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On the solubility of a class of two-dimensional integral equations on a quarter plane with monotone nonlinearity

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Abstract. In the paper we study a class of two-dimensional integral equations on a quarter-plane with monotone nonlinearity and substochastic kernel. With specific representations of the kernel and nonlinearity, an equation of this kind arises in various fields of natural science. In particular, such equations occur in the dynamical theory of p-adic open-closed strings for the scalar field of tachyons, in the mathematical theory of the geographical spread of a pandemic, in the kinetic theory of gases, and in the theory of radiative transfer in inhomogeneous media.
We prove constructive theorems on the existence of a nontrivial nonnegative and bounded solution. For one important particular case, the existence of a one-parameter family of nonnegative and bounded solutions is also established. Moreover, the asymptotic behavior at infinity of each solution from the given family os studied. At the end of the paper, specific particular examples (of an applied nature) of the kernel and nonlinearity that satisfy all the conditions of the proven statements are given.

Mathematics subject classification: 45G10.
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1 Introduction

Consider the following class of two-dimensional integral equations on the first quarter of the plane with monotone nonlinearity:

$$\mathscr{F}(x_1,x_2)=\int\limits_0^\infty\int\limits_0^\infty \mathcal{P}(x_1,y_1,x_2,y_2)\,G(x_1,x_2,\mathscr{F}(\rho_1(x_1,y_1),\rho_2(x_2,y_2)))\,dy_1\,dy_2, \tag{1}$$
$$(x_1,x_2)\in\mathbb{R}_2^+:=\mathbb{R}^+\times\mathbb{R}^+,\quad \mathbb{R}^+:=[0,+\infty)$$

with respect to an unknown measurable and bounded function $\mathscr{F}(x_1,x_2)$ on $\mathbb{R}_2^+$.

In the equation (1), the kernel $\mathcal{P}(x_1,y_1,x_2,y_2)$ is a measurable real-valued function on $\mathbb{R}_4^+:=\mathbb{R}^+\times\mathbb{R}^+\times\mathbb{R}^+\times\mathbb{R}^+$ satisfying the following conditions:

a) (*minorant condition*)

there exist continuous on $\mathbb{R}_2^+$ functions $K(y_1,\,y_2)$ and $\lambda(x_1,x_2)$ with properties

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$a_1)$ $K(y_1,\, y_2) > 0,\ (y_1,\, y_2) \in \mathbb{R}_2^+, \quad K \in L_1(\mathbb{R}_2^+) \cap M(\mathbb{R}_2^+),$

$$\int_0^\infty \int_0^\infty K(y_1,\, y_2)\, dy_1\, dy_2 = 1, \tag{2}$$

$a_2)$ $0 \le \lambda(x_1, x_2) \le 1,\ (x_1, x_2) \in \mathbb{R}_2^+, \quad \lambda \uparrow$ by x_j on $\mathbb{R}^+, \quad j = 1, 2,$

$$(1 - \lambda(x_1, x_2))\, x_1^m x_2^\ell \in L_1\left(\mathbb{R}_2^+\right),\ m, \ell = 0, 1, \tag{3}$$

such that

$$\mathcal{P}(x_1, y_1, x_2, y_2) \ge \lambda(x_1, x_2) K(y_1,\, y_2), \tag{4}$$

$b)$ (*substochasticity condition*)

$$\mu(x_1, x_2) := \int_0^\infty \int_0^\infty \mathcal{P}(x_1, y_1, x_2, y_2)\, dy_1\, dy_2 \le 1, \quad \mu(x_1, x_2) \not\equiv 1,\ (x_1, x_2) \in \mathbb{R}_2^+$$

and $\sup\limits_{(x_1, x_2) \in \mathbb{R}_2^+} \mu(x_1, x_2) = 1.$

Nonlinearity $G(x_1, x_2, u)$ is a measurable real-valued function on $\mathbb{R}_2^+ \times \mathbb{R}$ $(\mathbb{R} := (-\infty, +\infty))$ satisfying Carathéodory condition with respect to the argument u (i.e., for every $u \in \mathbb{R}$ the function G is measurable in $(x_1, x_2) \in \mathbb{R}_2^+$ and for almost every $(x_1, x_2) \in \mathbb{R}_2^+$ this function is continuous in u on set $\mathbb{R}$) and some other conditions (see the statement of the main result).

The functions $\{\rho_j(u, v)\}_{j=1,2}$ in the right side of (1) satisfy the following conditions:

1) $\rho_j(u, v) \ge 0,\ (u, v) \in \mathbb{R}_2^+, \qquad \rho_j \in C(\mathbb{R}_2^+),\ j = 1, 2,$

2) $\rho_j(u, v) \uparrow$ in u on $\mathbb{R}^+$ and $\rho_j(u, v) \uparrow$ in v on $\mathbb{R}^+,\ j = 1, 2,$

3) $\rho_j(u, 0) \ge u, \quad \rho_j(u, 1) \ge u + 1,\ u \in \mathbb{R}^+,\ j = 1, 2.$

The equation (1), apart from its purely mathematical interest, has numerous important applications. First of all, we should single out the problems of mathematical physics and mathematical biology. So, very important in practical terms is a special case of the equation when $\rho_j(u, v) = u + v,\ j = 1, 2,\ (u, v) \in \mathbb{R}_2^+$ with specific representations of the kernel $\mathcal{P}$ and the nonlinearity G. Such equations arise in the dynamical theory of p-adic open-closed strings for the scalar field of tachyons, in the mathematical theory of space-time (geographical) propagation of pandemics, in the kinetic theory of gases, in the theory of radiative transfer in inhomogeneous media [1–8].

In the particular case $\rho_j(u, v) = u + v,\ j = 1, 2,\ (u, v) \in \mathbb{R}_2^+$, when the functions G and P do not depend on the variables (x_1, x_2), the equation (1) was studied in [8–10] under various restrictions on nonlinearity. It should be noted that in the one-dimensional case the corresponding nonlinear integral equation with the difference kernel $\mathcal{P}(x-y)$ on the semiaxis, for various representations of the nonlinearity was studied in detail in the papers [11–13]. We also note there are scientific papers devoted to the study of one-dimensional nonlinear integral equations on a semiaxis with a sum-difference kernel $\mathcal{P}(x,y) = \mathcal{P}_0(x-y) - \mathcal{P}_0(x+y)$, $(x,y) \in \mathbb{R}_2^+$ and with convex nonlinearity (see for instance [2, 14–16] and references therein).

In the present paper, under sufficiently general restrictions on the nonlinearity G, we prove a constructive theorem on the existence of a nonnegative nontrivial (nonzero) bounded solution on the set $\mathbb{R}_2^+$. In one important particular case, we also construct a one-parameter family of bounded solutions and establish the integral asymptotics of the constructed solutions. The proofs of the formulated theorems are based on the construction of invariant cone segments for the corresponding nonlinear monotone integral operator in the space of essentially bounded functions on the set $\mathbb{R}_2^+$, as well as on the methods developed during the systematic study of corresponding homogeneous and non-homogeneous linear integral equations on $\mathbb{R}_2^+$ with operators of almost Volterra type (when $\rho_j(u,v) = u+v,\ j=1,2,\ (u,v) \in \mathbb{R}_2^+$ these operators turn into two-dimensional Volterra operators with variable lower limits). At the end of the paper, we provide concrete particular examples of the functions $\mathcal{P}$, K, λ and G, which are of both applied and purely theoretical interest.

2 Auxiliary facts and notations

Before we prove the main result, we first study auxiliary equations and establish important and useful results for them, which will be used later.

2.1 Summable solution of a linear inhomogeneous auxiliary integral equation on a quarter-plane

Consider the following linear inhomogeneous two-dimensional integral equation:

$$f(x_1,x_2) = g(x_1,x_2) + \int_0^\infty\int_0^\infty K(y_1,y_2)\, f\left(\rho_1(x_1,y_1), \rho_2(x_2,y_2)\right) dy_1 dy_2, \quad (x_1,x_2) \in \mathbb{R}_2^+, \tag{5}$$

with respect to a nonnegative and measurable on $\mathbb{R}_2^+$ function $f(x_1,x_2)$. Here $g(x_1,x_2)$ is a measurable function on $\mathbb{R}_2^+$ and

$$g(x_1,x_2) \geq 0,\ (x_1,x_2) \in \mathbb{R}_2^+, \quad g(x_1,x_2) \downarrow \text{ in } x_j \text{ on } \mathbb{R}^+, \quad j=1,2,$$
$$\int_0^\infty\int_0^\infty g(x_1,x_2)\, x_1^m x_2^\ell\, dx_1\, dx_2 < +\infty, \quad m,\ell = 0,1. \tag{6}$$

For the equation (5) we consider the following simple iterations:

$$f_{n+1}(x_1,x_2)=g(x_1,x_2)+\int_0^\infty\int_0^\infty K(y_1,y_2)\,f_n\left(\rho_1(x_1,y_1),\rho_2(x_2,y_2)\right)dy_1\,dy_2, \tag{7}$$
$$f_0(x_1,x_2)=g(x_1,x_2),\quad n=0,1,2,\ldots,\quad (x_1,x_2)\in\mathbb{R}_2^+.$$

Applying the method of mathematical induction it is easy to check that

$$f_n(x_1,x_2)\uparrow\quad\text{in}\quad n. \tag{8}$$

Now we prove that

$$f_n(x_1,x_2)\downarrow\ \text{in}\ x_j\ \text{on}\ \mathbb{R}^+,\quad j=1,2,\quad n=0,1,2,\ldots. \tag{9}$$

Indeed, the monotonicity of the zero approximation immediately follows from (6). Assume that (9) holds for some positive integer n. Then taking into account the conditions (6), a_1) and 2), from (7) for arbitrary $x_1,\widetilde{x}_1\in\mathbb{R}^+,\ x_1>\widetilde{x}_1$ we will have

$$f_{n+1}(x_1,x_2)\le g(\widetilde{x}_1,x_2)+\int_0^\infty\int_0^\infty K(y_1,y_2)\,f_n\left(\rho_1(\widetilde{x}_1,y_1),\rho_2(x_2,y_2)\right)\,dy_1\,dy_2=$$
$$=f_{n+1}(\widetilde{x}_1,x_2),\quad x_2\in\mathbb{R}^+.$$

By analogy, for arbitrary $x_2,\widetilde{x}_2\in\mathbb{R}^+,\ x_2>\widetilde{x}_2$ we get $f_{n+1}(x_1,x_2)\le f_{n+1}(x_1,\widetilde{x}_2)$, $x_1\in\mathbb{R}^+$. Therefore, (9) is valid.

Applying again induction on n we prove that

$$f_n\in L_1(\mathbb{R}_2^+),\quad n=0,1,2,\ldots. \tag{10}$$

In the case when $n=0$ the validity of (10) follows obviously from definition of zero approximation and its property (6). Assume that $f_n\in L_1(\mathbb{R}_2^+)$ for some $n\in\mathbb{N}$, then $g+f_n\in L_1(\mathbb{R}_2^+)$. On the other hand, taking into account (9), 2) and a_1), from (7) we derive the following estimation:

$$g(x_1,x_2)\le f_{n+1}(x_1,x_2)\le g(x_1,x_2)+$$
$$+\int_0^\infty\int_0^\infty K(y_1,y_2)f_n\left(\rho_1(x_1,0),\rho_2(x_2,0)\right)dy_1dy_2\le$$
$$\le g(x_1,x_2)+f_n(x_1,x_2)\int_0^\infty\int_0^\infty K(y_1,y_2)dy_1\,dy_2=g(x_1,x_2)+f_n(x_1,x_2),\ (x_1,x_2)\in\mathbb{R}_2^+,$$

whence it follows that $f_{n+1}\in L_1(\mathbb{R}_2^+)$.

Next we prove the existence of a such constant $C>0$ that

$$\int_0^\infty\int_0^\infty f_n(x_1,x_2)\,dx_1\,dx_2\le C,\quad n=1,2,\ldots. \tag{11}$$

Let $r_1 \geq 0$, $r_2 \geq 0$ be arbitrary numbers. Then taking into account the conditions $a_1), a_2), 1) - 3)$ and (6), from (7) we get

$$\int_{r_1}^{\infty}\int_{r_2}^{\infty} f_{n+1}(x_1,x_2)dx_1\,dx_2 \leq \int_{r_2}^{\infty}\int_{r_1}^{\infty} g(x_1,x_2)dx_1\,dx_2 +$$

$$+\int_{r_2}^{\infty}\int_{r_1}^{\infty}\int_{0}^{\infty}\int_{0}^{\infty} K(y_1,y_2)\,f_{n+1}\left(\rho_1(x_1,y_1),\rho_2(x_2,y_2)\right)dy_1\,dy_2 dx_1\,dx_2 =$$

$$= \int_{r_2}^{\infty}\int_{r_1}^{\infty} g(x_1,x_2)dx_1\,dx_2 + \int_{0}^{\infty}\int_{0}^{\infty} K(y_1,y_2)\times$$

$$\times\int_{r_2}^{\infty}\int_{r_1}^{\infty} f_{n+1}\left(\rho_1(x_1,y_1),\rho_2(x_2,y_2)\right)dx_1\,dx_2 dy_1\,dy_2 \leq \int_{r_2}^{\infty}\int_{r_1}^{\infty} g(x_1,x_2)dx_1\,dx_2 +$$

$$+\int_{0}^{1}\int_{0}^{\infty} K(y_1,y_2)\int_{r_2}^{\infty}\int_{r_1}^{\infty} f_{n+1}\left(\rho_1(x_1,y_1),\rho_2(x_2,0)\right)dx_1\,dx_2\,dy_1\,dy_2 +$$

$$+\int_{1}^{\infty}\int_{0}^{\infty} K(y_1,y_2)\int_{r_2}^{\infty}\int_{r_1}^{\infty} f_{n+1}\left(\rho_1(x_1,y_1),\rho_2(x_2,1)\right)dx_1\,dx_2\,dy_1\,dy_2 \leq$$

$$\leq \int_{r_2}^{\infty}\int_{r_1}^{\infty} g(x_1,x_2)\,dx_1\,dx_2 + \int_{0}^{1}\int_{0}^{1} K(y_1,y_2)\times$$

$$\times\int_{r_2}^{\infty}\int_{r_1}^{\infty} f_{n+1}\left(\rho_1(x_1,0),\rho_2(x_2,0)\right)dx_1\,dx_2\,dy_1\,dy_2 +$$

$$+\int_{0}^{1}\int_{1}^{\infty} K(y_1,y_2)\int_{r_2}^{\infty}\int_{r_1}^{\infty} f_{n+1}\left(\rho_1(x_1,1),\rho_2(x_2,0)\right)\,dx_1\,dx_2\,dy_1\,dy_2 +$$

$$+\int_{1}^{\infty}\int_{0}^{1} K(y_1,y_2)\int_{r_2}^{\infty}\int_{r_1}^{\infty} f_{n+1}\left(\rho_1(x_1,0),\rho_2(x_2,1)\right)\,dx_1\,dx_2\,dy_1\,dy_2 +$$

$$+\int_{1}^{\infty}\int_{1}^{\infty} K(y_1,y_2)\int_{r_2}^{\infty}\int_{r_1}^{\infty} f_{n+1}\left(\rho_1(x_1,1),\rho_2(x_2,1)\right)\,dx_1\,dx_2\,dy_1\,dy_2 \leq$$

$$\leq \int_{r_2}^{\infty}\int_{r_1}^{\infty} g(x_1,x_2)\,dx_1\,dx_2 + \int_0^1\int_0^1 K(y_1,y_2)\,dy_1\,dy_2 \int_{r_2}^{\infty}\int_{r_1}^{\infty} f_{n+1}(x_1,x_2)\,dx_1\,dx_2 +$$

$$+\int_0^1\int_1^{\infty} K(y_1,y_2)\,dy_1\,dy_2 \int_{r_2}^{\infty}\int_{r_1+1}^{\infty} f_{n+1}(x_1,x_2)\,dx_1\,dx_2 +$$

$$+\int_1^{\infty}\int_0^1 K(y_1,y_2)\,dy_1\,dy_2 \int_{r_2+1}^{\infty}\int_{r_1}^{\infty} f_{n+1}(x_1,x_2)\,dx_1\,dx_2 +$$

$$+\int_1^{\infty}\int_1^{\infty} K(y_1,y_2)\,dy_1\,dy_2 \int_{r_2+1}^{\infty}\int_{r_1+1}^{\infty} f_{n+1}(x_1,x_2)\,dx_1\,dx_2\,.$$

Hence, combining similar integrals and taking into account (2), we obtain

$$\int_{r_2}^{\infty}\int_{r_1}^{\infty} f_{n+1}(x_1,x_2)dx_1\,dx_2\Bigg(\int_0^1\int_0^1 K(y_1,y_2)\,dy_1\,dy_2 + \int_0^1\int_1^{\infty} K(y_1,y_2)\,dy_1\,dy_2 +$$

$$+\int_1^{\infty}\int_0^1 K(y_1,y_2)dy_1\,dy_2 + \int_1^{\infty}\int_1^{\infty} K(y_1,y_2)\,dy_1\,dy_2 - \int_0^1\int_0^1 K(y_1,y_2)\,dy_1\,dy_2\Bigg) \leq$$

$$\leq \int_{r_2}^{\infty}\int_{r_1}^{\infty} g(x_1,x_2)\,dx_1\,dx_2 + \int_0^1\int_1^{\infty} K(y_1,y_2)\,dy_1\,dy_2 \int_{r_2}^{\infty}\int_{r_1+1}^{\infty} f_{n+1}(x_1,x_2)\,dx_1\,dx_2 +$$

$$+\int_1^{\infty}\int_0^1 K(y_1,y_2)\,dy_1\,dy_2 \int_{r_2+1}^{\infty}\int_{r_1}^{\infty} f_{n+1}(x_1,x_2)\,dx_1\,dx_2 +$$

$$+\int_1^{\infty}\int_1^{\infty} K(y_1,y_2)\,dy_1\,dy_2 \int_{r_2+1}^{\infty}\int_{r_1+1}^{\infty} f_{n+1}(x_1,x_2)\,dx_1\,dx_2.$$

We introduce the following notations

$$\alpha_0 := \int_0^1\int_0^1 K(y_1,\,y_2)\,dy_1\,dy_2, \qquad \alpha_1 := \int_0^1\int_1^{\infty} K(y_1,\,y_2)\,dy_1\,dy_2,$$

$$\alpha_2 := \int_1^{\infty}\int_0^1 K(y_1,\,y_2)\,dy_1\,dy_2, \qquad \alpha_3 := \int_1^{\infty}\int_1^{\infty} K(y_1,\,y_2)\,dy_1\,dy_2.$$

Then the last inequality in the above notations can be written as follows:

$$(\alpha_1+\alpha_2+\alpha_3)\int_{r_2}^{\infty}\int_{r_1}^{\infty} f_{n+1}(x_1,x_2)\,dx_1\,dx_2 \le \int_{r_2}^{\infty}\int_{r_1}^{\infty} g(x_1,x_2)\,dx_1\,dx_2+$$
$$+\alpha_1\int_{r_2}^{\infty}\int_{r_1+1}^{\infty} f_{n+1}(x_1,x_2)\,dx_1\,dx_2+\alpha_2\int_{r_2+1}^{\infty}\int_{r_1}^{\infty} f_{n+1}(x_1,x_2)\,dx_1\,dx_2+$$
$$+\alpha_3\int_{r_2+1}^{\infty}\int_{r_1+1}^{\infty} f_{n+1}(x_1,x_2)\,dx_1\,dx_2.$$

After some transformations we get

$$\alpha_1\int_{r_2}^{\infty}\int_{r_1}^{r_1+1} f_{n+1}(x_1,x_2)\,dx_1\,dx_2+\alpha_2\int_{r_2}^{r_2+1}\int_{r_1}^{\infty} f_{n+1}(x_1,x_2)\,dx_1\,dx_2+$$
$$+\alpha_3\int_{r_2}^{r_2+1}\int_{r_1}^{r_1+1} f_{n+1}(x_1,x_2)\,dx_1\,dx_2+\alpha_3\int_{r_2}^{r_2+1}\int_{r_1+1}^{\infty} f_{n+1}(x_1,x_2)\,dx_1\,dx_2+ \qquad (12)$$
$$+\alpha_3\int_{r_2+1}^{\infty}\int_{r_1}^{r_1+1} f_{n+1}(x_1,x_2)\,dx_1\,dx_2 \le \int_{r_2}^{\infty}\int_{r_1}^{\infty} g(x_1,x_2)\,dx_1\,dx_2.$$

By virtue of (9), from (12) it follows, in particular, that

$$\alpha_1\int_{r_2}^{\infty} f_{n+1}(r_1+1,x_2)dx_2+\alpha_2\int_{r_1}^{\infty} f_{n+1}(x_1,r_2+1)dx_1+$$
$$+\alpha_3 f_{n+1}(r_1+1,r_2+1)+\alpha_3\int_{r_1+1}^{\infty} f_{n+1}(x_1,r_2+1)dx_1+ \qquad (13)$$
$$+\alpha_3\int_{r_2+1}^{\infty} f_{n+1}(r_1+1,x_2)dx_2 \le \int_{r_2}^{\infty}\int_{r_1}^{\infty} g(x_1,x_2)\,dx_1\,dx_2.$$

Taking into account the condition (6), by Fubini's theorem [17] we can state that

$$\int_0^{\infty}\int_0^{\infty}\int_{r_2}^{\infty}\int_{r_1}^{\infty} g(x_1,x_2)\,dx_1\,dx_2\,dr_1\,dr_2=\int_0^{\infty}\int_0^{\infty} g(x_1,x_2)\int_0^{x_1} dr_1\int_0^{x_2} dr_2\,dx_1\,dx_2=$$
$$=\int_0^{\infty}\int_0^{\infty} x_1\,x_2\,g(x_1,x_2)\,dx_1\,dx_2:=M_{11}<+\infty,$$

$$\int_0^\infty\int_{r_2}^\infty\int_{r_1}^\infty g(x_1,x_2)\,dx_1\,dx_2\,dr_1=\int_{r_2}^\infty\int_0^\infty\int_{r_1}^\infty g(x_1,x_2)\,dx_1\,dr_1\,dx_2\leq$$

$$\leq\int_0^\infty\int_0^\infty g(x_1,x_2)\int_0^{x_1}dr_1\,dx_1\,dx_2=\int_0^\infty\int_0^\infty x_1\,g(x_1,x_2)\,dx_1\,dx_2:=M_{10}<+\infty,$$

$$\int_0^\infty\int_{r_2}^\infty\int_{r_1}^\infty g(x_1,x_2)\,dx_1\,dx_2\,dr_2\leq\int_0^\infty\int_0^\infty x_2\,g(x_1,x_2)\,dx_1\,dx_2:=M_{01}<+\infty.$$

Therefore, from (13) we get

$$\int_1^\infty\int_1^\infty f_{n+1}(x_1,x_2)\,dx_1\,dx_2\leq\frac{M_{11}}{\alpha_3},\tag{14}$$

$$\int_0^\infty\int_1^\infty f_{n+1}(x_1,x_2)\,dx_1\,dx_2\leq\frac{M_{10}}{\alpha_1},\tag{15}$$

$$\int_1^\infty\int_0^\infty f_{n+1}(x_1,x_2)\,dx_1\,dx_2\leq\frac{M_{01}}{\alpha_2}.\tag{16}$$

Integrating both parts of (7) over the set $[0,\,1]\times[0,\,1]$ and then using the estimates (14)–(16), we have

$$\int_0^1\int_0^1 f_{n+1}(x_1,x_2)\,dx_1\,dx_2\leq\int_0^1\int_0^1 g(x_1,x_2)\,dx_1\,dx_2+$$

$$+\int_0^\infty\int_0^\infty K(y_1,y_2)\int_0^1\int_0^1 f_{n+1}\left(\rho_1(x_1,y_1),\rho_2(x_2,y_2)\right)\,dx_1\,dx_2\,dy_1\,dy_2\leq$$

$$\leq\int_0^1\int_0^1 g(x_1,x_2)dx_1dx_2+\int_0^1\int_0^\infty K(y_1,y_2)\int_0^1\int_0^1 f_{n+1}\left(x_1,\rho_2(x_2,y_2)\right)dx_1dx_2dy_1dy_2+$$

$$+\int_1^\infty\int_0^\infty K(y_1,y_2)\int_0^1\int_0^1 f_{n+1}\left(x_1+1,\rho_2(x_2,y_2)\right)\,dx_1\,dx_2\,dy_1\,dy_2\leq$$

$$\leq\int_0^1\int_0^1 g(x_1,x_2)\,dx_1\,dx_2+\int_0^1\int_0^1 K(y_1,y_2)\int_0^1\int_0^1 f_{n+1}(x_1,x_2)\,dx_1\,dx_2\,dy_1\,dy_2+$$

$$+\int_0^1\int_1^\infty K(y_1,y_2)\int_0^1\int_0^1 f_{n+1}(x_1,x_2+1)\,dx_1\,dx_2\,dy_1\,dy_2+$$

$$+\int_1^\infty\int_0^1 K(y_1,y_2)\int_0^1\int_0^1 f_{n+1}(x_1+1,x_2)\,dx_1\,dx_2\,dy_1\,dy_2+$$

$$+\int_1^\infty\int_1^\infty K(y_1,y_2)\int_0^1\int_0^1 f_{n+1}(x_1+1,x_2+1)\,dx_1\,dx_2\,dy_1\,dy_2\leq$$

$$\leq\int_0^1\int_0^1 g(x_1,x_2)\,dx_1\,dx_2+\alpha_0\int_0^1\int_0^1 f_{n+1}(x_1,x_2)\,dx_1\,dx_2+$$

$$+\alpha_1\int_1^\infty\int_0^1 f_{n+1}(x_1,x_2)\,dx_1\,dx_2+\alpha_2\int_0^1\int_1^\infty f_{n+1}(x_1,x_2)\,dx_1\,dx_2+$$

$$+\alpha_3\int_1^\infty\int_1^\infty f_{n+1}(x_1,x_2)\,dx_1\,dx_2\leq\int_0^1\int_0^1 g(x_1,x_2)\,dx_1\,dx_2+$$

$$+\alpha_0\int_0^1\int_0^1 f_{n+1}(x_1,x_2)\,dx_1\,dx_2+\frac{\alpha_1}{\alpha_2}M_{01}+\frac{\alpha_2}{\alpha_1}M_{10}+M_{11},$$

from which we get

$$\int_0^1\int_0^1 f_{n+1}(x_1,x_2)\,dx_1\,dx_2\leq(1-\alpha_0)^{-1}\left\{\int_0^1\int_0^1 g(x_1,x_2)\,dx_1\,dx_2+\right.$$
$$\left.+\frac{\alpha_1}{\alpha_2}M_{01}+\frac{\alpha_2}{\alpha_1}M_{10}+M_{11}\right\}:=C^*<+\infty,\quad n=0,1,2,\ldots\,. \tag{17}$$

Finally, summing the inequalities (14)–(17) we obtain

$$\int_0^\infty\int_0^\infty f_{n+1}(x_1,x_2)\,dx_1\,dx_2\leq C^*+\frac{M_{10}}{\alpha_1}+\frac{M_{01}}{\alpha_2}+\frac{M_{11}}{\alpha_3}<+\infty,\ \ n=0,1,2,\ldots, \tag{18}$$

i.e. the proving inequality (11), where $C=C^*+\dfrac{M_{10}}{\alpha_1}+\dfrac{M_{01}}{\alpha_2}+\dfrac{M_{11}}{\alpha_3}$.

Consequently, the sequence of summable and monotone functions $\{f_n(x_1,x_2)\}_{n=0}^\infty$ as $n\to\infty$ almost everywhere on $\mathbb{R}_2^+$ converges to the summable function $f(x_1,x_2)$. This fact follows from (8)–(10) and (18) by B. Levi's theorem [17]. Using again B. Levi's theorem it can be stated that limit function $f(x_1,x_2)$ satisfies the equation (5) almost everywhere on $\mathbb{R}_2^+$.

From (8), (9) and (18) we also get

$$f(x_1,x_2)\geq g(x_1,x_2),\quad (x_1,x_2)\in\mathbb{R}_2^+, \tag{19}$$

$$f(x_1, x_2) \downarrow \text{ in } x_j \text{ on } \mathbb{R}^+, \quad j = 1, 2, \tag{20}$$

$$\int_0^\infty \int_0^\infty f(x_1, x_2)\, dx_1\, dx_2 \le C^* + \frac{M_{10}}{\alpha_1} + \frac{M_{01}}{\alpha_2} + \frac{M_{11}}{\alpha_3}. \tag{21}$$

The foregoing implies

Theorem 1. *Let the function g satisfy the conditions* (6), *and let the kernel K have the properties a_1). Then under conditions* $1) - 3)$ *the equation* (5) *has a nonnegative and monotonically non-increasing in each argument and summable solution. Moreover, the estimates* (19) *and* (21) *hold for the solution.*

2.2 A nontrivial solution of a linear homogeneous auxiliary integral equation on a quarter-plane

Let us introduce into consideration the inhomogeneous auxiliary integral equation

$$\begin{aligned} f^*(x_1, x_2) &= 1 - \lambda(x_1, x_2) + \\ &+ \lambda(x_1, x_2) \int_0^\infty \int_0^\infty K(y_1, y_2)\, f^*\left(\rho_1(x_1, y_1), \rho_2(x_2, y_2)\right)\, dy_1\, dy_2, \quad (x_1, x_2) \in \mathbb{R}_2^+ \end{aligned} \tag{22}$$

with respect to the unknown measurable function $f^*(x_1, x_2)$, where the functions λ and K possess the properties a_2) and a_1) respectively.

Due to a_2) the function $1 - \lambda(x_1, x_2)$ satisfies the conditions (6). Therefore, according to Theorem 1, the equation (5) with the free term $g(x_1, x_2) = 1 - \lambda(x_1, x_2)$ has a nonnegative and monotone (with respect to each argument) and summable solution on $\mathbb{R}_2^+$. We denote this solution by $f_\lambda(x_1, x_2)$.

For the equation (22) consider the following iterations:

$$\begin{aligned} f^*_{n+1}(x_1, x_2) &= 1 - \lambda(x_1, x_2) + \\ &+ \lambda(x_1, x_2) \int_0^\infty \int_0^\infty K(y_1, y_2)\, f^*_n\left(\rho_1(x_1, y_1), \rho_2(x_2, y_2)\right)\, dy_1\, dy_2, \\ f^*_0(x_1, x_2) &= 1 - \lambda(x_1, x_2), \quad n = 0, 1, 2, \ldots, \quad (x_1, x_2) \in \mathbb{R}_2^+. \end{aligned} \tag{23}$$

By induction it is easy to show that

$$f^*_n(x_1, x_2) \uparrow \text{ in } n, \quad (x_1, x_2) \in \mathbb{R}_2^+, \tag{24}$$

$$f^*_n(x_1, x_2) \le \min\{1, f_\lambda(x_1, x_2)\}, \quad n = 0, 1, 2, \ldots, \quad (x_1, x_2) \in \mathbb{R}_2^+. \tag{25}$$

Therefore, the sequence of functions $\{f^*_n(x_1, x_2)\}_{n=0}^\infty$ has a pointwise limit as $n \to \infty$: $\lim\limits_{n\to\infty} f^*_n(x_1, x_2) = f^*(x_1, x_2)$. In accordance with B. Levi's theorem, the limit function $f^*(x_1, x_2)$ satisfies the equation (22). It follows from (24) and (25) that

$$1 - \lambda(x_1, x_2) \le f^*(x_1, x_2) \le \min\{1, f_\lambda(x_1, x_2)\}, \quad (x_1, x_2) \in \mathbb{R}_2^+, \tag{26}$$

whence, in particular, we obtain

$$f^* \in L_1(\mathbb{R}_2^+) \cap M(\mathbb{R}_2^+). \tag{27}$$

Further, we consider the corresponding homogeneous integral equation

$$S(x_1, x_2) = \lambda(x_1, x_2) \int_0^\infty \int_0^\infty K(y_1, y_2) S\left(\rho_1(x_1, y_1), \rho_2(x_2, y_2)\right) dy_1 dy_2, \tag{28}$$

$(x_1, x_2) \in \mathbb{R}_2^+$, with respect to the measurable and bounded function $S(x_1, x_2)$. Using a_1), we can check directly that $f^*_{\text{triv}}(x_1, x_2) \equiv 1$ is a solution of the equation (22). On the other hand, we have proved that the equation (22), in addition to such a trivial solution, also has an integrable and bounded solution $f^*(x_1, x_2)$ (with the property (26)). It is obvious that

$$S(x_1, x_2) = f^*_{\text{triv}}(x_1, x_2) - f^*(x_1, x_2) = 1 - f^*(x_1, x_2), \quad (x_1, x_2) \in \mathbb{R}_2^+$$

is a solution of the homogeneous equation (28). From (26), in particular, we get

$$1 \geq S(x_1, x_2) \geq 0, \quad S(x_1, x_2) \not\equiv 0, \quad S(x_1, x_2) \not\equiv 1, \quad (x_1, x_2) \in \mathbb{R}_2^+, \tag{29}$$

and from (27)

$$1 - S \in L_1(\mathbb{R}_2^+) \cap M(\mathbb{R}_2^+). \tag{30}$$

Thus, for the auxiliary linear homogeneous equation (28), the following theorem holds:

Theorem 2. *Under the conditions* $a_1), a_2)$ *and* $1) - 3)$ *the linear homogeneous integral equation* (28) *has a nonnegative nontrivial measurable and bounded solution* $S(x_1, x_2)$ *on* $\mathbb{R}_2^+$. *In addition,* $S(x_1, x_2)$ *possesses the* (29) *and* (30) *properties.*

Remark 1. It is interesting to note that the proved Theorem 2 generalizes and supplements the corresponding result from [18], devoted to the study of one-dimensional integral equations with $\rho(u, v) = u + v, \quad (u, v) \in \mathbb{R}_2^+$.

3 Solubility of the main nonlinear equation. Examples

In this section, we begin to study the initial nonlinear integral equation (1), first highlighting one special case (important in applications).

3.1 One-parameter family of bounded solutions of the equation (1) in one particular case

Let the nonlinearity $G(x_1, x_2, u)$ admit a representation of the form

$$G(x_1, x_2, u) = u + \omega(x_1, x_2, u), \quad (x_1, x_2, u) \in \mathbb{R}_2^+ \times \mathbb{R}, \tag{31}$$

where $\omega(x_1, x_2, u)$ satisfies the following conditions:

I) $\omega(x_1,x_2,u)\uparrow$ in u on $\mathbb{R}^+$,

II) $\omega(x_1,x_2,u)$ satisfies the Carathéodory condition with respect to the argument u on $\mathbb{R}_2^+\times\mathbb{R}$ (see the introduction about the Carathéodory condition),

III) $\omega(x_1,x_2,u)\geq 0,\quad (x_1,x_2,u)\in\mathbb{R}_3^+$,

IV) the supremum of ω with respect to u on $\mathbb{R}_2^+$:

$$\beta(x_1,x_2):=\sup_{u\in\mathbb{R}^+}\omega(x_1,x_2,u),\quad (x_1,x_2)\in\mathbb{R}_2^+, \tag{32}$$

possesses following properties: $\beta(x_1,x_2)\downarrow$ in x_j on $\mathbb{R}^+$, $j=1,2$

$$x_1^m x_2^\ell\,\beta(x_1,x_2)\in L_1\left(\mathbb{R}_2^+\right),\ m,\ell=0,1.$$

Suppose also that the kernel $\mathcal{P}(x_1,y_1,x_2,y_2)$ is linked with the functions λ and K by the relation

$$\mathcal{P}(x_1,y_1,x_2,y_2)=\lambda(x_1,x_2)K(y_1,\,y_2),\quad (x_1,y_1,x_2,y_2)\in\mathbb{R}_4^+. \tag{33}$$

Then the equation (1) will take the following form:

$$\begin{gathered}\mathscr{F}(x_1,x_2)=\lambda(x_1,x_2)\int\limits_0^\infty\int\limits_0^\infty K(y_1,\,y_2)\big\{\mathscr{F}(\rho_1(x_1,y_1),\rho_2(x_2,y_2))+\\ +\,\omega(x_1,x_2,\mathscr{F}(\rho_1(x_1,y_1),\rho_2(x_2,y_2)))\big\}\,dy_1\,dy_2,\ (x_1,x_2)\in\mathbb{R}_2^+.\end{gathered} \tag{34}$$

We construct special successive approximations

$$\begin{gathered}\mathscr{F}_{n+1}^\gamma(x_1,x_2)=\lambda(x_1,x_2)\int\limits_0^\infty\int\limits_0^\infty K(y_1,\,y_2)\big\{\mathscr{F}_n^\gamma(\rho_1(x_1,y_1),\rho_2(x_2,y_2))+\\ +\,\omega\big(x_1,x_2,\mathscr{F}_n^\gamma(\rho_1(x_1,y_1),\rho_2(x_2,y_2))\big)\big\}\,dy_1\,dy_2,\\ \mathscr{F}_0^\gamma(x_1,x_2)=\gamma S(x_1,x_2),\quad n=0,1,2,\ldots,\quad (x_1,x_2)\in\mathbb{R}_2^+,\end{gathered} \tag{35}$$

where $\gamma>0$ is an arbitrary numeric parameter.

Along with iterations (35), consider a linear inhomogeneous integral equation (5) with a free term of the form

$$g(x_1,x_2)=\beta(x_1,x_2),\quad (x_1,x_2)\in\mathbb{R}_2^+. \tag{36}$$

Due to conditions III) and IV), according to Theorem 1 the equation (5) with a free term of the form (36) has a nonnegative monotonically non-increasing and summable on $\mathbb{R}_2^+$ solution $f_\beta(x_1,x_2)$.

Below we establish several important properties that characterize the sequence $\{\mathscr{F}_n^\gamma(x_1,x_2)\}_{n=0}^\infty$ both for each value of the parameter $\gamma>0$.

By induction on n we prove

$$\mathscr{F}_n^\gamma(x_1,x_2)\uparrow\ \text{in}\ \ n,\ \gamma>0,\ (x_1,x_2)\in\mathbb{R}_2^+, \tag{37}$$

$$\mathscr{F}_n^{\gamma}(x_1,x_2) \le \gamma S(x_1,x_2) + f_\beta(x_1,x_2),\ \gamma>0,\ n=0,1,2,\dots,\ (x_1,x_2)\in\mathbb{R}_2^+. \quad (38)$$

We first prove that $\mathscr{F}_1^{\gamma}(x_1,x_2) \ge \mathscr{F}_0^{\gamma}(x_1,x_2)$ and $\mathscr{F}_1^{\gamma}(x_1,x_2) \le \gamma S(x_1,x_2) + f_\beta(x_1,x_2)$, $(x_1,x_2)\in\mathbb{R}_2^+$, $\gamma>0$. Indeed, taking into account (28), (32), as well as the conditions $a_1), a_2), III)$, from (35) we have

$$\mathscr{F}_1^{\gamma}(x_1,x_2) \ge \lambda(x_1,x_2)\int_0^\infty\int_0^\infty K(y_1,\ y_2)\mathscr{F}_0^{\gamma}(\rho_1(x_1,y_1),\rho_2(x_2,y_2))\,dy_1\,dy_2 =$$

$$= \gamma\lambda(x_1,x_2)\int_0^\infty\int_0^\infty K(y_1,\ y_2)S(\rho_1(x_1,y_1),\rho_2(x_2,y_2))\,dy_1\,dy_2 =$$

$$= \gamma S(x_1,x_2) = \mathscr{F}_0^{\gamma}(x_1,x_2),$$

$$\mathscr{F}_1^{\gamma}(x_1,x_2) = \lambda(x_1,x_2)\times$$

$$\times\int_0^\infty\int_0^\infty K(y_1,\ y_2)\Big\{\gamma S\big(\rho_1(x_1,y_1),\rho_2(x_2,y_2)\big) + f_\beta\big(\rho_1(x_1,y_1),\rho_2(x_2,y_2)\big)+$$

$$+\omega\Big(x_1,x_2,\gamma S\big(\rho_1(x_1,y_1),\rho_2(x_2,y_2)\big) + f_\beta\big(\rho_1(x_1,y_1),\rho_2(x_2,y_2)\big)\Big)\Big\}\,dy_1\,dy_2 \le$$

$$\le \gamma\lambda(x_1,x_2)\int_0^\infty\int_0^\infty K(y_1,\ y_2)S\big(\rho_1(x_1,y_1),\rho_2(x_2,y_2)\big)\,dy_1\,dy_2+$$

$$+\lambda(x_1,x_2)\int_0^\infty\int_0^\infty K(y_1,\ y_2)f_\beta\big(\rho_1(x_1,y_1),\rho_2(x_2,y_2)\big)\,dy_1\,dy_2+$$

$$+\beta(x_1,x_2)\lambda(x_1,x_2)\int_0^\infty\int_0^\infty K(y_1,\ y_2)\,dy_1\,dy_2 \le \gamma S(x_1,x_2) + \beta(x_1,x_2)+$$

$$+\int_0^\infty\int_0^\infty K(y_1,\ y_2)f_\beta\big(\rho_1(x_1,y_1),\rho_2(x_2,y_2)\big)\,dy_1\,dy_2 = \gamma S(x_1,x_2) + f_\beta(x_1,x_2).$$

Assume that the statements (37) and (38) are true for some $n\in\mathbb{N}$. We use again (28), (32), $a_1), a_2)$ and III). Then from (35) by virtue of I) we obtain

$$\mathscr{F}_{n+1}^{\gamma}(x_1,x_2) \ge \lambda(x_1,x_2)\int_0^\infty\int_0^\infty K(y_1,\ y_2)\Big\{\mathscr{F}_{n-1}^{\gamma}(\rho_1(x_1,y_1),\rho_2(x_2,y_2))+$$

$$+\,\omega(x_1,x_2,\mathscr{F}_{n-1}^{\gamma}(\rho_1(x_1,y_1),\rho_2(x_2,y_2)))\Big\}\,dy_1\,dy_2 = \mathscr{F}_n^{\gamma}(x_1,x_2),$$

$$\mathscr{F}_{n+1}^{\gamma}(x_1,x_2)\le \gamma S(x_1,x_2)+f_\beta(x_1,x_2),\ n=0,1,2,\dots,\quad (x_1,x_2)\in\mathbb{R}_2^+,\ \gamma>0,$$

whence the required assertions (37) and (38) follow.

Based on the Carathéodory condition for the function ω (see II)) it is easy to prove that for every $\gamma>0$ each element of the sequence $\{\mathscr{F}_n^{\gamma}(x_1,x_2)\}_{n=0}^{\infty}$ is a measurable function on $\mathbb{R}_2^+$.

Thus, in view of (37) and (38) we can assert that the sequence of measurable functions on $\mathbb{R}_2^+$ $\{\mathscr{F}_n^{\gamma}(x_1,x_2)\}_{n=0}^{\infty}$ has a pointwise limit as $n\to\infty$: $\lim\limits_{n\to\infty}\mathscr{F}_n^{\gamma}(x_1,x_2)=\mathscr{F}^{\gamma}(x_1,x_2)$. By Levy's theorem, the limit function $\mathscr{F}^{\gamma}(x_1,x_2)$ satisfies the equation (34) for every $\gamma>0$. Moreover, from (37) and (38) we get that $\mathscr{F}^{\gamma}(x_1,x_2)$ satisfies the following double inequality:

$$\gamma S(x_1,x_2)\le \mathscr{F}^{\gamma}(x_1,x_2)\le \gamma S(x_1,x_2)+f_\beta(x_1,x_2),\quad (x_1,x_2)\in\mathbb{R}_2^+,\ \gamma>0. \tag{39}$$

Now we note one more important and useful property of the sequence of functions $\{\mathscr{F}_n^{\gamma}(x_1,x_2)\}_{n=0}^{\infty}$ on $\mathbb{R}_2^+$ for different values of the parameter $\gamma>0$. We prove by induction that if $\gamma_1,\ \gamma_2\in(0,\,+\infty)$, $\gamma_1>\gamma_2$ are arbitrary parameters, then

$$\mathscr{F}_n^{\gamma_1}(x_1,x_2)-\mathscr{F}_n^{\gamma_2}(x_1,x_2)\ge(\gamma_1-\gamma_2)S(x_1,x_2),\ n=0,1,2,\dots,\ (x_1,x_2)\in\mathbb{R}_2^+. \tag{40}$$

Indeed, when $n=0$ the required inequality is obvious. Suppose (40) is satisfied for some $n\in\mathbb{N}$. Then, using the conditions $I), a_1), a_2)$ and taking into account (28), from (35) we have

$$\mathscr{F}_{n+1}^{\gamma_1}(x_1,x_2)-\mathscr{F}_{n+1}^{\gamma_2}(x_1,x_2)=\lambda(x_1,x_2)\int\limits_0^{\infty}\int\limits_0^{\infty}K(y_1,\,y_2)\Big\{\mathscr{F}_n^{\gamma_1}(\rho_1(x_1,y_1),\rho_2(x_2,y_2))-$$
$$-\mathscr{F}_n^{\gamma_2}(\rho_1(x_1,y_1),\rho_2(x_2,y_2))+\omega(x_1,x_2,\mathscr{F}_n^{\gamma_1}(\rho_1(x_1,y_1),\rho_2(x_2,y_2)))-$$
$$-\omega(x_1,x_2,\mathscr{F}_n^{\gamma_2}(\rho_1(x_1,y_1),\rho_2(x_2,y_2)))\Big\}\,dy_1\,dy_2\ge\lambda(x_1,x_2)\times$$
$$\times\int\limits_0^{\infty}\int\limits_0^{\infty}K(y_1,\,y_2)\Big\{\mathscr{F}_n^{\gamma_1}(\rho_1(x_1,y_1),\rho_2(x_2,y_2))-\mathscr{F}_n^{\gamma_2}(\rho_1(x_1,y_1),\rho_2(x_2,y_2))\Big\}dy_1dy_2\ge$$
$$\ge(\gamma_1-\gamma_2)\lambda(x_1,x_2)\int\limits_0^{\infty}\int\limits_0^{\infty}K(y_1,\,y_2)S\big(\rho_1(x_1,y_1),\rho_2(x_2,y_2)\big)dy_1dy_2=(\gamma_1-\gamma_2)S(x_1,x_2).$$

Letting the number $n\to\infty$ into (40), we get

$$\mathscr{F}^{\gamma_1}(x_1,x_2)-\mathscr{F}^{\gamma_2}(x_1,x_2)\ge(\gamma_1-\gamma_2)S(x_1,x_2),\quad (x_1,x_2)\in\mathbb{R}_2^+. \tag{41}$$

Since $1-S\in L_1(\mathbb{R}_2^+)\cap M(\mathbb{R}_2^+)$ and $f_\beta\in L_1(\mathbb{R}_2^+)$, then in view of (39) from the estimate below

$$|\gamma-\mathscr{F}^{\gamma}(x_1,x_2)|=|\gamma-\gamma S(x_1,x_2)+\gamma S(x_1,x_2)-\mathscr{F}^{\gamma}(x_1,x_2)|\le$$
$$\le\gamma(1-S(x_1,x_2))+f_\beta(x_1,x_2),\ \gamma>0,\ (x_1,x_2)\in\mathbb{R}_2^+$$

we obtain the following important fact: for each $\gamma>0$ the function $\gamma-\mathscr{F}^{\gamma}\in L_1(\mathbb{R}_2^+)$.

Thus the following theorem is true.

Theorem 3. *Under conditions* $a_1), a_2), I) - IV)$ *and* $1) - 3)$, *the nonlinear integral equation* (34) *has a one-parameter family of nonnegative nontrivial measurable solutions* $\{\mathscr{F}^\gamma(x_1, x_2)\}_{\gamma\in(0,+\infty)}$ *and*

- *for all* $\gamma \in (0, +\infty)$ *the inequalities* (39) *hold,*
- *for all* $\gamma_1, \gamma_2 \in (0, +\infty)$, $\gamma_1 > \gamma_2$, (41) *takes place,*
- *for all* $\gamma \in (0, +\infty)$ *functions* $\gamma - \mathscr{F}^\gamma(x_1, x_2)$ *are summable on* $\mathbb{R}_2^+$.

Remark 2. Under the assumptions of Theorem 3, if moreover the following conditions are fulfilled

$p_1)$ $\rho_j(0, v) \geq v, \; v \in \mathbb{R}^+, \; j = 1, 2,$
$p_2)$ $\beta \in M(\mathbb{R}_2^+)$,

then for any $\gamma > 0$ the solution $\mathscr{F}^\gamma(x_1, x_2)$ is bounded on the set $\mathbb{R}_2^+$.

Proof. First, we verify that $f_\beta \in M(\mathbb{R}_2^+)$. Indeed, given the monotonicity of $f_\beta(x_1, x_2)$ in x_j on $\mathbb{R}^+$, $j = 1, 2$, and also conditions $2), a_1), p_1), p_2)$, from the equation (5) with free term $g(x_1, x_2) = \beta(x_1, x_2)$ we get

$$f_\beta(x_1, x_2) \leq \sup_{(x_1,x_2)\in\mathbb{R}_2^+} \beta(x_1, x_2) +$$
$$+ \sup_{(y_1,y_2)\in\mathbb{R}_2^+} K(y_1, y_2) \int_0^\infty\!\!\int_0^\infty f_\beta\big(\rho_1(x_1, y_1), \rho_2(x_2, y_2)\big)\, dy_1\, dy_2 \leq$$
$$\leq \sup_{(x_1,x_2)\in\mathbb{R}_2^+} \beta(x_1, x_2) + \sup_{(y_1,y_2)\in\mathbb{R}_2^+} K(y_1, y_2) \int_0^\infty\!\!\int_0^\infty f_\beta\big(\rho_1(0, y_1), \rho_2(0, y_2)\big)\, dy_1\, dy_2 \leq$$
$$\leq \sup_{(x_1,x_2)\in\mathbb{R}_2^+} \beta(x_1, x_2) + \sup_{(y_1,y_2)\in\mathbb{R}_2^+} K(y_1, y_2) \int_0^\infty\!\!\int_0^\infty f_\beta(y_1, y_2)\, dy_1\, dy_2 < +\infty,$$

whence it follows that $f_\beta \in M(\mathbb{R}_2^+)$. Consequently, from (29) and (39) we have

$$0 \leq \mathscr{F}^\gamma(x_1, x_2) \leq \gamma + \sup_{(x_1,x_2)\in\mathbb{R}_2^+} f_\beta(x_1, x_2) < +\infty, \; \gamma > 0, \; (x_1, x_2) \in \mathbb{R}_2^+.$$

□

3.2 Main result

Let us turn to the study of the original equation (1) with a common kernel $\mathcal{P}$ and a common nonlinearity $G(x_1, x_2, u)$.

First, to represent the main conditions imposed on the function G, we introduce a new function. Let $G_0(u)$ be a continuous on the set $\mathbb{R}^+$ function and

$c_1)$ $G_0(u) \uparrow$ u on $\mathbb{R}^+$, $G_0(0) = 0$,

$c_2)$ $G_0(u)$ is upward convex on $\mathbb{R}^+$, $G_0 \in C(\mathbb{R}^+)$,

c_3) there exists a number $\eta > \sup\limits_{(x_1,x_2)\in\mathbb{R}_2^+} f_\beta(x_1,x_2) := B_0$ such that

$G_0(u) \geq u,\ u \in [0,\ \eta]$.

The properties of $c_1) - c_3)$ imply the existence of a single number $\xi > \eta$ such that

$$G_0(\xi) = \xi - B_0. \tag{42}$$

The approximate graph of the function G_0 is shown in the figure.

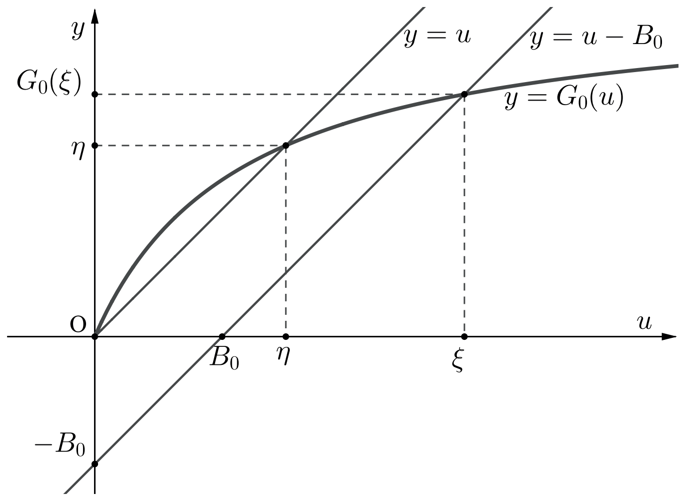

Figure. The approximate graph of the function G_0 on $[0,\ \xi]$.

Regarding the nonlinearity of $G(x_1,x_2,u)$, we assume that the following conditions are satisfied:

n_1) $G(x_1,x_2,u) \uparrow$ in u on $\mathbb{R}^+$ and $G(x_1,x_2,u)$ satisfies the Carathéodory condition on $\mathbb{R}_2^+ \times \mathbb{R}$ by argument u,

n_2) $G(x_1,x_2,u) \geq u + \omega(x_1,x_2,u),\ (x_1,x_2,u) \in \mathbb{R}_3^+$,
where ω has properties $I) - IV)$ and p_2),

n_3) $G(x_1,x_2,u) \leq G_0(u) + \beta(x_1,x_2),\ (x_1,x_2,u) \in \mathbb{R}_2^+ \times [0,\ \xi]$.

The next theorem is valid.

Theorem 4. *Let conditions* $a), b), 1) - 3), p_1), c_1) - c_3)$ *and* $n_1) - n_3)$ *be satisfied. Then the nonlinear integral equation* (1) *has a nonnegative nontrivial solution bounded on* $\mathbb{R}_2^+$.

Proof. Let $\gamma^* := \eta - B_0 > 0$. By Theorem 3 and Remark 2, to the number γ^* the bounded solution $\mathscr{F}^{\gamma^*}(x_1,x_2)$ of the equation (34) corresponds, where $\gamma^* - \mathscr{F}^{\gamma^*} \in L_1(\mathbb{R}_2^+)$ and the double inequality takes place

$$\gamma^* S(x_1,x_2) \leq \mathscr{F}^{\gamma^*}(x_1,x_2) \leq \gamma^* S(x_1,x_2) + f_\beta(x_1,x_2), \quad (x_1,x_2) \in \mathbb{R}_2^+. \tag{43}$$

By the definition of the number γ^* and the inequality $S(x_1, x_2) \leq 1,\ (x_1, x_2) \in \mathbb{R}_2^+$ from (43) it follows that

$$\mathscr{F}^{\gamma^*}(x_1, x_2) \leq \gamma^* + B_0 = \eta,\ (x_1, x_2) \in \mathbb{R}_2^+. \tag{44}$$

Let us proceed to the construction of a solution to the equation (1) by successive approximations

$$\begin{gathered}\mathscr{F}_{(n+1)}(x_1, x_2) = \int_0^\infty \int_0^\infty \mathcal{P}(x_1, y_1, x_2, y_2) \times \\ \times G(x_1, x_2, \mathscr{F}_{(n)}(\rho_1(x_1, y_1), \rho_2(x_2, y_2)) dy_1 dy_2, \\ \mathscr{F}_{(0)}(x_1, x_2) = \mathscr{F}^{\gamma^*}(x_1, x_2),\ n = 0, 1, 2, \ldots,\ (x_1, x_2) \in \mathbb{R}_2^+.\end{gathered} \tag{45}$$

We prove by induction that

$$\mathscr{F}_{(n)}(x_1, x_2) \uparrow \text{ in } n,\ (x_1, x_2) \in \mathbb{R}_2^+. \tag{46}$$

First, note that, based on (4), (34) and condition n_2), the following chain of inequalities holds:

$$\mathscr{F}_{(1)}(x_1, x_2) \geq \int_0^\infty \int_0^\infty \mathcal{P}(x_1, y_1, x_2, y_2) \Big(\mathscr{F}^{\gamma^*}(\rho_1(x_1, y_1), \rho_2(x_2, y_2)) +$$

$$+\omega(x_1, x_2, \mathscr{F}^{\gamma^*}(\rho_1(x_1, y_1), \rho_2(x_2, y_2)))\Big)\, dy_1\, dy_2 \geq \lambda(x_1, x_2) \int_0^\infty \int_0^\infty K(y_1,\, y_2) \times$$

$$\times \Big(\mathscr{F}^{\gamma^*}(\rho_1(x_1, y_1), \rho_2(x_2, y_2)) + \omega(x_1, x_2, \mathscr{F}^{\gamma^*}(\rho_1(x_1, y_1), \rho_2(x_2, y_2)))\Big)\, dy_1\, dy_2 =$$

$$= \mathscr{F}^{\gamma^*}(x_1, x_2) = \mathscr{F}_{(0)}(x_1, x_2),\ (x_1, x_2) \in \mathbb{R}_2^+.$$

Assuming $\mathscr{F}_{(n)}(x_1, x_2) \geq \mathscr{F}_{(n-1)}(x_1, x_2),\ (x_1, x_2) \in \mathbb{R}_2^+$ for some positive integer n, due to the non-negativity of the kernel $\mathcal{P}$ and the condition n_1) from (45) we obtain that $\mathscr{F}_{(n+1)}(x_1, x_2) \geq \mathscr{F}_{(n)}(x_1, x_2),\ (x_1, x_2) \in \mathbb{R}_2^+$.

Let now prove that

$$\mathscr{F}_{(n)}(x_1, x_2) \leq \xi,\ n = 0, 1, 2, \ldots,\ (x_1, x_2) \in \mathbb{R}_2^+. \tag{47}$$

When $n = 0$ the inequality (47) is an obvious consequence of the inequalities (44) and $\eta < \xi$. Suppose (47) holds for some $n \in \mathbb{N}$. Then, in view of the conditions $b), n_1), n_3)$ and the definition of the number ξ (see (42)), from (45) we will have

$$\mathscr{F}_{(n+1)}(x_1, x_2) \leq \int_0^\infty \int_0^\infty \mathcal{P}(x_1, y_1, x_2, y_2)\, G(x_1, x_2, \xi)\, dy_1\, dy_2 \leq$$

$$\leq (G_0(\xi)+\beta(x_1,x_2))\int_0^\infty\int_0^\infty \mathcal{P}(x_1,y_1,x_2,y_2)\,dy_1\,dy_2 \leq$$

$$\leq (G_0(\xi)+B_0)\,\mu(x_1,x_2) \leq G_0(\xi)+B_0=\xi,\quad (x_1,x_2)\in\mathbb{R}_2^+.$$

Thus, given that (46) and (47) hold, one can assert that the sequence of measurable on $\mathbb{R}_2^+$ functions $\{\mathscr{F}_{(n)}(x_1,x_2)\}_{n=0}^\infty$ has a pointwise limit as $n\to\infty$: $\lim\limits_{n\to\infty}\mathscr{F}_{(n)}(x_1,x_2)=\mathscr{F}(x_1,x_2)$, and the limit function $\mathscr{F}(x_1,x_2)$ satisfies the equation (1) (due to B. Levi's theorem) and the double inequality

$$\mathscr{F}^{\gamma^*}(x_1,x_2)\leq \mathscr{F}(x_1,x_2)\leq \xi,\ (x_1,x_2)\in\mathbb{R}_2^+.$$

This completes the proof. □

3.3 Examples

In the end of the work, we provide concrete illustrative examples of the functions $\{\rho_j\}_{j=1,2}$, ω, λ, K, G_0, G and $\mathcal{P}$ satisfying all assumptions of the formulated theorems.

Examples of functions $\{\rho_j\}_{j=1,2}$:

$A_1)$ $\rho_j(u,v)=u+v,\ (u,v)\in\mathbb{R}_2^+,\ j=1,2,$

$A_2)$ $\rho_j(u,v)=u\,(1+\alpha_j v)+\beta_j v,\ (u,v)\in\mathbb{R}_2^+,\ j=1,2,$
where $\alpha_j\geq 0,\ \beta_j\geq 1$ are numerical parameters, $j=1,2,$

$A_3)$ $\rho_j(u,v)=(u+\varepsilon_j)e^v+2(1-e^{-v}),\ (u,v)\in\mathbb{R}_2^+,\ j=1,2,$
where $\varepsilon_j\geq 1$ is a numerical parameter, $j=1,2.$

Examples of functions ω:

$B_1)$ $\omega(x_1,x_2,u)=\beta(x_1,x_2)(1-e^{-u}),\quad (x_1,x_2,u)\in\mathbb{R}_3^+,$

$B_2)$ $\omega(x_1,x_2,u)=\beta(x_1,x_2)\dfrac{u}{u+1},\quad (x_1,x_2,u)\in\mathbb{R}_3^+.$

Examples of functions λ:

$D_1)$ $\lambda(x_1,x_2)=1-e^{-(x_1+x_2)},\quad (x_1,x_2)\in\mathbb{R}_2^+,$

$D_2)$ $\lambda(x_1,x_2)=1-\varepsilon e^{-(x_1^2+x_2^2)},\quad (x_1,x_2)\in\mathbb{R}_2^+,\quad 0<\varepsilon\leq 1$ is a parameter.

Examples of kernel K:

$E_1)$ $K(y_1,\,y_2)=\dfrac{4}{\pi}e^{-(y_1^2+y_2^2)},\quad (y_1,\,y_2)\in\mathbb{R}_2^+,$

$E_2)$ $K(y_1,\,y_2)=\int\limits_a^b e^{-(y_1+y_2)s}\,Q(s)\,ds,\quad (y_1,\,y_2)\in\mathbb{R}_2^+,$
where $Q(s)>0$ is a continuous function on $[a,\,b), 0<a<b\leq+\infty$, and $\int\limits_a^b \frac{G(s)}{s^2}ds=1.$

Examples of nonlinearity G_0:

$H_1)$ $G_0(u) = \sqrt[p]{u},\ u \in \mathbb{R}^+,\ p > 2$ is an arbitrary odd number,

$H_2)$ $G_0(u) = d(1 - e^{-u}),\ u \in \mathbb{R}^+,\ d > 1$ is a numeric parameter.

Examples of kernel $\mathcal{P}$:

$L_1)$ $\mathcal{P}(x_1, y_1, x_2, y_2) = \lambda(x_1, x_2)K(y_1,\, y_2),\quad (x_1, y_1, x_2, y_2) \in \mathbb{R}_4^+,$

$L_2)$ $\mathcal{P}(x_1, y_1, x_2, y_2) = \lambda(x_1, x_2)K(y_1,\, y_2) + K_0(x_1, y_1, x_2, y_2),\ (x_1, y_1, x_2, y_2) \in \mathbb{R}_4^+,$ where $K_0(x_1, y_1, x_2, y_2) \geq 0,\ (x_1, y_1, x_2, y_2) \in \mathbb{R}_4^+$ and $\int\limits_0^\infty \int\limits_0^\infty K_0(x_1, y_1, x_2, y_2)\, dy_1\, dy_2 = \varepsilon(1 - \lambda(x_1, x_2)),\ 0 < \varepsilon < 1$ is a parameter.

Examples of nonlinearity G :

$U_1)$ $G(x_1, x_2, u) = G_0(u) + \omega(x_1, x_2, u),\quad (x_1, x_2, u) \in \mathbb{R}_2^+ \times \mathbb{R},$

$U_2)$ $G(x_1, x_2, u) = \sqrt{(u + \omega(x_1, x_2, u))(G_0(u) + \omega(x_1, x_2, u))},\ (x_1, x_2, u) \in \mathbb{R}_2^+ \times \mathbb{R},$

$U_3)$ $G(x_1, x_2, u) = \frac{1}{2}(G_0(u) + u) + \omega(x_1, x_2, u),\quad (x_1, x_2, u) \in \mathbb{R}_2^+ \times \mathbb{R}.$

In conclusion, we note that among the above examples, the most important and most frequently encountered in applications of mathematical physics and mathematical biology are $A_1)$, $B_1)$, $D_1)$, $E_1)$, $E_2)$, $L_1)$, $H_1)$, $H_2)$ and $U_1)$.

Remark 3. Unfortunately, the question of the uniqueness of the solution of the general nonlinear integral equation (1) in certain cone segments (functions bounded on $\mathbb{R}_2^+$) is still open problem.

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