdot y = a \cdot xy \ \forall\ x, y \in Q\}$.
2. The right nucleus of Q is $N_\rho = \{a \in Q : y \cdot xa = yx \cdot a \ \forall\ x, y \in Q\}$.
3. The middle nucleus of Q is $N_\mu = \{a \in Q : ya \cdot x = y \cdot ax \ \forall\ x, y \in Q\}$.
4. The nucleus of Q is $N(Q, \ \cdot) = N_\lambda \cap N_\rho \cap N_\mu$.
5. The centrum or commutant of Q is $C(Q, \cdot) = \{a \in Q : ax = xa \ \forall x \in Q\}$.
6. The centre of Q is $Z(Q, \cdot) = N(Q, \cdot) \cap C(Q, \cdot)$.

Theorem 3. *[35] Let $(Q, \cdot)$ be an inverse property loop or MBL. Then, for any $a \in Q$:*

1. $I_\lambda R_a I_\rho = L_{a^\lambda}$.
2. $I_\rho R_a I_\rho = L_{a^\rho}$.
3. $I_\rho L_a I_\rho = R_{a^\rho}$.
4. $I_\lambda L_a I_\rho = R_{a^\lambda}$.

Lemma 1. *[35]*

1. *Let θ be a right (left) pseudo-automorphism of a loop, then $e\theta = e$.*
2. *Let θ be a right (left) pseudo-automorphism of a LIP (RIP) loop. Then, $I\theta = \theta I$.*

Here are some existing results on some isostrophy invariants of Bol loops.

Theorem 4. *(Grecu and Syrbu [17])*
Let $(Q,\circ)$ be a middle Bol loop and let $(Q,\cdot)$ and $(Q,*)$ be the corresponding right and left Bol loops, respectively.

1. $AUM(Q,\circ) = AUM(Q,\cdot) = AUM(Q,*)$.
2. $AUT(Q,\circ) \cong AUT(Q,\cdot) \cong AUT(Q,*)$.
3. $PS_\lambda(Q,\circ) \cong PS_\rho(Q,\cdot) \cong PS_\lambda(Q,*)$.

Theorem 5. *(Syrbu and Grecu [44])*
Let $(Q,\circ)$ be a middle Bol loop and let $(Q,\cdot)$ and $(Q,*)$ be the corresponding right and left Bol loops, respectively.

1. $PS_\rho(Q,\circ) = PS_\mu(Q,\cdot)$.
2. $PS_\mu(Q,\circ) = PS_\lambda(Q,\cdot)$.
3. $PS_\rho(Q,\circ) = PS_\rho(Q,\cdot)$.
4. $\alpha \in PS_\lambda(Q,\circ) \Leftrightarrow I\alpha I \in PS_\rho(Q,\circ)$.

In the current work, we shall be linking the Bryant-Schneider group of a middle Bol loop with some of the isostrophy-group invariance results in Theorem 4 and Theorem 5. In particular, it will be shown that some subgroups of the Bryant-Schneider group of a middle Bol loop are equal (or isomorphic) to the automorphism and pseudo-aumorphism groups of its corresponding right (left) Bol loop.

2 Main Results

Lemma 2. *Let (α,β,γ) be an autotopism of a middle Bol loop $(Q,\circ)$. Then $(I\beta I, I\alpha I, I\gamma I)$ is also an autotopism of $(Q,\circ)$.*

Proof. Let $(Q,\circ)$ be a middle Bol loop and (α,β,γ) be the autotopism of $(Q,\circ)$, then for all $x,y \in Q$, we have

$$x\alpha \circ y\beta = (x\circ y)\gamma \Longrightarrow [x\alpha \circ y\beta]I = (x\circ y)\gamma I \Longrightarrow [(y\beta)I \circ (x\alpha)I] = (x\circ y)\gamma I.$$

Doing $y \mapsto yI$ and $x \mapsto xI$ in the last equation, we get

$$yI\beta I \circ xI\alpha I = [(xI \circ yI)\gamma]I \implies yI\beta I \circ xI\alpha I = [(y\circ x)I\gamma]I.$$

Thus, $(I\beta I, I\alpha I, I\gamma I) \in AUT(Q,\circ)$. □

Theorem 6. *Let $(Q,\circ)$ be a middle Bol loop and let $\theta \in BS(Q,\circ)$ be such that $\theta : e \mapsto e$. For some $f, g \in Q$:*

1. $L_f^{-1} = P_g^{-1}R_gR_{g^2}^{-1}P_g$ *and* $R_g^{-1} = P_f^{-1}R_fL_{f^2}^{-1}P_f^{-1}$.
2. $\theta = \theta(f,g) \equiv \theta(f,f^{-1})$ *and* $\theta = \theta(f,g) \equiv \theta(g^{-1},g)$.

Proof. Suppose that $(Q,\circ)$ is a middle Bol loop, then
$B = (IP_x^{-1}, IP_x, IP_x^{-1}R_x)$ is an autotopism of $(Q,\circ)$ for all $x \in Q$. Since $A = (\theta R_g^{-1}, \theta L_f^{-1}, \theta) \in AUT(Q,\circ)$ for some $f, g \in Q$, then

$$A = (I\theta L_f^{-1}I, I\theta R_g^{-1}I, I\theta I) \in AUT(Q,\circ) \text{ for some } f, g \in Q. \tag{4}$$

Thus,

$$AB = (I\theta L_f^{-1}IIP_x^{-1}, I\theta R_g^{-1}IIP_x, I\theta IIP_x^{-1}R_x)$$

$$= (I\theta L_f^{-1}P_x^{-1}, I\theta R_g^{-1}P_x, I\theta P_x^{-1}R_x) \in AUT(Q,\circ). \tag{5}$$

Writing this in identical relation, for all $z, y \in Q$, we have
$yI\theta L_f^{-1}P_x^{-1} \circ zI\theta R_g^{-1}P_x = (y \circ z)I\theta P_x^{-1}R_x$
$\Longrightarrow y^{-1}\theta L_f^{-1}P_x^{-1} \circ z^{-1}\theta R_g^{-1}P_x = (y \circ z)^{-1}\theta P_x^{-1}R_x$

$$\Longrightarrow x/(f\backslash(y^{-1})\theta) \circ ((z^{-1}\theta)/g)\backslash x = (x/(z^{-1} \circ y^{-1})\theta) \circ x. \tag{6}$$

Here, setting $y = e$ and $x = f$ in (6), we have

$$\begin{aligned}
[f/(f\backslash e)] \circ (z^{-1}\theta)/g)\backslash f &= (f/(z^{-1}\theta)f \\
\Longrightarrow f/f^\rho \circ zR_g^{-1}P_f &= zP_f^{-1}R_f \\
\Longrightarrow R_g^{-1}P_fL_{f/f^\rho} &= P_f^{-1}R_f \\
\Longrightarrow R_g^{-1} &= P_f^{-1}R_fL_{f/f^\rho}^{-1}P_f^{-1} \\
\Longrightarrow R_g &= P_fR_f^{-1}L_{f^2}P_f
\end{aligned}$$

So, $x \circ g = \{[f^2(x\backslash f)]/f\}\backslash f$. With $x = g$, we get $g = f^{-1}$.
Thus, $\theta = \theta(f,g) \equiv \theta(f,f^{-1})$.

Analogously, if we repeat the same procedure by setting $z = e$ and $x = g$ in (6), we have

$$\begin{aligned}
g/(f\backslash(y^{-1})\theta) \circ (e/g)\backslash g &= (g/(y^{-1})\theta)g \\
\Longrightarrow yL_f^{-1}P_g^{-1}R_{g^\lambda}\backslash g &= yP_g^{-1}R_g \\
\Longrightarrow L_f^{-1}P_g^{-1}R_g^2 &= P_g^{-1}R_g \\
\Longrightarrow L_f^{-1} &= P_g^{-1}R_gR_{g^2}^{-1}P_g
\end{aligned}$$

So, $f\backslash x = \{[(g/x)g]/g^2\}\backslash g$. With $x = f$, we get $f = g^{-1}$.
Thus, $\theta \equiv \theta(f,g) = \theta(g^{-1},g)$. □

Corollary 1. *Let $(Q,\circ)$ be a middle Bol loop. Any $\theta\in BS(Q,\circ)$ such that $\theta: e\mapsto e$ induces $\Phi = I\theta P_g^{-1}R_g\in SYM(Q)$ for some $g\in Q$ and the following hold:*

1. *$\Phi\in BS(Q,\circ)$.*
2. *Φ is a middle pseudo-automorphism with a square companion.*

Proof. Replacing $R_g^{-1} = P_f^{-1}R_fL_{f^2}^{-1}P_f^{-1}$ and $L_f^{-1} = P_g^{-1}R_gR_{g^2}^{-1}P_g$ in (5) gives $(I\theta P_g^{-1}R_gR_{g^2}^{-1}P_gP_x^{-1}, I\theta P_f^{-1}R_fL_{f^2}^{-1}P_f^{-1}P_x, I\theta P_x^{-1}R_x)$ which is an autotopism of $(Q,\circ)$.

Put $x=g$ to get $(I\theta P_g^{-1}R_gR_{g^2}^{-1}, I\theta P_f^{-1}R_fL_{f^2}^{-1}P_f^{-1}P_g, I\theta P_g^{-1}R_g)\in AUT(Q,\circ)$. Setting $f=g$ gives $(I\theta P_g^{-1}R_gR_{g^2}^{-1}, I\theta P_g^{-1}R_gL_{g^2}^{-1}, I\theta P_g^{-1}R_g)\in AUT(Q,\circ)$. Letting $\Phi = I\theta P_g^{-1}R_g$, gives $(\Phi R_{g^2}^{-1},\Phi L_{g^2}^{-1},\Phi)$ is also autotopism of $(Q,\circ)$. □

Corollary 2. *Let $(Q,\circ)$ be a middle Bol loop and $\theta\equiv\theta(f,g)\in BS(Q,\circ)$ for some $f,g\in Q$ (in which either is of order 2 i.e. $|f|=2$ or $|g|=2$) such that $\theta: e\mapsto e$. Then, θ induces an automorphism $\Phi = I\theta P_g^{-1}R_g\in SYM(Q)$ for some $g\in Q$.*

Proof. This follows from Corollary 1. □

Theorem 7. *Let $(Q,\circ)$ be a middle Bol loop. Then,*

$$BS'(Q,\circ) = \Big\{\theta\in BS(Q,\circ)\mid \theta: e\mapsto e \textit{ and } (x\theta)^{-1} = \big(x^{-1}\big)\theta\Big\}$$
$$=\Big\{\theta\in SYM(Q)|\exists\ f\in Q\ \ni\ \big(\theta R_{f^{-1}}^{-1},\theta L_f^{-1},\theta\big)\in AUT(Q),\ e\theta = e \textit{ and}$$
$$(x\theta)^{-1} = (x^{-1})\theta\ \forall\ x\in Q\Big\} = \Big\{\theta\in SYM(Q)|\exists\ g\in Q\ \ni\ \big(\theta R_g^{-1},\theta L_{g^{-1}}^{-1},\theta\big)$$
$$\in AUT(Q),\ e\theta = e \textit{ and } (x\theta)^{-1} = (x^{-1})\theta\ \forall\ x\in Q\Big\}\le BS(Q,\circ).$$

Proof. Let
$BS'(Q,\circ) = \Big\{\theta\in BS(Q,\circ)\mid \theta: e\mapsto e \text{ and } (x\theta)^{-1} = \big(x^{-1}\big)\theta\Big\}\subseteq BS(Q,\circ)$.
Going by Theorem 6,

$$BS'(Q,\circ) = \Big\{\theta\in BS(Q,\circ)\mid \theta: e\mapsto e \text{ and } (x\theta)^{-1} = \big(x^{-1}\big)\theta\Big\}$$
$$=\Big\{\theta\in SYM(Q)|\exists\ f\in Q\ \ni\ \big(\theta R_{f^{-1}}^{-1},\theta L_f^{-1},\theta\big)\in AUT(Q),\ e\theta = e \text{ and}$$
$$(x\theta)^{-1} = (x^{-1})\theta\ \forall\ x\in Q\Big\} = \Big\{\theta\in SYM(Q)|\exists\ g\in Q\ \ni\ \big(\theta R_g^{-1},\theta L_{g^{-1}}^{-1},\theta\big)$$
$$\in AUT(Q),\ e\theta = e \text{ and } (x\theta)^{-1} = (x^{-1})\theta\ \forall\ x\in Q\Big\}.$$

Suppose that $\mathbb{I}$ is the identity mapping on Q, then, $e\mathbb{I} = e$ and $(g\mathbb{I})^{-1} = (g^{-1})\mathbb{I}$ $\forall g\in Q$ and $(\mathbb{I}R_e^{-1},\mathbb{I}L_e^{-1},\mathbb{I}) = (\mathbb{I},\mathbb{I},\mathbb{I})\in AUT(Q,\circ)$. So, $\mathbb{I}\in BS'(Q,\circ)$. Thus, $BS'(Q,\circ)\neq\emptyset$.

Let $\alpha,\beta\in BS'(Q,\circ)$. Then, $\alpha,\beta\in BS(Q,\circ)$ and $e\alpha = e$ and $(x\alpha)^{-1} = (x^{-1})\alpha$, $e\beta = e$ and $(x\beta)^{-1} = (x^{-1})\beta$, $\forall x\in Q$.

Furthermore, there exist $f_1, g_1, f_2, g_2 \in Q$ with $g_1 = f_1^{-1}, g_2 = f_2^{-1}$ such that

$$A = (\alpha R_{g_1}^{-1}, \alpha L_{f_1}^{-1}, \alpha), B = (\beta R_{g_2}^{-1}, \beta L_{f_2}^{-1}, \beta), B^{-1} =$$
$$(R_{g_2}\beta^{-1}, L_{f_2}\beta^{-1}, \beta^{-1}) \in AUT(Q, \circ).$$
$$AB^{-1} = (\alpha R_{g_1}^{-1}, \alpha L_{f_1}^{-1}, \alpha)(R_{g_2}\beta^{-1}, L_{f_2}\beta^{-1}, \beta^{-1}) =$$
$$(\alpha R_{g_1}^{-1} R_{g_2}\beta^{-1}, \alpha L_{f_1}^{-1} L_{f_2}\beta^{-1}, \alpha\beta^{-1}) \in AUT(Q, \circ).$$

Let $\rho = \beta R_{g_1}^{-1} R_{g_2}\beta^{-1}$ and $\sigma = \beta L_{f_1}^{-1} L_{f_2}\beta^{-1}$ so that $(\alpha\beta^{-1}\rho, \alpha\beta^{-1}\sigma, \alpha\beta^{-1}) \in AUT(Q, \circ)$ if and only if for all $x, y \in Q$

$$x\alpha\beta^{-1}\rho \circ y\alpha\beta^{-1}\sigma = (x \circ y)\alpha\beta^{-1}. \tag{7}$$

Setting $x = e$ in Q and replacing y by $y\beta\alpha^{-1}$ in (7), we have

$$(e\alpha\beta^{-1}\rho) \circ (y\sigma) = y \Longrightarrow y\sigma L_{(e\alpha\beta^{-1}\rho)} = y \Longrightarrow \sigma = L_{(e\alpha\beta^{-1}\rho)}^{-1}.$$

Similarly, setting $y = e$ in Q and replacing x by $x\beta\alpha^{-1}$ in (7), we have

$$(x\rho) \circ (e\alpha\beta^{-1}\sigma) = x \Longrightarrow x\rho R_{(e\alpha\beta^{-1}\sigma)} = x \Longrightarrow \rho = R_{(e\alpha\beta^{-1}\sigma)}^{-1}.$$

Thus, $g = e\alpha\beta^{-1}\sigma = e\sigma = e\beta L_{f_1}^{-1} L_{f_2}\beta^{-1} = [f_2 \circ (f_1 \backslash e)]\beta^{-1} = [f_2 \circ f_1^{-1}]\beta^{-1}$ and $f = e\alpha\beta^{-1}\rho = e\rho = e\beta R_{f_1^{-1}}^{-1} R_{f_2^{-1}}\beta^{-1} = eR_{f_1^{-1}}^{-1} R_{f_2^{-1}}\beta^{-1} = [(e/f_1^{-1}) \circ f_2^{-1}]\beta^{-1} = (f_1 \circ f_2^{-1})\beta^{-1}$. Then, $f^{-1} = [(f_1 \circ f_2^{-1})\beta^{-1}]^{-1} = (f_1 \circ f_2^{-1})^{-1}\beta^{-1} = (f_2 \circ f_1^{-1})\beta^{-1} = g$. Hence,

$$AB^{-1} = (\alpha\beta^{-1}\rho, \alpha\beta^{-1}\sigma, \alpha\beta^{-1}) = (\alpha\beta^{-1}R_{f^{-1}}^{-1}, \alpha\beta^{-1}L_f^{-1}, \alpha\beta^{-1}) \in AUT(Q, \circ),$$
$$e\alpha\beta^{-1} = e \text{ and } (x^{-1})\alpha\beta^{-1} = (x\alpha\beta^{-1})^{-1}\ \forall\ x \in Q. \text{ So, } \alpha\beta^{-1} \in BS'(Q, \circ).$$

Therefore, $BS'(Q, \circ) \leq BS(Q, \circ)$. □

Corollary 3. *Let $(Q, \circ)$ be a middle Bol loop. Then,*

$$AUM(Q, \circ) \leq BS'(Q, \circ) \leq BS(Q, \circ).$$

Proof. This follows from Theorem 7. □

Theorem 8. *Let $(Q, \circ)$ be a middle Bol loop and $(Q, \cdot)$ be the corresponding right Bol loop. Then, $BS'(Q, \circ) = PS_\lambda(Q, \cdot)$.*

Proof. We shall show that $\theta \in BS'(Q, \circ)$ if and only if $\theta \in PS_\lambda(Q, \cdot)$. Let $\theta \in BS'(Q, \circ)$, then $\theta \in BS(Q, \circ)$ such that $e\theta = e$. Thus, for some $f, g \in Q$, we have $\big(\theta\mathbb{R}_g^{-1}, \theta\mathbb{L}_f^{-1}, \theta\big) \in AUT(Q)$. For all $x, y \in Q$, we have

$$x\theta\mathbb{R}_g^{-1} \circ y\theta\mathbb{L}_f^{-1} = (x \circ y)\theta$$
$$\Leftrightarrow x\theta L_{g^{-1}} \circ y\theta(IP_f)^{-1} = (x \circ y)\theta$$

$$\Leftrightarrow \big(y\theta(IP_f)^{-1}\big)I\backslash x\theta L_{g^{-1}} = (y^{-1}\backslash x)\theta.$$

Set $z = y^{-1}\backslash x \Leftrightarrow x = y^{-1}\cdot z$. Then we have

$$\big(y\theta(IP_f)^{-1}\big)I\cdot z\theta = (y^{-1}\cdot z)\theta L_{g^{-1}} \Leftrightarrow \big(yI\theta(IP_f)^{-1}\big)I\cdot z\theta = (y\cdot z)\theta L_{g^{-1}}.$$

Putting $z = e$, we have $\big(yI\theta(IP_f)^{-1}\big)I\cdot e\theta = y\theta L_{g^{-1}} \Leftrightarrow \big(yI\theta(IP_f)^{-1}\big)I = y\theta L_{g^{-1}} \Leftrightarrow yI\theta(IP_f)^{-1}I = y\theta L_{g^{-1}}$. Thus, $(\theta L_{g^{-1}},\theta,\theta L_{g^{-1}}) \in AUT(Q,\cdot)$ which means that θ is a left pseudo-automorphism with companion g^{-1}.

Conversely, suppose that $\theta \in SYM(Q)$ is a left pseudo-automorphism of $(Q,\cdot)$ with companion g, then $(\theta L_g,\theta,\theta L_g) \in AUT(Q,\cdot)$. Note that $e\theta = e$ by Lemma 1. For all $x, y \in Q$, we have

$$x\theta L_g\cdot y\theta = (x\cdot y)\theta L_g$$
$$\Leftrightarrow x\theta\mathbb{R}^{-1}_{g^{-1}}\cdot y\theta = (xy)\theta\mathbb{R}^{-1}_{g^{-1}}$$
$$\Leftrightarrow y\theta//\big(x\theta\mathbb{R}^{-1}_{g^{-1}}\big)I = \big(y//x^{-1}\big)\theta\mathbb{R}^{-1}_{g^{-1}}.$$

Set $y//x^{-1} = z \Leftrightarrow y = z\circ x^{-1}$ for $z \in Q$. This leads us to

$$(z\circ xI)\theta = z\theta\mathbb{R}^{-1}_{g^{-1}}\circ x\theta\mathbb{R}^{-1}_{g^{-1}}I \Leftrightarrow (z\circ xI)\theta = z\theta\mathbb{R}^{-1}_{g^{-1}}\circ x\theta I\mathbb{L}^{-1}_g. \quad (8)$$

Substituting $z = e$, $xI\theta = e\mathbb{R}^{-1}_{g^{-1}}\circ x\theta I\mathbb{L}^{-1}_g \Leftrightarrow xI\theta = g\circ x\theta I\mathbb{L}^{-1}_g \Leftrightarrow xI\theta = x\theta I\mathbb{L}^{-1}_g\mathbb{L}_g \Leftrightarrow xI\theta = x\theta I$. So, (8) becomes $(z\circ xI)\theta = z\theta\mathbb{R}^{-1}_{g^{-1}}\circ xI\theta\mathbb{L}^{-1}_g \Leftrightarrow (\theta\mathbb{R}^{-1}_{g^{-1}},\theta\mathbb{L}^{-1}_g,\theta) \in AUT(Q,\circ) \Rightarrow \theta \in BS(Q,\circ)$. Thus, $\theta \in BS'(Q,\circ)$. □

Lemma 3. *Let $(Q,\cdot)$ be a loop.*

1. $BS_\rho(Q,\cdot) \le BS(Q,\cdot)$ *and* $BS_\lambda(Q,\cdot) \le BS(Q,\cdot)$.
2. $BS'_\rho(Q,\cdot) = \Big\{\theta \in BS_\rho(Q,\cdot)\mid \theta : e\mapsto e\Big\} \le BS_\rho(Q,\cdot) \le BS(Q,\cdot)$.
3. $BS'_\lambda(Q,\cdot) = \Big\{\theta \in BS_\lambda(Q,\cdot)\mid \theta : e\mapsto e\Big\} \le BS_\lambda(Q,\cdot) \le BS(Q,\cdot)$.
4. $BS''_\rho(Q,\cdot) = \Big\{\theta \in BS_\rho(Q,\cdot)\mid \theta : e\mapsto e$ *and* $(x\theta)^{-1} = \big(x^{-1}\big)\theta\ \forall\ x \in Q\Big\} \le BS'_\rho(Q,\cdot)$.
5. $BS''_\lambda(Q,\cdot) = \Big\{\theta \in BS_\lambda(Q,\cdot)\mid \theta : e\mapsto e$ *and* $(x\theta)^{-1} = \big(x^{-1}\big)\theta\ \forall\ x \in Q\Big\} \le BS'_\lambda(Q,\cdot)$.

Proof.

1. The proof is similar to that of Theorem 1.
2. This follows from 1.

3. This follows from 1.

4. This follows from 2.

5. This follows from 3.

□

Theorem 9. *Let $(Q,\circ)$ be a middle Bol loop and let $(Q,\cdot)$ and $(Q,*)$ be its corresponding right and left Bol loops respectively. Then,*

1. $BS'_\rho(Q,\circ)=AUM(Q,\circ)=AUM(Q,\cdot)=AUM(Q,*)$.

2. $BS''_\lambda(Q,\circ)=AUM(Q,\circ)=AUM(Q,\cdot)=AUM(Q,*)$.

Proof. 1. Let $\theta\in BS'_\rho(Q,\circ)$, then $\theta\in BS(Q,\circ)$ i.e. for some $f\in Q$, $(\theta,\theta\mathbb{L}_f^{-1},\theta)\in AUT(Q,\circ)$ and $\theta:e\mapsto e$. So, for all $x,y\in Q$, we have

$$x\theta\circ y\theta\mathbb{L}_f^{-1}=(x\circ y)\theta$$
$$\Leftrightarrow x\theta\circ y\theta(IP_f)^{-1}=(x\circ y)\theta$$
$$\Leftrightarrow \big(y\theta(IP_f)^{-1}\big)I\backslash x\theta=(y^{-1}\backslash x)\theta.$$

Set $z=y^{-1}\backslash x\Leftrightarrow x=y^{-1}\cdot z$ for $z\in Q$ in order to get

$$yI\theta(IP_f)^{-1}I\cdot z\theta=(yz)\theta. \tag{9}$$

Substitute $z=e$ into (9), then we have $yI\theta(IP_f)^{-1}I=y\theta\Leftrightarrow\theta=I\theta(IP_f)^{-1}I$. Put this into (9) to have $(\theta,\theta,\theta)\in AUT(Q,\cdot)$. Thus, θ is an automorphism of right Bol loop $(Q,\cdot)$. Thus, $BS'_\rho(Q,\circ)\le AUM(Q,\cdot)$. By Theorem 4, $AUM(Q,\cdot)=AUM(Q,\circ)$, so, $BS'_\rho(Q,\circ)\le AUM(Q,\circ)$. But, $AUM(Q,\circ)\le BS'_\rho(Q,\circ)$ by Corollary 3. Thus, $BS'_\rho(Q,\circ)=AUM(Q,\circ)=AUM(Q,\cdot)=AUM(Q,*)$.

2. Let $\theta\in BS''_\lambda(Q,\circ)$, then $\theta\in BS(Q,\circ)$ i.e. for some $f\in Q$, $(\theta\mathbb{R}_g^{-1},\theta,\theta)\in AUT(Q,\circ)$, $\theta:e\mapsto e$ and $I\theta=\theta I$. So, for all $x,y\in Q$, we have

$$x\theta\mathbb{R}_g^{-1}\circ y\theta=(x\circ y)\theta$$
$$\Leftrightarrow x\theta L_{f^{-1}}\circ y\theta=(x\circ y)\theta$$
$$\Leftrightarrow y\theta I\backslash x\theta L_{f^{-1}}=(y^{-1}\backslash x)\theta.$$

Set $z=y^{-1}\backslash x\Leftrightarrow x=y^{-1}\cdot z$ for $z\in Q$ in order to get

$$y\theta I\cdot z\theta=(y^{-1}\cdot z)\theta L_{f^{-1}}$$
$$\Leftrightarrow yI\theta I\cdot z\theta=(y\cdot z)\theta L_{f^{-1}}$$
$$\Leftrightarrow y\theta\cdot z\theta=(y\cdot z)\theta L_{f^{-1}}. \tag{10}$$

Substitute $z=e$ into (10), then we have $\theta L_{f^{-1}}=\theta$. Put this into (10) to have $(\theta,\theta,\theta)\in AUT(Q,\cdot)$. Thus θ is an automorphism of right Bol loop $(Q,\cdot)$.

Thus, $BS''_\lambda(Q,\circ)\le AUM(Q,\cdot)$. By Theorem 4, $AUM(Q,\cdot)=AUM(Q,\circ)$, so, $BS''_\lambda(Q,\circ)\le AUM(Q,\circ)$. But, $AUM(Q,\circ)\le BS''_\lambda(Q,\circ)$. Thus, $BS''_\lambda(Q,\circ)=AUM(Q,\circ)=AUM(Q,\cdot)=AUM(Q,*)$.

□

Theorem 10. *Let $(Q,\circ)$ be a middle Bol loop and $(Q,\cdot)$ be the corresponding right Bol loop. Then, $PS_\rho(Q,\cdot)=PS_\lambda(Q,\circ)=PS_\rho(Q,\circ)$.*

Proof. If θ is right pseudo-automorphism of $(Q,\cdot)$ with companion g, then $(\theta,\theta R_g,\theta R_g)\in AUT(Q,\cdot)$. For all $x,y\in Q$, we have

$$\begin{aligned} x\theta\cdot y\theta R_g=(x\cdot y)\theta R_g &\Rightarrow x\theta\cdot y\theta I\mathbb{P}_g^{-1}=(x\cdot y)\theta I\mathbb{P}_g^{-1}\\ &\Rightarrow y\theta I\mathbb{P}_g^{-1}//x\theta I=(y//xI)\theta I\mathbb{P}_g^{-1}. \end{aligned}\tag{11}$$

Set $z=y//xI\Longrightarrow y=z\circ xI$. So, (11) becomes

$$\begin{aligned} (z\circ xI)\theta I\mathbb{P}_g^{-1}=z\theta I\mathbb{P}_g^{-1}\circ x\theta I &\Rightarrow (z\circ x)\theta I\mathbb{P}_g^{-1}=z\theta I\mathbb{P}_g^{-1}\circ xI\theta I\\ &\Rightarrow (z\circ x)\theta I\mathbb{P}_g^{-1}=z\theta I\mathbb{P}_g^{-1}\circ x. \end{aligned}\tag{12}$$

Set $z=e$ in (12) to get $\theta I\mathbb{P}_g^{-1}=\theta\mathbb{L}_{e\theta I\mathbb{P}_g^{-1}}=\theta\mathbb{L}_{g'}$. Thus, (12) becomes $(z\circ x)\theta\mathbb{L}_{g'}=z\theta\mathbb{L}_{g'}\circ x\Rightarrow(\theta\mathbb{L}_{g'},\theta,\theta\mathbb{L}_{g'})\in AUT(Q,\circ)$. Thence, θ is left pseudo-automorphism of $(Q,\circ)$ with companion g'.

Conversely, if θ is a left pseudo-automorphism of $(Q,\circ)$ with companion g, then $(\theta,\theta\mathbb{L}_g,\theta\mathbb{L}_g)\in AUT(Q,\circ)$. For all $x,y\in Q$, we have

$$\begin{aligned} x\theta\circ y\theta\mathbb{L}_g=(x\circ)\theta\mathbb{L}_g &\Rightarrow x\theta IP_g\circ y\theta=(x\circ y)\theta IP_g\\ &\Rightarrow y\theta I\backslash x\theta IP_g=(yI\backslash x)\theta IP_g. \end{aligned}\tag{13}$$

Set $z=yI\backslash x\Longrightarrow x=yI\cdot z$. So, (13) becomes

$$\begin{aligned} (yI\cdot z)\theta IP_g=y\theta I\cdot z\theta IP_g &\Rightarrow (y\cdot z)\theta IP_g=zI\theta I\cdot z\theta IP_g\\ &\Rightarrow (y\cdot z)\theta IP_g=z\theta\cdot z\theta IP_g. \end{aligned}\tag{14}$$

Set $z=e$ in (14) to get $\theta IP_g=\theta R_{e\theta IP_g}=\theta R_{g'}$. Thus, (14) becomes $(y\cdot z)\theta R_{g'}=z\theta\cdot z\theta R_{g'}\Rightarrow(\theta,\theta R_g,\theta R_{g'})\in AUT(Q,\cdot)$. Thence, θ is right pseudo-automorphism of $(Q,\cdot)$ with companion g'. So, $PS_\rho(Q,\cdot)=PS_\lambda(Q,\circ)=PS_\rho(Q,\circ)$ by Theorem 5. □

Theorem 11. *Let $(Q,\circ)$ be a middle Bol loop and let $(Q,\cdot)$ and $(Q,*)$ be its corresponding right and left Bol loops respectively. Then, $BS'(Q,\cdot)=PS_\rho(Q,\circ)=PS_\rho(Q,\cdot)=PS_\mu(Q,\cdot)=PS_\lambda(Q,\cdot)=PS_\mu(Q,\circ)=PS_\lambda(Q,\circ)\cong PS_\lambda(Q,*)$.*

Proof. We shall show that $\theta\in BS'(Q,\cdot)$ if and only if $\theta\in PS_\rho(Q,\circ)$. Let $\theta\in BS'(Q,\cdot)$, then $\theta\in BS(Q,\cdot)$ such that $e\theta=e$. Thus, for some $f,g\in Q$, $(\theta R_g^{-1},\theta L_f^{-1},\theta)\in AUT(Q)$. For all $x,y,\in Q$, we have

$$x\theta R_g^{-1}\cdot y\theta L_f^{-1}=(xy)\theta\Leftrightarrow x\theta\mathbb{P}_gI\cdot y\theta\mathbb{R}_{f^{-1}}=(x\cdot y)\theta$$

$$\Leftrightarrow (y\theta\mathbb{R}_{f^{-1}})//(x\theta\mathbb{P}_g I)I = \left(y//x^{-1}\right)\theta$$
$$\Leftrightarrow y\theta\mathbb{R}_{f^{-1}} = \left(y//x^{-1}\right)\theta\circ x\theta\mathbb{P}_g. \quad (15)$$

Set $z = y//x^{-1} \Leftrightarrow y = z\circ x^{-1}$. So, (15) becomes

$$\left(z\circ x^{-1}\right)\theta\mathbb{R}_{f^{-1}} = z\theta\circ x\theta\mathbb{P}_g$$
$$\Rightarrow (z\circ x)\,\theta\mathbb{R}_{f^{-1}} = z\theta\circ xI\theta\mathbb{P}_g. \quad (16)$$

Put $z = e$ in (16) to get $\theta\mathbb{R}_{f^{-1}} = I\theta\mathbb{P}_g$. Hence, (16) becomes $(z\circ x)\theta\mathbb{R}_{f^{-1}} = z\theta\circ x\theta\mathbb{R}_{f^{-1}} \Leftrightarrow (\theta,\theta\mathbb{R}_{f^{-1}},\theta\mathbb{R}_{f^{-1}})\in PS_\rho(Q,\circ)$.

Conversely, suppose that $\theta\in SYM(Q)$ is a right pseudo-automorphism of $(Q,\circ)$ with companion f^{-1}, then $(\theta,\theta\mathbb{R}_{f^{-1}},\theta\mathbb{R}_{f^{-1}})\in PS_\rho(Q,\circ)$. Note that $e\theta = e$. For all $x,y\in Q$, we have

$$x\theta\circ y\theta\mathbb{R}_{f^{-1}} = (x\circ y)\theta\mathbb{R}_{f^{-1}} \Leftrightarrow x\theta\circ y\theta L_f^{-1} = (x\circ y)\theta L_f^{-1}$$
$$\Leftrightarrow (y\theta L_f^{-1})I\backslash x\theta = (yI\backslash x)\theta L_f^{-1} \Leftrightarrow x\theta = (y\theta L_f^{-1})I\cdot(yI\backslash x)\theta L_f^{-1}.$$

Put $z = yI\backslash x \Leftrightarrow x = yI\cdot z$. Thus, the last equality is true

$$\Leftrightarrow (y\theta L_f^{-1})I\cdot z\theta L_f^{-1} = (yI\cdot z)\theta \Leftrightarrow yI\theta L_f^{-1}I\cdot z\theta L_f^{-1} = (y\cdot z)\theta.$$

Putting $z = e$ in the last equation, we get $I\theta L_f^{-1}I = \theta R_{f^{-1}}f^{-1}$ and consequently, $y\theta R_{f^{-1}}^{-1}\cdot z\theta L_f^{-1} = (y\cdot z)\theta \Leftrightarrow (\theta R_{f^{-1}}^{-1},\theta L_f^{-1},\theta)\in AUT(Q,\cdot) \Rightarrow \theta\in BS(Q,\cdot)$. Thus, $\theta\in BS'(Q.\cdot)$. So, by Theorem 4, Theorem 5, Theorem 8 and Theorem 10, $BS'(Q,\cdot) = PS_\rho(Q,\circ) = PS_\rho(Q,\cdot) = PS_\mu(Q,\cdot) = PS_\lambda(Q,\cdot) = PS_\mu(Q,\circ) = PS_\lambda(Q,\circ)\cong PS_\lambda(Q,*)$. □

Theorem 12. *Let $(Q,\circ)$ be a middle Bol loop of exponent 2 and let $(Q,\cdot)$ and $(Q,*)$ be its corresponding right and left Bol loops respectively. $(Q,\cdot)$ and $(Q,*)$ are left G-loop and right G-loop respectively.*

Proof. $(Q,\circ)$ is a middle Bol loop if and only if

$$(I\mathbb{P}_x^{-1}, I\mathbb{P}_x, I\mathbb{P}_x\mathbb{L}_x)\in AUT(Q,\circ). \quad (17)$$

Let $I\mathbb{P}_x\mathbb{L}_x = \theta$, then this implies that $I\mathbb{P}_x = \theta\mathbb{L}_x^{-1}$ and $yI\mathbb{P}_xI = y\theta\mathbb{L}_x^{-1}I \Rightarrow (y^{-1}\backslash\backslash x)^{-1} = (x\backslash\backslash y\theta)^{-1} \Rightarrow x^{-1}//y = (y\theta)I//x^{-1} \Rightarrow \mathbb{P}_x^{-1} = \theta I\mathbb{R}_x^{-1} \Rightarrow I\mathbb{P}_x^{-1} = I\theta I\mathbb{R}_x^{-1}$. Thus, $I\mathbb{P}_x^{-1} = I\theta I\mathbb{R}_x^{-1}$. Thence, (17) becomes $(I\theta I\mathbb{R}_x^{-1},\theta\mathbb{L}_x^{-1},\theta)\in AUT(Q,\circ)$. For all $a,b\in Q$, we have

$$aI\theta I\mathbb{R}_x^{-1}\circ b\theta\mathbb{L}_x^{-1} = (a\circ b)\theta$$
$$\Longrightarrow aI\theta IL_{x^{-1}}\circ b\theta(IP_x)^{-1} = (a\circ b)\theta$$
$$\Longrightarrow b\theta((IP_x)^{-1})I\backslash aI\theta IL_{x^{-1}} = (b^{-1}\backslash a)\theta$$
$$\Longrightarrow b\theta((IP_x)^{-1})I\cdot(b^{-1}\backslash x)\theta = aI\theta IL_{x^{-1}}.$$

Put $c = b^{-1}\backslash a \Longrightarrow a = b^{-1} \cdot c$ for $c \in Q$. So, from the last equation,

$$b\theta((IP_x)^{-1})I \cdot c\theta = (b^{-1} \cdot c)I\theta IL_{x^{-1}} \Longrightarrow bI\theta((IP_x)^{-1})I \cdot c\theta = (b \cdot c)I\theta IL_{x^{-1}}.$$

Note that $e\theta = e \Leftrightarrow (Q, \circ)$ is of exponent 2. Thus, setting $c = e$, then $bI\theta((IP_x)^{-1})I = bI\theta IL_{x^{-1}} \Longrightarrow ((IP_x)^{-1})I = IL_{x^{-1}}$. Thence, $bI\theta IL_{x^{-1}} \cdot c\theta = (b \cdot c)I\theta IL_{x^{-1}}$. Now, set $b = e$ to get $eL_{x^{-1}} \cdot c\theta = cI\theta IL_{x^{-1}}$, which implies that $\theta = I\theta I$. Hence, $b\theta L_x \cdot c\theta = (b \cdot c)\theta L_x \Longrightarrow (\theta L_x, \theta, \theta L_x) \in AUT(Q, \cdot)$ for all $x \in Q$. Thus, $\theta \in PS_\lambda(Q, \cdot)$ with companion $x \in Q$. Therefore, $(Q, \cdot)$ is a left G-loop.

The proof for $(Q, *)$ is similar. □

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