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Asymptotic Behavior of Homogeneous Linear Recurrent Processes and Their Perturbations

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Abstract. In this paper the impact of small perturbations on asymptotic evolution of homogeneous linear recurrent processes is investigated. Analytical methods for describing homogeneous linear recurrent systems, from convergence, periodicity and boundedness perspective, are presented. These methods are based on Jury Stability Criterion and the classification of the roots of minimal characteristic polynomial in relation to unit disc.

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1 Introduction

The main goal of this paper is to study the impact of small perturbations on asymptotic evolution of homogeneous linear recurrent processes.

It is started with definitions and main properties of homogeneous linear recurrent processes. The direct formula for the states and the formula for generating function are given. Also, the linear combination and the product are presented as algebraic operations over the set of homogeneous linear recurrences.

Next, the definition of minimality, over a given set, is introduced. Inequalities for the dimension of the linear combination and product are presented. We formulate the minimization method based on matrix rank definition and the minimization method by elimination of characteristic zeros.

After that, we are interested in asymptotic behavior of homogeneous linear recurrences. The convergence criteria and the efficient formula for calculating the limit are given. The Jury Stability Criterion is proposed as alternative, for the case when the characteristic roots are not known.

Next, we continue with investigation of the main probabilistic characteristics of homogeneous linear recurrent distributions. The top of interest is represented by efficient methods for finding the expectation, the variance, the standard deviation, the moments of order n, the median and the mode of these distributions.

The last section is devoted to the perturbations generated by deviations in initial state or deviations in generating vector components. Also, mixed perturbations are considered. The asymptotic stability is studied and the maximal perturbation impact is estimated.

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2 Homogeneous Linear Recurrent Processes

The homogeneous linear recurrences and their main properties were intensively studied in [3] and [4]. Next, they will be briefly recalled and new extensions will be presented. These results will represent the ground of a new analytical method for studying the small perturbations and their impact on asymptotic evolution.

2.1 Main Definitions and Properties

A non-degenerate homogeneous linear m-recurrence over a set K is defined as a sequence $a = \{a_n\}_{n=0}^{\infty} \subseteq \mathbb{C}$ that satisfies the recurrence

$$a_n = \sum_{k=0}^{m-1} q_k a_{n-1-k}, \ \forall n \geq m,$$

for a given positive integer m, generating vector $q = (q_k)_{k=0}^{m-1} \in K^m$ and initial state $I_m^{[a]} = (a_n)_{n=0}^{m-1}$, where $q_{m-1} \neq 0$.

The function $G^{[a]}(z) = \sum\limits_{n=0}^{\infty} a_n z^n$ is the generating function and the function $G_t^{[a]}(z) = \sum\limits_{n=0}^{t-1} a_n z^n$ is the partial generating function of order t of the sequence a.

For this sequence a with generating vector q, the unit characteristic polynomial $H_m^{[q]}(z) = 1 - zG_m^{[q]}(z)$ and the characteristic equation $H_m^{[q]}(z) = 0$ are defined. Every polynomial $H_{m,\alpha}^{[q]}(z) = \alpha H_m^{[q]}(z)$ is, also, considered a characteristic polynomial of a.

The set $G[K][m](a)$ represents the set of all generating vectors of length m and $H[K][m](a)$ represents the set of characteristic polynomials of degree m of the sequence a. The set $Rol[K][m]$ is the set of all non-degenerate homogeneous linear m-recurrences over K.

Additionally, the sets $Rol[K] = \bigcup\limits_{m=1}^{\infty} Rol[K][m]$, $G[K](a) = \bigcup\limits_{m=1}^{\infty} G[K][m](a)$ and $H[K](a) = \bigcup\limits_{m=1}^{\infty} H[K][m](a)$ are considered.

Next, it is considered that the set K is a subfield of $\mathbb{C}$. The following theorem, theoretically grounded in [4], describes the generating function as a simple formula:

Theorem 1. *Let $a \in Rol[K][m]$ and $q = (q_k)_{k=0}^{m-1} \in G[K][m](a)$. The generating function is a rational fraction for which the following formula holds:*

$$G^{[a]}(z) = \frac{G_m^{[a]}(z) - z \sum\limits_{k=0}^{m-1} q_k z^k G_{m-1-k}^{[a]}(z)}{H_m^{[q]}(z)}.$$

Also, the following result presents us the direct formula for calculating the terms of a homogeneous linear recurrence:

Theorem 2. *Let* $a \in Rol[K][m]$ *with generating vector* $q \in G[K][m](a)$ *and characteristic polynomial* $H^{[q]}_{m,\alpha}(z) = \prod_{k=0}^{p-1}(z - z_k)^{s_k}$*, where* $z_i \neq z_j$*,* $\forall i \neq j$*. Considering for convenience* $0^0 = 1$*, the direct formula for calculating the terms of sequence* a *is*

$$a_n = I^{[a]}_m \cdot ((B^{[a]})^T)^{-1} \cdot (\beta^{[a]}_n)^T, \ \forall n \in \mathbb{N},$$

where $\beta^{[a]}_n = \left(n^j z_k^{-n}\right)_{k=\overline{0,p-1},\ j=\overline{0,s_k-1}}$*,* $\forall n \in \mathbb{N}$*, and* $B^{[a]} = (\beta^{[a]}_i)_{i=0}^{m-1}$.

Another important result from [4] is the fact that the linear combination and product are algebraic operations over $Rol[K]$. More exactly, the next theorems hold.

Theorem 3. *Let* $a^{(j)} \in Rol[K]$*,* $P_j(z) \in H[K](a^{(j)})$ *and* $\alpha_j \in \mathbb{C}$*,* $j = \overline{1,t}$*. Then* $a = \sum_{k=1}^{t} \alpha_k a^{(k)} \in Rol[K]$ *and* $P(z) = lcm(P_1(z), P_2(z), \ldots, P_t(z)) \in H[K](a)$*.*

Theorem 4. *Consider that* $a \in Rol[K][m]$*,* $b \in Rol[K][1]$*,* $(q_0) \in G[K][1](b)$ *and* $P(z) \in H[K][m](a)$*. Then,* $ab = (a_n b_n)_{n=0}^{\infty} \in Rol[K][m]$ *and* $P(q_0 z) \in H[K](ab)$*.*

Theorem 5. *Consider* $a \in Rol[\mathbb{C}][m_1]$*,* $b \in Rol[\mathbb{C}][m_2]$*,* $u \in G[\mathbb{C}][m_1](a)$ *and* $v \in G[\mathbb{C}][m_2](b)$*. Let* $z_0, z_1, \ldots, z_{p-1}$ *be all distinct complex roots, of multiplicity* $s_0, s_1, \ldots, s_{p-1}$ *correspondingly, of the polynomial* $H^{[u]}_{m_1}(z)$*;* $z^*_0, z^*_1, \ldots, z^*_{p^*-1}$ *be all distinct complex roots, of multiplicity* $s^*_0, s^*_1, \ldots, s^*_{p^*-1}$ *correspondingly, of the polynomial* $H^{[v]}_{m_2}(z)$*. Then,* $ab \in Rol[\mathbb{C}]$ *and*

$$P(z) = lcm(\{(z - z_k z^*_r)^{s_k + s^*_r - 1} | k = \overline{0,p-1}, r = \overline{0,p^*-1}\}) \in H[\mathbb{C}](ab).$$

2.2 Minimization Methods

The non-zero sequence (with at least one non-zero element) $a \in Rol[K]$ is called m-minimal over K if $a \in Rol[K][m]$ and $a \notin Rol[K][t]$, $\forall t < m$. In this case, the number m represents the dimension of the sequence a over K and it is denoted $dim[K](a) = m$. The dimension of the zero sequence is considered 0.

It is obvious that $dim[K](a) \leq m$, $\forall a \in Rol[K][m]$. Also, if $K_1 \subseteq K_2$ and $a \in Rol[K_1]$, then $a \in Rol[K_2]$ and $dim[K_2](a) \leq dim[K_1](a)$.

According to Theorem 3, if $a^{(k)} \in Rol[K]$ and $\alpha_k \in \mathbb{C}$, $k = \overline{1,t}$, then

$$dim[K]\left(\sum_{k=1}^{t} \alpha_k a^{(k)}\right) \leq \sum_{k=1}^{t} dim[K](a^{(k)}).$$

Additionally, from Theorem 5, for $\forall a^{(k)} \in Rol[\mathbb{C}]$, $k = \overline{1,t}$, we have the inequality

$$dim[\mathbb{C}]\left(\prod_{k=1}^{t} a^{(k)}\right) \leq \prod_{k=1}^{t} dim[\mathbb{C}](a^{(k)}).$$

It is known from [4] that the minimal generating vector is unique, i.e.

$$|G[K][dim[K](a)](a)| = 1.$$

This unique minimal generating vector determines the unique minimal unit characteristic polynomial $P(z) \in H[K][dim[K](a)](a)$. We may omit the word "unit" and consider $P(z)$ as the minimal characteristic polynomial of a. This polynomial allows us to describe the set of all characteristic polynomials in the following way:

$$H[K](a) = \{Q(z) \in K[z] \mid Q(z) \vdots P(z),\ Q(0) \neq 0\}.$$

The minimization problem consists in finding the dimension of the non-zero sequence a and its minimal generating vector over K. According to [4], there are two minimization methods over $\mathbb{C}$: the minimization method based on matrix rank definition and the minimization method by elimination of characteristic zeros.

Theorem 6. *If $a \in Rol[\mathbb{C}][m]$, then $dim[\mathbb{C}](a) = R = rank(A_m^{[a]})$ and the minimal generating vector is $q = (q_0, q_1, \ldots, q_{R-1}) \in G[\mathbb{C}][R](a)$, where the reverse vector $x = (q_{R-1}, q_{R-2}, \ldots, q_0)$ is the unique solution of the system with linear equations $A_R^{[a]} x^T = (f_R^{[a]})^T$ with free terms $f_R^{[a]} = (a_R, a_{R+1}, \ldots, a_{2R-1})$ and the system matrix $A_R^{[a]} = (a_{i+j})_{i,j=\overline{0,R-1}}$.*

Theorem 7. *Let $a \in Rol[\mathbb{C}][m]$, $x = I_m^{[a]}((B^{[a]})^T)^{-1} = (A_{k,j})_{k=\overline{0,p-1},\ j=\overline{0,s_k-1}}$, t_k be the number of zeros from the end of $(A_{k,j})_{j=\overline{0,s_k-1}}$, $k = \overline{0,p-1}$ and $t = \sum_{k=0}^{p-1} t_k$. Then $dim[\mathbb{C}](a) = m - t$ and $Q(z) = \dfrac{P(z)}{\prod_{k=0}^{p-1}(z-z_k)^{t_k}} \in H[\mathbb{C}][m-t](a)$, where z_k, $k = \overline{0,p-1}$, are all distinct roots of the polynomial $P(z) \in H[\mathbb{C}][m](a)$.*

These methods also can be used for minimization over a subset K of $\mathbb{C}$. Having determined the minimal characteristic polynomial over $\mathbb{C}$, the second step is to find a multiple of minimal degree for it, through the divisors of characteristic polynomial over K, which has the free term -1 and the rest of coefficients belonging to K.

The minimization method based on matrix rank definition is more applicable than the minimization method by elimination of characteristic zeros, because it does not suppose to know the complex roots of the characteristic polynomial.

3 Asymptotic Behavior of Homogeneous Linear Recurrences

In this section, the asymptotic behavior of homogeneous linear recurrences is studied. The convergence criteria and the efficient formula for calculating the limit are given. The Jury Stability Criterion is proposed as alternative, for the case when the characteristic roots are not known.

3.1 Convergence Criterion Based on Characteristic Zeros

According to [4], the convergence criterion is given by the following theorem. Practically, the classification of the roots of minimal characteristic polynomial gives us the information about the asymptotic behavior of given homogeneous linear recurrent process.

Theorem 8. *Consider* $a \in Rol[\mathbb{C}][m]$ *a non-zero sequence with* $dim[\mathbb{C}](a) = m$ *and* $P(z) \in H[\mathbb{C}][m](a)$. *Let* $z_0, z_1, \ldots, z_{p-1}$ *be all distinct roots of the polynomial* $P(z)$, *of corresponding multiplicity* $s_0, s_1, \ldots, s_{p-1}$. *The sequence* a *is convergent if and only if* $|z_k| > 1$ *or* ($z_k = 1$ *and* $s_k = 1$), $k = \overline{0, p-1}$.

In other words, the minimal characteristic polynomial of the convergent sequence $a \in Rol[\mathbb{C}][m]$ has at most one simple root equal to 1. The rest of the roots lie outside of the unit disc.

Moreover, if a is convergent, the limit can be easily calculated. We have $\lim\limits_{n\to\infty} a_n = 0$ in the case when $P(1) \neq 0$, and $\lim\limits_{n\to\infty} a_n = (I_m^{[a]}((B^{[a]})^T)^{-1})_{t_0}$ in the case when $P(1) = 0$. Next, according to minimization method by elimination of characteristic zeros, we have $\lim\limits_{n\to\infty} a_n \neq 0$ when $P(1) = 0$. In this situation, to avoid the need for knowing the roots of minimal characteristic polynomial, the sequence a is transformed into a linear $(m-1)$ – recurrence with a constant inhomogeneity.

Theorem 9. *Let*

$$a \in Rol[\mathbb{C}][m],\ P(z) = H_m^{[p]}(z) \in H[\mathbb{C}][m](a),\ P(1) = 0,$$

where $m = dim[\mathbb{C}](a) \geq 2$. *Then, the sequence* a *is a linear* $(m-1)$ – *recurrence over* $\mathbb{C}$, *generated by vector* $q = (q_0, q_1, \ldots, q_{m-2})$ *and inhomogeneity*

$$r_{m-1} = a_{m-1} - \sum_{k=0}^{m-2} q_k a_{m-2-k},$$

where

$$q_k = \sum_{j=0}^{k} p_j - 1,\ k = \overline{0, m-2}.$$

If, additionally, a *is convergent, then*

$$\lim_{n\to\infty} a_n = \frac{r_{m-1}}{1 - \sum\limits_{k=0}^{m-2} q_k} \neq 0.$$

If there is at least one root, of the minimal characteristic polynomial, which lies inside of the unit disc, then a diverges to infinity. The same thing happens when there is at least one multiple root on the unit circle. Instead, if all the roots are simple roots of unity, then a is periodic. When all the roots are simple roots of unity or lie outside of the unit disc, then a is bounded.

3.2 Jury Stability Criterion

Let $a \in Rol[\mathbb{C}][m]$ with minimal characteristic polynomial

$$P(z) = H_m^{[p]}(z) \in H[\mathbb{C}][m](a).$$

The Jury Stability Criterion, described in [1] and [2], can be applied for studying the localization of the roots of reciprocal polynomial $P^*(z)$ of $P(z)$ in relation to unit circle, without finding the roots. Basically, the calculations are organized as a table, where

- the columns correspond to monomials of $P^*(z)$, ordered in descending order by exponent;
- the first row contains the coefficients of $P^*(z)$;
- each further even row $2k+2$ contains the numbers from previous row in reverse order;
- each further odd row $2k+3$ is calculated by subtracting α times the previous even row from the previous odd row, where $\alpha = \beta_{2k+2}/\beta_{2k+1}$, β_{2k+2} is the first element from previous even row $2k+2$ and β_{2k+1} is the first element from previous odd row $2k+1$;
- the table is expanded until the last row of the table contains only one non-zero element.

Since $\beta_1 = 1 > 0$, then for every negative value from the sequence $\beta_1, \beta_3, \beta_5, \ldots$ the polynomial $P^*(z)$ has one root outside of the unit disc, i.e. the polynomial $P(z)$ has one root inside the unit disc. So, for stability, it is needed all these values $\beta_1, \beta_3, \beta_5, \ldots$ to be non-negative.

A particular additional result, which is involved from [1], is the fact that we need to have at least $P(1) > 0$, $P(-1) > 0$ and $|p_{m-1}| < 1$ in order all the roots of $P(z)$ lie outside of unit disc. For instance, based on [3], this does not happen when $P(z) \in \mathbb{Z}[z]$. Instead, the homogeneous linear recurrent distributions satisfy this property.

4 Homogeneous Linear Recurrent Distributions

Let consider a nonnegative integer random variable ξ with probabilistic distribution $rep(\xi) = a = (a_n)_{n=0}^{\infty}$. This means that a_n represents the probability that random variable ξ has the value n, for each $n = 0, 1, 2, \ldots$, i.e. $a_n = \mathbb{P}(\xi = n)$, $n = \overline{0, \infty}$.

According to [4], the main probabilistic characteristics of random variable ξ are: the expectation $\mathbb{E}(\xi)$, the moments $\nu_n(\xi) = \mathbb{E}(\xi^n)$ $(n = \overline{1, \infty})$, the variance $\mathbb{V}(\xi) = \nu_2(\xi) - \nu_1^2(\xi)$ and the standard deviation $\sigma(\xi) = \sqrt{\mathbb{V}(\xi)}$. Two additional probabilistic characteristics, that are useful for solving various stochastic problems, are the mode μ, for which $a_\mu = \max_{n\in\mathbb{N}} a_n$, and the median m_0, that satisfies the double inequality $\mathbb{P}(\xi < m_0) < \frac{1}{2} \le P(\xi \le m_0)$, equivalent with $\sum_{k=0}^{m_0-1} a_k < \frac{1}{2} \le \sum_{k=0}^{m_0} a_k$.

Both, the mode μ and the median m_0, can be found by successive search algorithm, i.e. by checking consecutively the values $a_0,\ a_1,\ a_2,\ \ldots$, until the median is found or the maximum number of iterations for finding the mode is reached.

For finding the median, the maximum number of iterations of the successive search algorithm is $N(\xi) = [\mathbb{E}(\xi) + \sigma(\xi)\sqrt{2}]$. The algorithm starts with setting $\psi_0 = a_0$ and continues with calculation of the value $\psi_n = \psi_{n-1} + a_n$ at each step $n = 1, 2, \ldots$, until the inequality $\psi_n \ge \frac{1}{2}$ becomes true.

Similarly, for finding the mode, the maximum number of iterations of the successive search algorithm is $n_s(\xi) = \left[\mathbb{E}(\xi) + \frac{\sigma(\xi)}{\sqrt{a_s}}\right]$, where s is the smallest index for which $a_s > 0$. The mode μ is that index which satisfies the equality $a_\mu = \max_{s\le n\le n_s(\xi)} a_n$.

We can easily note that the successive search algorithm for finding the mode of the random variable ξ depends on the main probabilistic characteristics $\mathbb{E}(\xi)$ and $\sigma(\xi)$. In general case, these values can be obtained from generating function $G_\xi(z) = G^{[a]}(z)$ using the formulas:

$$\mathbb{E}(\xi) = G'_\xi(1),\ \mathbb{V}(\xi) = G''_\xi(1) + G'_\xi(1) - (G'_\xi(1))^2,\ \sigma(\xi) = \sqrt{\mathbb{V}(\xi)}.$$

Next, we consider the homogeneous linear recurrent distributions, i.e. the case when $a = rep(\xi) \in Rol[\mathbb{C}]$. It is known that $a \in Rol[\mathbb{R}]$ and $dim[\mathbb{R}][a] = dim[\mathbb{C}][a]$. Moreover, since distributions are convergent to 0, the minimal characteristic polynomial does not have the root $z = 1$. In this case, the moments can be found in an easier way, using the following theorem from [4]:

Theorem 10. *Let ξ be a random variable with distribution $a = rep(\xi) \in Rol[\mathbb{R}][m]$ and generating vector $q \in G[\mathbb{R}][m](a)$. Then $c^{(k)} = (n^k a_n)_{n=0}^\infty \in Rol[\mathbb{R}][M_k]$, $q^{(k)} \in G[\mathbb{R}][M_k](c^{(k)})$ and*

$$\nu_k(\xi) = G^{[c^{(k)}]}(1),\ \forall k \ge 1,$$

where $M_k = m(k+1)$ and

$$H_{M_k}^{[q^{(k)}]}(z) = (H_m^{[q]}(z))^{k+1} \in H[\mathbb{R}][M_k](c^{(k)}).$$

In consequence, $\mathbb{E}(\xi)$ and $\sigma(\xi)$ can be calculated too, using the relations

$$\mathbb{E}(\xi) = \nu_1(\xi),\ \mathbb{V}(\xi) = \nu_2(\xi) - \nu_1^2(\xi),\ \sigma(\xi) = \sqrt{\mathbb{V}(\xi)}.$$

5 Perturbations and Their Asymptotic Behavior

We consider the homogeneous linear recurrence $a \in Rol[\mathbb{R}][m]$ with initial state $I_m^{[a]} = (a_n)_{n=0}^{m-1}$, generating vector $q \in G[\mathbb{R}][m](a)$ and the corresponding characteristic polynomial $H_m^{[q]}(z) \in H[\mathbb{R}][m](a)$. Perturbations are defined as deviations in the evolution of a, caused by small deviations in the parameters, i.e. deviations of initial state elements and deviations of generating vector components.

5.1 Perturbations Generated by Deviations in Initial State

Initially, we consider only deviations in initial state $I_m^{[a]}$ of the homogeneous linear recurrence a, without any change in generating vector q. In this case, the perturbed recurrence represents a new homogeneous linear recurrence $b \in Rol[\mathbb{R}][m]$ with initial state $I_m^{[b]} = (b_n)_{n=0}^{m-1}$ and the same generating vector $q \in G[\mathbb{R}][m](b)$, where

$$b_n = a_n + \Delta_n,\ n = \overline{0, m-1}.$$

The perturbation is given by the sequence $\epsilon = (\epsilon_n)_{n=0}^{\infty}$, where $\epsilon_n = b_n - a_n$, $n = \overline{0, \infty}$. We have $\epsilon_n = \Delta_n$, $n = \overline{0, m-1}$. Also, applying Theorem 3, we obtain $\epsilon \in Rol[\mathbb{R}][m]$ and $q \in G[\mathbb{R}][m](\epsilon)$. So,

$$\epsilon \in Rol[\mathbb{R}][m],\ q \in G[\mathbb{R}][m](\epsilon),\ I_m^{[\epsilon]} = (\Delta_n)_{n=0}^{m-1}.$$

The perturbation $\epsilon = (\epsilon_n)_{n=0}^{\infty}$ is considered asymptotically stable if and only if $\lim_{n\to\infty} \epsilon_n = 0$. The convergence of ϵ can be studied according to Section 3.

As a remark, the asymptotical stability of perturbation ϵ does not depend on deviation in initial state. Since the components of generating vector q are not changed, the characteristic roots are not changed too. This means that the asymptotic behavior of the perturbed recurrence is exactly the same as asymptotic behavior of the original recurrence.

The maximal impact of the perturbation ϵ is represented by the positive value $\epsilon^* = \max_{n=\overline{0,\infty}} |\epsilon_n|$. Even if ϵ is asymptotically stable, it might have a big enough maximal perturbation impact.

In order to study the maximal impact of the asymptotically stable perturbation ϵ, we can consider the sequence $\epsilon^2 = (\epsilon_n^2)_{n=0}^{\infty}$. Since $\epsilon^2 = \epsilon \cdot \epsilon$ and $\epsilon \in Rol[\mathbb{R}][m]$, we have

$$\epsilon^2 \in Rol[\mathbb{R}],\ dim[\mathbb{R}][\epsilon^2] \le (dim[\mathbb{R}][\epsilon])^2 \le m^2.$$

In consequence, $\epsilon^2 \in Rol[\mathbb{R}][m^2]$ and its minimal generating vector can be obtained using the minimization method based on matrix rank definition.

Next, using Theorem 1, the value $s = \sum_{n=0}^{\infty} \epsilon_n^2 = G^{[\epsilon^2]}(1)$ can be calculated. If ξ is a random variable with distribution $p = \epsilon^2/s$, then its mode μ and its probability p_μ can be found using the successive search algorithm, described in Section 4. In the end, we obtain the maximal perturbation impact $\epsilon^* = \sqrt{sp_\mu}$.

5.2 Perturbations Generated by Deviations in Generating Vector

Now, we consider only deviations in generating vector q of the homogeneous linear recurrence a, without any change in initial state $I_m^{[a]}$. In this case, the perturbed recurrence represents a new homogeneous linear recurrence $b \in Rol[\mathbb{R}][m]$ with initial state $I_m^{[b]} = I_m^{[a]}$ and the generating vector $r = (r_n)_{n=0}^{\infty} \in G[\mathbb{R}][m](b)$, where

$$r_n = q_n + \delta_n,\ n = \overline{0, m-1}.$$

The perturbation is given by the sequence $\varepsilon = (\varepsilon_n)_{n=0}^{\infty}$, where $\varepsilon_n = b_n - a_n$, $n = \overline{0, \infty}$. Applying Theorem 3, we obtain

$$\varepsilon \in Rol[\mathbb{R}],\ dim[\mathbb{R}](\varepsilon) \leq dim[\mathbb{R}](a) + dim[\mathbb{R}](b) \leq m + m = 2m.$$

In consequence, $\varepsilon \in Rol[\mathbb{R}][2m]$ and its minimal generating vector can be obtained using the minimization method based on matrix rank definition.

The perturbation $\varepsilon = (\varepsilon_n)_{n=0}^{\infty}$ is considered asymptotically stable if and only if $\lim_{n \to \infty} \varepsilon_n = 0$. The convergence of ε can be studied according to Section 3.

As a remark, the perturbation ε can be asymptotically stable even if the initial recurrence a is not convergent. This happens when $dim[\mathbb{R}](\varepsilon) < dim[\mathbb{R}](a)$ and all roots of the minimal characteristic polynomial of a over $\mathbb{R}$ which are not greater than 1 in absolute value disappear from the list of all roots of the minimal characteristic polynomial of ε over $\mathbb{R}$.

Similarly to Section 5.1, in order to study the maximal impact $\varepsilon^* = \max_{n=\overline{0,\infty}} |\varepsilon_n|$ of the asymptotically stable perturbation ε, we can consider the sequence ε^2, obtaining $\varepsilon^2 \in Rol[\mathbb{R}][4m^2]$. Its minimal generating vector can be obtained using the minimization method based on matrix rank definition too.

5.3 Mixed Perturbations

Mixed perturbations are generated by both types of deviations: deviations in initial state $I_m^{[a]}$ and deviations in generating vector q of the homogeneous linear recurrence a. The perturbed recurrence represents a new homogeneous linear recurrence $c \in Rol[\mathbb{R}][m]$ with initial state $I_m^{[c]}$ and the generating vector $r = (r_n)_{n=0}^{\infty} \in G[\mathbb{R}][m](c)$, where

$$c_n = a_n + \Delta_n,\ r_n = q_n + \delta_n,\ n = \overline{0, m-1}.$$

The perturbation is given by the sequence $\varepsilon = (\varepsilon_n)_{n=0}^{\infty}$, where

$$\varepsilon_n = c_n - a_n,\ n = \overline{0, \infty}.$$

We can study mixed perturbations using results from Section 5.1 and Section 5.2. The deviation in initial state and the deviation in generating vector can be performed consecutively, one by one, in the following way.

Let $b \in Rol[\mathbb{R}][m]$ be the perturbed recurrence, generated by deviation in initial state, i.e.

$$q = (q_n)_{n=0}^{m-1} \in G[\mathbb{R}][m](b),$$
$$b_n = a_n + \Delta_n,\ n = \overline{0, m-1}.$$

Its perturbation is represented by the sequence $\epsilon = (\epsilon_n)_{n=0}^{\infty}$, where

$$\epsilon_n = b_n - a_n,\ n = \overline{0, \infty}.$$

Next, the perturbed recurrence $c \in Rol[\mathbb{R}][m]$ is obtained from $b \in Rol[\mathbb{R}][m]$ by applying the given deviation in generating vector $q = (q_n)_{n=0}^{\infty} \in G[\mathbb{R}][m](b)$:

$$c_n = b_n,\ r_n = q_n + \delta_n,\ n = \overline{0, m-1}.$$

The corresponding perturbation is represented by the sequence $\zeta = (\zeta_n)_{n=0}^{\infty}$, where

$$\zeta_n = c_n - b_n,\ n = \overline{0, \infty}.$$

The mixed perturbation $\varepsilon = (\varepsilon_n)_{n=0}^{\infty}$ represents the sum of these two perturbations from decomposition:

$$\varepsilon_n = c_n - a_n = (c_n - b_n) + (b_n - a_n) = \zeta_n + \epsilon_n,\ n = \overline{0, \infty}.$$

So, based on Theorem 3, it is also a homogeneous linear recurrence. As consequence, the asymptotic behavior and the maximal perturbation impact can be studied similarly.

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