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Morita Contexts, Preradicals and Closure Operators in Modules

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Abstract. The preradicals and closure operators in module categories are studied. The concordance is shown between the mappings connecting the classes of preradicals and of closure operators of two module categories R-Mod and S-Mod in the case of a Morita context $(R,\ {}_RU_S,\ {}_SV_R, S)$, using the functors $Hom_R(U,\text{-})$ and $Hom_S(V,\text{-})$.

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1 Introduction. Preliminary definitions and constructions.

In this article the preradicals and the closure operators of module categories are studied in the case of a Morita context $(R,\ {}_RU_S,\ {}_SV_R, S)$.

The associated functors $R\text{-Mod} \underset{H^V=Hom_S(V,\text{-})}{\overset{H^U=Hom_R(U,\text{-})}{\rightleftarrows}} S\text{-Mod}$ define two pairs of the mappings:

a) between the classes of preradicals $\mathbb{PR}(R) \underset{(-)^*}{\overset{(-)^*}{\rightleftarrows}} \mathbb{PR}(S)$ and

b) between the classes of closure operators $\mathbb{CO}(R) \underset{(-)^*}{\overset{(-)^*}{\rightleftarrows}} \mathbb{CO}(S)$ of the categories R-Mod and S-Mod ([6]).

Moreover, in R-Mod we have the mappings:

$$\begin{tikzcd}
\mathbb{CO}(R) \arrow[r, "\Phi^R"] & \mathbb{PR}(R) \arrow[l, bend right, "\Psi_1^R"'] \arrow[l, bend left, "\Psi_1^S"]
\end{tikzcd},$$

as well as the similar mappings Φ^S, Ψ_1^S and Ψ_2^S for S-Mod [5].

The purpose of this work is the elucidation of the concordance (compatibility) between the investigated ten mappings. For that six cases are analyzed $\big($by two for every pair (Φ^R, Φ^S), (Ψ_1^R, Ψ_1^S) and $(\Psi_2^R, \Psi_2^S)\big)$, when we have the combinations from four mappings, which form the square diagrams. The required concordance is understood in the sense of commutativity of the respective diagrams.

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Now we recall the main necessary notions and facts. A *preradical* r of the category R-Mod of left R-modules is a subfunctor of identity functor of R-Mod, i.e. r is a function which separates in every module $M \in R$-Mod a submodule $r(M) \subseteq M$ such that $[r(M)]f \subseteq r(M')$ ([1]) (the R-morphisms are written on the right). Let $\mathbb{PR}(R)$ be the class of all preradicals of R-Mod. The partial order in $\mathbb{PR}(R)$ is defined by the rule: $r \leq s \Leftrightarrow r(M) \subseteq s(M)$ for every $M \in R$-Mod. Denote by $\mathbb{L}(M)$ the lattice of submodules of $M \in R$-Mod.

A preradical $r \in \mathbb{PR}(R)$ is *hereditary* (or: r is a *pretorsion*) if $r(N) = r(M) \cap N$ for every $N \in \mathbb{L}(M)$.

Let $r \in \mathbb{PR}(R)$ and $f : M \to M'$ be an arbitrary R-morphism. Then by the definition f implies the morphisms $r(f)$ and $(1/r)(f)$ such that the following diagram is commutative:

$$\begin{tikzcd}
0 \arrow[r] & r(M) \arrow[r, "i^r_M", "\subseteq"'] \arrow[d, dashed, "r(f)"] & M \arrow[r, "\pi^r_M", "\text{nat}"'] \arrow[d, "f"] & M/r(M) \arrow[r] \arrow[d, dashed, "(1/r)(f)"] & 0 \\
0 \arrow[r] & r(M') \arrow[r, "i^r_{M'}", "\subseteq"'] & M' \arrow[r, "\pi^r_{M'}", "\text{nat}"'] & M'/r(M') \arrow[r] & 0,
\end{tikzcd}$$

where $r(f)$ is a restriction of f and $\big(m + r(M)\big)\big[(1/r)(f)\big] \overset{\text{def}}{=\!=} mf + r(M')$ for every $m + r(M) \in M/r(M)$.

If f is a monomorphism, then is obvious that $r(f)$ is a monomorphism. It is useful to show when $(1/r)(f)$ is a monomorphism.

We remark that if $f = i : M \xrightarrow{\subseteq} M'$ is an *inclusion* and r is a *pretorsion*, then $(1/r)(i)$ is a monomorphism : if $m + r(M') = \overline{0}$, then $m \in r(M') \cap M = r(M)$, i.e. $m + r(M) = \overline{0}$. This fact can be generalized for every monomorphism.

Lemma 1.1. *If* $f : M \to M'$ *is a monomorphism and* $r \in \mathbb{PR}(R)$ *is a pretorsion, then* $(1/r)(l)$ *is a monomorphism.*

Proof. We represent $f : M \to M'$ in the form $M \xrightarrow[\cong]{\overline{f}} Im\,f \xrightarrow[\subseteq]{i} M'$ and by heredity of r we have: $r(Im\,f) = Imf \cap r(M')$, therefore $Im\,f/r(Im\,f) = Im\,f/\big(Im\,f \cap r(M')\big) \cong \big(Im\,f + r(M')\big)/r(M') \subseteq M'/r(M')$. Since $(1/r)(i)$ is a monomorphism, the composition

$$(1/r)(f)\colon \quad M/r(M) \xrightarrow[\cong]{(1/r)(\overline{f})} Im\,f/r(Im\,f) \xrightarrow{(1/r)(i)} M'/r(M')$$

also is a monomorphism.

Lemma 1.1 is proved. ☐

A *closure operator* of R-Mod is a function C which associates to every pair $N \subseteq M$, where $N \in \mathbb{L}(M)$, a submodule $C_M(N) \subseteq M$ such that:

(c_1) $N \subseteq C_M(N)$ (*extension*);

(c_2) $N_1, N_2 \in \mathbb{L}(M),\ N_1 \subseteq N_2 \Rightarrow C_M(N_1) \subseteq C_M(N_2)$ (*monotony*);

(c_3) $\big[C_M(N)\big]f \subseteq C_{M'}(Nf)$ for every monomorphism $f : M \to M'$ and $N \subseteq M$ (*continuity*) ([3–5]).

Let $\mathbb{CO}(R)$ be the class of all closure operators of R-Mod. The partial order in $\mathbb{CO}(R)$ is defined by the rule: $C \leq D \Leftrightarrow C_M(N) \subseteq D_M(N)$ for every pair $N \subseteq M$ of R-Mod.

The classes $\mathbb{PR}(R)$ and $\mathbb{CO}(R)$ of preradicals and of closure operators of R-Mod are related by the following three mappings:

$$\begin{tikzcd}
\mathbb{CO}(R) \arrow[r, "\Phi^R"] & \mathbb{PR}(R)\,, \arrow[l, bend right, "\Psi_1^R"'] \arrow[l, bend left, "\Psi_2^R"]
\end{tikzcd}$$

which are defined as follows [5].

a) $\Phi^R : \mathbb{CO}(R) \longrightarrow \mathbb{PR}(R)$. For every $C \in \mathbb{CO}(R)$ we denote $\Phi^R(C) = r_C$ and define

$$r_C(X) \overset{\text{def}}{=\!=} C_X(0) \tag{1.1}$$

for every $X \in R$-Mod.

b) $\Psi_1^R : \mathbb{PR}(R) \longrightarrow \mathbb{CO}(R)$. For $r \in \mathbb{PR}(R)$ we denote $\Psi_1^R(r) = C^r$ and define $C_X^r(M)$ by the relation

$$C_X^r(M)/M \overset{\text{def}}{=\!=} r(X/M) \tag{1.2}$$

for every $M \subseteq X$ of R-Mod.

c) $\Psi_2^R : \mathbb{PR}(R) \longrightarrow \mathbb{CO}(R)$. If $r \in \mathbb{PR}(R)$, then we denote $\Psi_2^R(r) = C_r$ and define:

$$(C_r)_X(M) \overset{\text{def}}{=\!=} r(X) + M \tag{1.3}$$

for every $M \subseteq X$ of R-Mod.

For the category S-Mod in a completely similar manner the mappings are defined:

$$\begin{tikzcd}
\mathbb{CO}(S) \arrow[r, "\Phi^S"] & \mathbb{PR}(S)\,. \arrow[l, bend right, "\Psi_1^S"'] \arrow[l, bend left, "\Psi_2^S"]
\end{tikzcd}$$

In continuation we consider that an arbitrary *Morita context* $(R,\ {}_RU_S,\ {}_SV_R, S)$ is given with the associated bimodular morphisms

$$(,) :\ U \otimes_S V \longrightarrow R, \qquad [,] :\ V \otimes_R U \longrightarrow S,$$

where $(u,v)u' = u[v,u']$ and $[v,u]v' = v(u,v')$ for every $u, u' \in U,\ v, v' \in V$ ([2]). In such situation we will study the functors

$$\begin{tikzcd}
R\text{-Mod} \arrow[r, shift left, "H^U = Hom_R\,(U,\text{-})"] & S\text{-Mod} \arrow[l, shift left, "H^V = Hom_S\,(V,\text{-})"]
\end{tikzcd}$$

with the natural transformations

$$\varphi \,:\; \mathbb{1}_{R\text{–Mod}} \longrightarrow H^V H^U, \qquad \psi \,:\; \mathbb{1}_{S\text{–Mod}} \longrightarrow H^U H^V,$$

which are defined as follows.

For $X \in R$-Mod the R-morphism $\varphi_X :\; X \longrightarrow H^V H^U(X)$ acts by the rule:

$$u\left(v(x\,\varphi_X)\right) \overset{\text{def}}{=\!=} (u,v)\,x,$$

where $x \in X$, $v \in V$, $u \in U$. Similarly, for $Y \in S$-Mod the S-morphism $\psi_Y : Y \xrightarrow{H\ \ ^{U}} H^V(Y)$ is defined by the rule:

$$v\left(u(y\,\psi_Y\right) \overset{\text{def}}{=\!=} [v,u]\,y,$$

where $y \in Y$, $u \in U$, $v \in V$. Moreover, the transformations φ and ψ are in concordance with the functors H^U and H^V in the sense of relations:

$$H^U(\varphi_X) \;=\; \psi_{H^U(X)}\,, \quad H^V(\psi_Y) \;=\; \varphi_{H^V(Y)} \tag{1.4}$$

for every $X \in R$-Mod and $Y \in S$-Mod.

In the studied situation two mappings can be defined between the *classes of preradicals* of R-Mod and S-Mod:

$$\begin{tikzcd} \mathbb{PR}(R) \arrow[r, shift left, "(-)^*"] & \mathbb{PR}(S) \arrow[l, shift left, "(-)^*"] \end{tikzcd}.$$

For $r \in \mathbb{PR}(R)$ the function r^* in S-Mod is defined as follows. If $Y \in S$-Mod we have in R-Mod the sequence:

$$0 \to r\big(H^V(Y)\big) \xrightarrow[\subseteq]{i^r_{H^V(Y)}} H^V(Y) \xrightarrow[\text{nat}]{\pi^r_{H^V(Y)}} H^V(Y)/r\big(H^V(Y)\big) \to 0\,.$$

Applying H^U and using ψ, we obtain in S-Mod the situation:

$$\begin{tikzcd} & & Y \arrow[d, "\psi_Y"] \arrow[dr, dashed] & \\ 0 \arrow[r] & H^U\Big(r\big(H^V(Y)\big)\Big) \arrow[r, "H^U\left(i^r_{H^V(Y)}\right)"] & H^U H^V(Y) \arrow[r, "H^U\left(\pi^r_{H^V(Y)}\right)"] & H^U\big[H^V(Y)/r\big(H^V(Y)\big)\big], \end{tikzcd}$$

where $Im\, H^U(i^r_{H^V(Y)}) = \operatorname{Ker} H^U(\pi^r_{H^V(Y)})$.

The preradical r^* of S-Mod is defined by the rule:

$$r^*(Y) \overset{\text{def}}{=\!=} \operatorname{Ker}\,[\psi_Y \cdot H^U(\pi^r_{H^V(Y)})] \tag{1.5}$$

$$\big(\text{or:}\;\; r^*(Y) \overset{\text{def}}{=\!=} [Im\, H^U(i^r_{H^V(Y)})]\,\psi_Y^{-1}\big).$$

The inverse mapping $s \rightsquigarrow s^*$ for $s \in \mathbb{PR}(S)$ is defined similarly. Namely, for $X \in R$-Mod we have the diagram:

$$\begin{tikzcd}
 & & H^VH^U(X) \arrow[from=1-3, d, phantom] & \\
\end{tikzcd}$$

$$\begin{tikzcd}
 & & X \arrow[d, "\varphi_X"] \arrow[dr, dashed] & \\
0 \arrow[r] & H^V\Big(s\big(H^U(X)\big)\Big) \arrow[r, "H^V\big(i^s_{H^U(X)}\big)"] & H^VH^U(X) \arrow[r, "H^V\big(\pi^s_{H^U(X)}\big)"] & H^V\big[H^U(X)/s\big(H^U(X)\big)\big],
\end{tikzcd}$$

where $i^s_{H^U(X)}$ is the inclusion, $\pi^s_{H^U(X)}$ is the natural epimorphism and $Im\, H^V\big(i^s_{H^U(X)}\big) = \operatorname{Ker} H^V\big(\pi^s_{H^U(X)}\big)$. Then by the definition:

$$s^*(X) \overset{\text{def}}{=\!=} \operatorname{Ker}\,[\varphi_X \cdot H^V(\pi^s_{H^U(X)})] \tag{1.6}$$

$$\big(\text{or: } s^*(X) \overset{\text{def}}{=\!=} [Im\, H^V(i^s_{H^U(X)})]\varphi_X^{-1}\big).$$

Another important for us construction in the studied situation is the pair of mappings

$$\begin{tikzcd}
\mathbb{CO}(R) \arrow[r, shift left, "(-)^*"] & \mathbb{CO}(S) \arrow[l, shift left, "(-)^*"]
\end{tikzcd}$$

between the classes of *closure operators* of the categories R-Mod and S-Mod ([6]).

They are defined as follows. If $C \in \mathbb{CO}(R)$ and $n:\ N \xrightarrow{\subseteq} Y$ is an inclusion of S-Mod, then by H^V we have in R-Mod the morphism $H^V(n):\ H^V(n) \longrightarrow H^V(Y)$ and its decomposition with respect to the closure operator C:

$$\begin{tikzcd}
H^V(N) \arrow[r, "\overline{H^V(n)}", "\cong"'] \arrow[rrr, bend left, "H^V(n)"] & Im\, H^V(n) \arrow[r, "j^n_C", "\subseteq"'] & C_{H^V(Y)}\big(Im\, H^V(n)\big) \arrow[r, "i^n_C", "\subseteq"'] & H^V(Y)\,.
\end{tikzcd}$$

We consider the exact sequence:

$$0 \to C_{H^V(Y)}\big(Im\, H^V(n)\big) \xrightarrow[\subseteq]{i^n_C} H^V(Y) \xrightarrow[\text{nat}]{\pi^n_C} H^V(Y)/C_{H^V(Y)}\big(Im\, H^V(n)\big) \to 0.$$

Using H^V and ψ, we have in S-Mod the diagram:

$$\begin{tikzcd}
 & & Y \arrow[d, "\psi_Y"] \arrow[dr, dashed] & \\
0 \arrow[r] & H^U\big[C_{H^V(Y)}\big(Im\, H^V(n)\big)\big] \arrow[r, "H^U(i^n_C)"] & H^UH^V(Y) \arrow[r, "H^U(\pi^n_C)"] & H^U\big[H^V(Y)/C_{H^V(Y)}\big(Im\, H^V(n)\big)\big]\,,
\end{tikzcd}$$

where $Im\, H^U(i^n_C) = \operatorname{Ker} H^U(\pi^n_C)$. The operator C^* of S-Mod is defined by the rule:

$$C^*_Y(N) \overset{\text{def}}{=\!=} \operatorname{Ker}\,[\psi_Y \cdot H^U(\pi^n_C)] \tag{1.7}$$

$$\big(\text{or: } \ C^*_Y(N) \overset{\text{def}}{=\!=} [Im\, H^U(i^n_C)]\, \psi_Y^{-1}\big)$$

for every $N \subseteq Y$ of S-Mod.

The inverse mapping $\mathbb{CO}(S) \xrightarrow{(-)^*} \mathbb{CO}(R)$ is defined similarly. For $D \in \mathbb{CO}(S)$ and inclusion $m:\ M \xrightarrow{\subseteq} X$ of R-Mod we have in S-Mod the morphism $H^U(m)$, its decomposition with respect to D and the sequence:

$$0 \rightarrow D_{H^U(X)}\big(Im\, H^U(m)\big) \xrightarrow[\subseteq]{i^m_D} H^U(X) \xrightarrow[\text{nat}]{\pi^m_D} H^U(X)/D_{H^U(X)}\big(Im\, H^U(m)\big) \rightarrow 0.$$

By H^U and φ now we obtain in R-Mod the situation:

$$\begin{tikzcd}
 & & X \arrow[d, "\varphi_X"] \arrow[dr, dashed] & \\
0 \arrow[r] & H^V\big[D_{H^U(X)}\big(Im\, H^U(m)\big)\big] \arrow[r, "H^V(i^m_D)"] & H^V H^U(X) \arrow[r, "H^V(\pi^m_D)"] & H^V\big[H^U(X)/D_{H^U(X)}\big(Im\, H^U(m)\big)\big],
\end{tikzcd}$$

where $Im\, H^V(i^m_D) = \operatorname{Ker} H^V(\pi^m_D)$. The closure operator D^* of R-Mod is defined by the rule:

$$D^*_X(M) \overset{\text{def}}{=\!=} \operatorname{Ker}\,[\varphi_X \cdot H^V(\pi^m_D)] \tag{1.8}$$

$$\big(\text{or: } \ D^*_X(M) \overset{\text{def}}{=\!=} [Im\, H^V(i^m_D)]\, \varphi_X^{-1}\big)$$

for every $M \subseteq X$.

Totalizing the indicated above constructions, we can now show the general situation, obtained in the case of a Morita context $(R,\ {}_RU_S,\ {}_SV_R,\ S)$ by the functors H^U and H^V. We have at all the following ten mappings:

$$\begin{tikzcd}
\mathbb{PR}(R) \arrow[r, shift left, "(-)^*"] \arrow[d, bend right, "\Psi^R_1"'] \arrow[d, bend left, "\Psi^R_2"] & \mathbb{PR}(S) \arrow[l, shift left, "(-)^*"] \arrow[d, bend right, "\Psi^S_1"'] \arrow[d, bend left, "\Psi^S_2"] \\
\mathbb{CO}(R) \arrow[u, "\Phi^R"] \arrow[r, shift left, "(-)^*"] & \mathbb{CO}(S) \arrow[u, "\Phi^S"] \arrow[l, shift left, "(-)^*"]\ ,
\end{tikzcd}$$

defined by (1.1)-(1.3) and (1.5)-(1.8). The following investigations consist in elucidation of concordance (compatibility) between these mappings. For that diverse combinations of them are studied, which contain four mappings, forming the square diagrams. The concordance is understood in the sense of commutativity of the respective diagrams.

We will study in continuation all cases related by the following pairs of mappings: (Φ^R, Φ^S), (Ψ^R_1, Ψ^S_1), (Ψ^R_2, Ψ^S_2).

2 Squares containing the mappings Φ^{R} and Φ^{S}.

We begin with the study of two cases related by the combinations of the mappings containing Φ^R and Φ^S.

1. Consider the diagram:

$$\begin{tikzcd}
\mathbb{PR}(R) \arrow[r, "(-)^*"] & \mathbb{PR}(S) \\
\mathbb{CO}(R) \arrow[u, "\Phi^R"'] \arrow[r, "(-)^*"] & \mathbb{CO}(S) \arrow[u, "\Phi^S"']
\end{tikzcd}$$

and analyze the relation between his mappings.

Theorem 2.1. *For every closure operator* $C \in \mathbb{CO}(R)$ *is true the equality* $(r_C)^* = r_{C^*}$*, i.e. the above diagram is commutative.*

Proof. a) Let $C \in \mathbb{CO}(R)$ and consider the transitions: $C \overset{\Phi^R}{\rightsquigarrow} r_C \overset{(-)^*}{\rightsquigarrow} (r_C)^*$.

The first step is defined by the rule (1.1): $r_C(X) \overset{\text{def}}{=\!=} C_X(0)$ for $X \in R$-Mod. The second step $r_C \overset{(-)^*}{\rightsquigarrow} (r_C)^*$ is defined by (1.5) for $r = r_C$:

$$(r_C)^*(Y) \overset{\text{def}}{=\!=} \text{Ker}\,[\psi_Y \cdot H^U(\pi^{r_C}_{H^V(Y)})], \tag{2.1}$$

where $Y \in S$-Mod and $\pi^{r_C}_{H^V(Y)} : H^V(Y) \to H^V(Y)/r_C\big(H^V(Y)\big)$ is a natural epimorphism.

b) For the same operator $C \in \mathbb{CO}(R)$ now we consider the way: $C \overset{(-)^*}{\rightsquigarrow} C^* \overset{\Phi^S}{\rightsquigarrow} r_{C^*}$. For the transition $C \overset{(-)^*}{\rightsquigarrow} C^*$ by the rule (1.7) we have $(C^*)_Y(N) \overset{\text{def}}{=\!=} \text{Ker}\,[\psi_Y \cdot$ $\cdot\, H^U(\pi^n_C)]$, where $n : N \xrightarrow{\subseteq} Y$ is a inclusion of S-Mod and $\pi^n_C : H^V(Y) \to$ $\to H^V(Y)/C_{H^V(Y)}\big(Im\, H^V(n)\big)$ is a natural epimorphism.

The second step $C^* \overset{\Phi^S}{\rightsquigarrow} r_{C^*}$ is defined by the rule: $r_{C^*}(Y) \overset{\text{def}}{=\!=} C^*_Y(0)$, where $Y \in S$-Mod. Now in the above construction of C^* we suppose that $N = 0$ (i.e. $n = 0 : 0 \xrightarrow{\subseteq} Y$). Then $H^V(n) = 0$, $Im\, H^V(n) = 0$ and in R-Mod we have the sequence:

$$0 \to C_{H^V(Y)}(0) \xrightarrow[\subseteq]{i^n_C} H^V(Y) \xrightarrow[\text{nat}]{\pi^n_C} H^V(Y)/C_{H^V(Y)}(0) \to 0.$$

By the definition of C^* for $N = 0$ we obtain $C^*_Y(0) \overset{\text{def}}{=\!=} \text{Ker}\,[\psi_Y \cdot$ $\cdot\, H^U(\pi^n_C)]$, therefore:

$$r_{C^*}(Y) = \text{Ker}\,[\psi_Y \cdot H^U(\pi^n_C)], \tag{2.2}$$

where $n = 0$.

c) Now we compare the expressions (2.1) and (2.2) for $n = 0$. Since $r_C\big(H^V(Y)\big) \overset{\text{def}}{=\!=} C_{H^V(Y)}(0)$, we have $H^V(Y)/r_C\big(H^V(Y)\big) = H^V(Y)/C_{H^V(Y)}(0)$.

Therefore the morphisms $\pi^{r_C}_{H^V(Y)}$ and π^n_C (for $n = 0$) coincide. Now from (2.1) and (2.2) it is clear that $(r_C)^*(Y) = r_{C^*}(Y)$ for every $Y \in S$-Mod, i.e. $(r_C)^* = r_{C^*}$.

Theorem 2.1 is proved. $\square$

2. Now we consider the second combination of the mappings, containing Φ^R and Φ^S:

$$\begin{tikzcd}
\mathbb{PR}(R) & \mathbb{PR}(S) \arrow[l, "(-)^*"'] \\
\mathbb{CO}(R) \arrow[u, "\Phi^R"'] & \mathbb{CO}(S)\ . \arrow[l, "(-)^*"] \arrow[u, "\Phi^S"']
\end{tikzcd}$$

Theorem 2.2. *For every closure operator $D \in \mathbb{CO}(S)$ the relation $(r_D)^* = r_{D^*}$ is true, i.e. the above diagram is commutative.*

Proof. This case is similar to the previous and its proof is symmetrical to the proof of the Theorem 2.1. Nevertheless, for the completeness of account we expose the proof in a short form.

a) Let $D \in \mathbb{CO}(S)$. For the route $D \overset{\Phi^S}{\rightsquigarrow} r_D \overset{(-)^*}{\rightsquigarrow} (r_D)^*$, using the rule $r_D\,(Y) \stackrel{\text{def}}{=}$ $\stackrel{\text{def}}{=} D_Y(0)$ and (1.6) for $s = r_D$, we obtain:

$$(r_D)^*(X) \stackrel{\text{def}}{=} \operatorname{Ker}\big[\varphi_X \cdot H^V\big(\pi^{r_D}_{H^U(X)}\big)\big] \tag{2.3}$$

and $r_D\big(H^U(X)\big) \stackrel{\text{def}}{=} D_{H^U(X)}(0)$.

b) For the transitions $D \overset{(-)^*}{\rightsquigarrow} D^* \overset{\Phi^R}{\rightsquigarrow} r_{D^*}$ we have $D^*{}_X(M) \stackrel{\text{def}}{=} \operatorname{Ker}[\varphi_X \cdot H^V(\pi^m_D)]$ and $r_{D^*}(X) \stackrel{\text{def}}{=} D^*{}_X(0)$ for $X \in R$-Mod. The module $D^*{}_X(M)$ for $M = 0$ is $D^*{}_X(0) \stackrel{\text{def}}{=} \operatorname{Ker}\big[\varphi_X \cdot H^V\big(\pi^m_D\big)\big]$, where $m = 0$. Therefore,

$$r_{D^*}(X) = \operatorname{Ker}[\varphi_X \cdot H^V(\pi^m_D)] \tag{2.4}$$

for $m = 0$.

c) Comparing the modules $(r_D)^*(X)$ and $r_{D^*}(X)$, since $r_D\big(H^U(X)\big) \stackrel{\text{def}}{=} D_{H^U(X)}(0)$, we see that the epimorphisms $\pi^{r_D}_{H^U(X)}$ and π^m_D (for $m = 0$) coincide. Now from the relations (2.3) and (2.4) it is clear that $(r_D)^*(X) = r_{D^*}(X)$ for every $X \in R$-Mod, i.e. $(r_D)^* = r_{D^*}$.

Theorem 2.2 is proved. $\square$

3 Squares containing the mappings $\mathbf{\Psi^R_1}$ and $\mathbf{\Psi^S_1}$.

In this section we will investigate two cases with the combinations of mappings containing Ψ^R_1 and Ψ^S_1.

1. Consider the diagram:

$$\begin{tikzcd}
\mathbb{PR}(R) \arrow[r, "(-)^*"] \arrow[d, "\Psi_1^R"] & \mathbb{PR}(S) \arrow[d, "\Psi_1^S"] \\
\mathbb{CO}(R) \arrow[r, "(-)^*"] & \mathbb{CO}(S)\ .
\end{tikzcd}$$

Theorem 3.1. *For any preradical* $r \in \mathbb{PR}(R)$ *the relation* $(C^r)^* \leq C^{r^*}$ *is true. If the preradical* $r \in \mathbb{PR}(R)$ *is hereditary, then the equality* $(C^r)^* = C^{r^*}$ *holds, i.e. for the pretorsions of* $\mathbb{PR}(R)$ *the above diagram is commutative.*

Proof. a) Let $r \in \mathbb{PR}(R)$ and we begin by the route: $r \overset{(-)^*}{\rightsquigarrow} r^* \overset{\Psi_1^S}{\rightsquigarrow} C^{r^*}$. The transition $r \overset{(-)^*}{\rightsquigarrow} r^*$ is defined by (1.5): $r^*(Y) \overset{\text{def}}{=\!=} \text{Ker}\,[\psi_Y \cdot H^U(\pi^r_{H^V(Y)})]$, where $Y \in S$-Mod and $\pi^r_{H^V(Y)}$ is the natural epimorphism. The second step $r^* \overset{\Psi_1^S}{\rightsquigarrow} C^{r^*}$ is defined by the rule (1.2), i.e. $[(C^{r^*})_Y(N)]/N \overset{\text{def}}{=\!=} r^*(Y/N)$ for every inclusion $n : N \xrightarrow{\subseteq} Y$ of S-Mod. If in the above construction of r^* we substitute Y by Y/N, we obtain $(C^{r^*})_Y(N)/N = \text{Ker}\,[\psi_{Y/N} \cdot H^U(\pi^r_{H^V(Y/N)})]$.

Denote by $\pi_N : Y \longrightarrow Y/N$ the natural epimorphism. Then from the above construction follows that

$$(C^{r^*})_Y(N) = \text{Ker}\,[\pi_N \cdot \psi_{Y/N} \cdot H^U(\pi^r_{H^V(Y/N)})]. \tag{3.1}$$

b) Now we will consider the way: $r \overset{\Psi_1^R}{\rightsquigarrow} C^r \overset{(-)^*}{\rightsquigarrow} (C^r)^*$ for $r \in \mathbb{PR}(R)$. The first step is defined by the rule: $C^r_X(M)/M \overset{\text{def}}{=\!=} r(X/M)$ for every inclusion $M \subseteq X$ of R-Mod. The second step $C^r \overset{(-)^*}{\rightsquigarrow} (C^r)^*$ is defined for every inclusion $n : N \xrightarrow{\subseteq} Y$ of S-Mod by the rule

$$(C^r)^*_Y(N) \overset{\text{def}}{=\!=} \text{Ker}\,[\psi_Y \cdot H^U(\pi^n_{C^r})], \tag{3.2}$$

where $\pi^n_{C^r} : H^V(Y) \to H^V(Y)/C^r_{H^V(Y)}\big(Im\, H^V(n)\big)$ is a natural epimorphism.

c) To compare the constructed above modules $(C^{r^*})_Y(N)$ and $(C^r)^*_Y(N)$ we consider in R-Mod the following diagram, having an arbitrary preradical $r \in \mathbb{PR}(R)$ and the inclusion $n : N \xrightarrow{\subseteq} Y$ of S-Mod:

$$\begin{tikzcd}
H^V(Y) \arrow[r, "\pi^n_{C^r}", "\text{nat}"'] \arrow[d, "\pi^{(k)}", "\text{nat}"'] \arrow[ddd, bend right=60, "H^V(\pi_N)"'] & H^V(Y)/C^r_{H^V(Y)}\big(Im\, H^V(n)\big) \arrow[d, "\cong"] \arrow[ddd, dashed, bend left=60, "\overline{(1/r)(l)}"] \\
A \arrow[r, "\pi^r", "\text{nat}"'] \arrow[d, "\cong"'] \arrow[dd, dashed, bend left, "l"] & A/r(A) \arrow[dd, dashed, "(1/r)(l)"] \\
Im\, H^V(\pi_N) \arrow[d, "\cap"'] & \\
H^V(Y/N) \arrow[r, "\pi^r_{H^V(Y/N)}", "\text{nat}"'] & \big(H^V(Y/N)\big)/r\big(H^V(Y/N)\big)\,,
\end{tikzcd}$$

where all morphisms denoted by π (with indexes) are natural epimorphisms and $A = H^V(Y)/Im\, H^V(n)$.

From the definition of C^r we have:

$$[C^r_{H^V(Y)}(Im\, H^V(n))]/Im\, H^V(n) = r[H^V(Y)/Im\, H^V(n)],$$

therefore $[H^V(Y)/Im\, H^V(n)]\Big/ r[H^V(Y)/Im\, H^V(n)] =$

$$= [H^V(Y)/Im\, H^V(n)]\Big/ [C^r_{H^V(Y)}(Im\, H^V(n))/Im\, H^V(n)] \cong$$

$$\cong H^V(Y)/C^r_{H^V(Y)}(Im\, H^V(n)).$$

The exact sequence $0 \to N \underset{\subseteq}{\xrightarrow{n}} Y \underset{\text{nat}}{\xrightarrow{\pi_N}} Y/N \to 0$ of S-Mod implies the exact sequence $0 \to H^V(N) \xrightarrow{H^V(n)} H^V(Y) \xrightarrow{H^V(\pi_N)} H^V(Y/N)$ of R-Mod, therefore $Im\, H^V(n) = \operatorname{Ker} H^V(\pi_N)$. By the first isomorphism theorem we have:

$$H^V(Y)/Im\, H^V(n) = H^V(Y)/\operatorname{Ker} H^V(\pi_N) \cong Im\, H^V(\pi_N) \subseteq H^V(Y/N).$$

We denote by l the composition $H^V(Y)/Im\, H^V(n) \xrightarrow{\cong} Im\, H^V(\pi_N) \xrightarrow{\subseteq} H^V(Y/N)$, which is a monomorphism. Then l defines the morphism $(1/r)(l)$, which together with the indicated above isomorphism defines the morphism $\overline{(1/r)(l)}$.

Applying H^U and using ψ now we obtain in S-Mod the following diagram:

$$\begin{tikzcd}
Y \arrow[r, "\psi_Y"] \arrow[dd, "\pi_N"] & H^U H^V(Y) \arrow[r, "H^U(\pi^n_{C^r})"] \arrow[d, "H^U(\pi^{(\kappa)})"] \arrow[dd, bend right, "H^U H^V(\pi_N)"'] & H^U\left[H^V(Y)/C^r_{H^V(Y)}(Im\, H^V(n))\right] \arrow[d, "\cong"] \arrow[dd, dashed, bend right, "H^U[\overline{(1/r)(l)}]"'] \\
 & H^U(A) \arrow[r, "H^U(\pi^r)"] \arrow[d, "H^U(l)"] & H^U(A/r(A)) \arrow[d, dashed, "H^U[(1/r)(l)]"] \\
Y/N \arrow[r, "\psi_{Y/N}"] & H^U H^V(Y/N) \arrow[r, "H^U(\pi^r_{H^V(Y/N)})"] & H^U\left[H^V(Y/N)/r(H^V(Y/N))\right].
\end{tikzcd}$$

As we mentioned above, $[(C^{r^*})_Y(N)] \,/\, N = r^*(Y/\,N) = \operatorname{Ker}[\psi_{Y/N} \cdot H^U(\pi^r_{H^V(Y/N)})]$, therefore $(C^{r^*})_Y(N) = \operatorname{Ker}[\pi_N \cdot \psi_{Y/N} \cdot H^U(\pi^r_{H^V(Y/N)})]$. Similarly, by the definition $(C^{r^*})_Y(N) \overset{\text{def}}{=\!=} \operatorname{Ker}[\psi_Y \cdot H^U(\pi^n_{C^r})]$. From the commutativity of the above diagram we obtain:

$$\operatorname{Ker}[\psi_Y \cdot H^U(\pi^n_{C^r})] \subseteq \operatorname{Ker}[\psi_Y \cdot H^U(\pi^n_{C^r}) \cdot H^U(\overline{(1/r)(l)})] =$$

$$= \operatorname{Ker}[\pi_N \cdot \psi_{Y/N} \cdot H^U(\pi^r_{H^V(Y/N)})],$$

which by (3.1) and (3.2) means that $(C^r)^*{}_Y(N) \subseteq (C^{r^*})_Y(N)$ for every $N \subseteq Y$, i.e. $(C^r)^* \leq C^{r^*}$.

d) Now we will prove the last statement of the theorem, supposing that $r \in$ $\in \mathbb{PR}(R)$ is a *pretorsion.* In the indicated above construction l is a monomorphism, therefore by the Lemma 1.1, using the heredity of r, we conclude that $(1/r)(l)$ is a monomorphism. But then by the definition $\overline{(1/r)(l)}$ also is a monomorphism, therefore $H^U\big(\overline{(1/r)(l)}\big)$ is a monomorphism.

Using this fact we obtain:

$$\operatorname{Ker}[\psi_Y \cdot H^U(\pi^n_{C^r})] = \operatorname{Ker}[\psi_Y \cdot H^U(\pi^n_{C^r}) \cdot H^U(\overline{(1/r)(l)})] =$$

$$= \operatorname{Ker}[\pi_N \cdot \psi_{Y/N} \cdot H^U(\pi^r_{H^V(Y/N)})].$$

This means that $(C^r)^*_Y(N) = (C^{r^*})_Y(N)$ for every $N \subseteq Y$. Therefore we have the equality $(C^r)^* = C^{r^*}$ for every pretorsion r of R-Mod.

Theorem 3.1 is proved. ☐

2. The second case of this section consists in the investigation of the following diagram containing the mappings Ψ_1^R and Ψ_1^S:

$$\begin{tikzcd}
\mathbb{PR}(R) \arrow[d, "\Psi_1^R"] & \mathbb{PR}(S) \arrow[l, "(-)^*"'] \arrow[d, "\Psi_1^S"] \\
\mathbb{CO}(R) & \mathbb{CO}(S) . \arrow[l, "(-)^*"']
\end{tikzcd}$$

Theorem 3.2. *For every preradical $s \in \mathbb{PR}(S)$ the relation $(C^s)^* \leq C^{s^*}$ is true. If the preradical $s \in \mathbb{PR}(S)$ is hereditary, then the equality $(C^s)^* = C^{s^*}$ holds, i.e. for the pretorsions of $\mathbb{PR}(S)$ the above diagram is commutative.*

Proof. This result is similar to the Theorem 3.1 and we preserve the scheme of its proof.

a) Let $s \in \mathbb{PR}(S)$ and consider the route: $s \overset{(-)^*}{\rightsquigarrow} s^* \overset{\Psi_1^R}{\rightsquigarrow} C^{s^*}$. For an inclusion $m : M \xrightarrow{\subseteq} X$ of R-Mod by the definition we have $[C^{s^*}_X(M)]/M \overset{\text{def}}{=\!=} s^*(X/M) \overset{\text{def}}{=\!=}$ $\overset{\text{def}}{=\!=} \operatorname{Ker}[\varphi_{X/M} \cdot H^V(\pi^s_{H^U(X/M)})]$. Therefore:

$$C^{s^*}_X(M) = \operatorname{Ker}[\pi_M \cdot \varphi_{X/M} \cdot H^V(\pi^s_{H^U(X/M)})], \tag{3.3}$$

where $\pi_M : X \longrightarrow X/M$ is the natural morphism.

b) Now we consider the way: $s \overset{\Psi_1^S}{\rightsquigarrow} C^s \overset{(-)^*}{\rightsquigarrow} (C^s)^*$ for $s \in \mathbb{PR}(S)$. The first step is defined by $[C^s_Y(N)]/N \overset{\text{def}}{=\!=} s(Y/N)$ for every $N \subseteq Y$ of S-Mod.

On the second step for every inclusion $m : M \xrightarrow{\subseteq} X$ of R-Mod using H^U and C^s we obtain in S-Mod the diagram:

$$\begin{tikzcd}
H^U(X) \arrow[r, "\pi^m_{C^s}", "\text{nat}"'] \arrow[d, "\pi^{(\kappa)}", "\text{nat}"'] & H^U(X)/C^s_{H^U(X)}\big(Im\, H^U(m)\big) \arrow[d, "\cong"] \arrow[ddd, dashed, bend left=60, "\overline{(1/s)(l)}"] \\
B \arrow[r, "\pi^s", "\text{nat}"'] \arrow[d, "\cong"] \arrow[dd, dashed, bend left=40, "l"] & B/s(B) \arrow[dd, dashed, "(1/s)(l)"] \\
Im\, H^U(\pi_M) \arrow[d, "\rotatebox{-90}{$\subseteq$}"] & \\
H^U(X/M) \arrow[r, "\pi^s_{H^U(X/M)}"] & H^U(X/M)/s\big(H^U(X/M)\big)\, ,
\end{tikzcd}$$

where $B = H^U(X)/Im\, H^U(m)$. As in the previous case (Theorem 3.1) we have a monomorphism l, which implies $(1/s)(l)$ and $\overline{(1/s)(l)}$.

Applying H^V to $\pi^m_{C^s}$ we obtain the operator $(C^s)^*$ defined by the rule:

$$(C^s)^*_X(M) \overset{\text{def}}{=\!=} \text{Ker}\,[\varphi_X \cdot H^V(\pi^m_{C^s})]. \tag{3.4}$$

c) To compare the modules $C^{s^*}_X(M)$ and $(C^s)^*_X(M)$ constructed above we use the previous diagram, from which by H^V we obtain in R-Mod such situation:

$$\begin{tikzcd}
 & (C^s)^*_X(M) \arrow[dl, dashed, "\rotatebox{30}{$\supseteq$}"'] \arrow[d, "\rotatebox{-90}{$\subseteq$}"] & & \\
C^{s^*}_X(M) \arrow[r, "\subseteq"] \arrow[d, dashed, "\pi'_M"] & X \arrow[r, "\varphi_X"] \arrow[d, "\pi_M"] & H^V H^U(X) \arrow[r, "H^V(\pi^m_{C^s})"] \arrow[d, dashed, "H^V H^U(\pi_M)"] & H^V\big[H^U(X)/C^s_{H^U(X)}\big(Im\, H^U(m)\big)\big] \arrow[d, "H^V\big(\overline{(1/s)(l)}\big)"] \\
C^{s^*}_X(M)/M \arrow[r, "\subseteq"] & X/M \arrow[r, "\varphi_{X/M}"] & H^V H^U(X/M) \arrow[r, "H^V(\pi^s_{H^V(X/M)})"] & H^V\big[H^U(X/M)/s\big(H^U(X/M)\big)\big],
\end{tikzcd}$$

where $C^{s^*}_X(M)$ and $(C^s)^*_X(M)$ are defined by (3.3) and (3.4). Since this diagram is commutative, we obtain that $(C^s)^*_X(M) \subseteq C^{s^*}_X(M)$ for every $X \subseteq M$, therefore $(C^s)^* \leq C^{s^*}$.

d) Now we suppose that $s \in \mathbb{PR}(S)$ is a *pretorsion*. From the previous construction it is clear that l is a monomorphism and by heredity of s, using the Lemma 1.1, we conclude that $(1/s)(l)$ is a monomorphism. Then by the definition $\overline{(1/s)(l)}$ is a monomorphism, therefore $H^V\big(\overline{(1/s)(l)}\big)$ also is a monomorphism. This fact implies that:

$$\begin{aligned} \text{Ker}\,[\varphi_X \cdot H^V(\pi^m_{C^s})] = {} & \text{Ker}\,\big[\varphi_X \cdot H^V(\pi^m_{C^s}) \cdot H^V\big(\overline{(1/s)(l)}\big)\big] = \\ = {} & \text{Ker}\,[\pi_M \cdot \varphi_{X/M} \cdot H^V(\pi^s_{H^V(X/M)})], \end{aligned}$$

which means that $(C^s)^*_X(M) = C^{s^*}_X(M)$ for every $M \subseteq X$. In such way we obtain $(C^s)^* = C^{s^*}$ for every pretorsion s of S-Mod.

Theorem 3.2 is proved. ☐

4 Squares containing the mappings Ψ_2^R and Ψ_2^S.

In continuation we investigate the last two cases, analyzing the squares with the mappings Ψ_2^R and Ψ_2^S.

1. We begin this part with the following diagram:

$$\begin{tikzcd}
\mathbb{PR}(R) \arrow[r, "(-)^*"] \arrow[d, "\Psi_2^R"] & \mathbb{PR}(S) \arrow[d, "\Psi_2^S"] \\
\mathbb{CO}(R) \arrow[r, "(-)^*"] & \mathbb{CO}(S) \, .
\end{tikzcd}$$

Theorem 4.1. *For every preradical* $r \in \mathbb{PR}(R)$ *the relation* $C_{r^*} \leq (C_r)^*$ *is true.*

Proof. a) Let $r \in \mathbb{PR}(R)$ and consider the way: $r \overset{(-)^*}{\rightsquigarrow} r^* \overset{\Psi_2^S}{\rightsquigarrow} C_{r^*}$. The step $r \overset{(-)^*}{\rightsquigarrow} r^*$ is defined by the rule: $r^*(Y) = \operatorname{Ker}[\psi_Y \cdot H^U(\pi^r_{H^V(Y)})]$, where $Y \in S$-Mod and $\pi^r_{H^V(Y)} : \ H^V(Y) \to H^V(Y)/r\big(H^V(Y)\big)$ is a natural epimorphism. The second step is defined by $(C_{r^*})_Y(N) \overset{\text{def}}{=\!=} r^*(Y) + N$, therefore:

$$(C_{r^*})_Y(N) = \operatorname{Ker}[\psi_Y \cdot H^V(\pi^r_{H^V(Y)})] + N \tag{4.1}$$

for every inclusion $n : N \xrightarrow{\subseteq} Y$ of S-Mod.

b) Now we follow the way: $r \overset{\Psi_2^R}{\rightsquigarrow} C_r \overset{(-)^*}{\rightsquigarrow} {C_r}^*$ for $r \in \mathbb{PR}(R)$. By the definition of Ψ_2^R we have $(C_r)_X(M) \overset{\text{def}}{=\!=} r(X) + M$ for every inclusion $M \subseteq X$ of R-Mod.

To define the second step $C_r \overset{(-)^*}{\rightsquigarrow} {C_r}^*$, let $n : \ N \xrightarrow{\subseteq} Y$ be an inclusion of S-Mod. Using H^V and C_r, we obtain in R-Mod such situation:

$$\begin{tikzcd}
r\big(H^V(Y)\big) \arrow[dr, "i^r_{H^V(Y)}", "\subseteq"'] \arrow[dd, dashed, "l", "\cap"'] & & H^V(Y)/r\big(H^V(Y)\big) \arrow[dd, dashed, "\pi^{(l)}"] \\
 & H^V(Y) \arrow[ur, "\pi^r_{H^V(Y)}", "nat"'] \arrow[dr, "\pi^n_{C_r}", "nat"'] & \\
(C_r)_{H^V(Y)}\big(Im\, H^V(n)\big) \arrow[ur, "i^n_{C_r}", "\subseteq"'] & & H^V(Y)/\big(C_r\big)_{H^V(Y)}\big(Im\, H^V(n)\big),
\end{tikzcd}$$

where by the definition of C_r we have:

$$(C_r)_{H^V(Y)}\big(Im\, H^V(n)\big) \overset{\text{def}}{=\!=} r\big(H^V(Y)\big) + Im\, H^V(n).$$

We denote by l the inclusion $r\big(H^V(Y)\big) \xrightarrow{\subseteq} (C_r)_{H^V(Y)}\big(Im\, H^V(n)\big)$ and by $\pi^{(l)}$ the corresponding epimorphism.

Applying the functor H^U we obtain in S-Mod the diagram:

$$\begin{tikzcd}
(C_{r^*})_Y(N) \arrow[dd, "\subseteq"] \arrow[dr, "\subseteq"] & & & H^U\big[H^V(Y)/r\big(H^V(Y)\big)\big] \arrow[dd, dashed, "H^U(\pi^{(l)})"] \\
& Y \arrow[r, "\psi_Y"] & H^U H^V(Y) \arrow[ur, "H^U\left(\pi^r_{H^V(Y)}\right)"] \arrow[dr, "H^U\left(\pi^n_{C_r}\right)"'] & \\
(C_r)^*_Y(N) \arrow[ur, "\subseteq"'] & & & H^U\big[H^V(Y)/\big(C_r\big)_{H^V(Y)}\big(Im\, H^V(n)\big)\big]
\end{tikzcd}$$

and by the definition we have:

$$(C_r)^*_Y(N) \overset{\text{def}}{=\!=} \text{Ker}\,\big[\psi_Y \cdot H^U\big(\pi^n_{C_r}\big)\big]. \tag{4.2}$$

c) Now we show the relation between the modules $(C_{r^*})_Y(N)$ and $(C_r)^*_Y(N)$, expressed by (4.1) and (4.2). From the last diagram we have:

$$\text{Ker}\,[\psi_Y \cdot H^U\big(\pi^r_{H^V(Y)}\big)] \subseteq \text{Ker}\,[\psi_Y \cdot H^U\big(\pi^r_{H^V(Y)}\big) \cdot H^U(\pi^{(l)})] =$$
$$= \text{Ker}\,\big[\psi_Y \cdot H^U\big(\pi^n_{C_r}\big)\big] \overset{\text{def}}{=\!=} (C_r)^*_Y(N).$$

Using the trivial inclusion $N \subseteq (C_r)^*_Y(N)$, we see that $\text{Ker}\,\big[\psi_Y \cdot$ $\cdot\, H^U\big(\pi^r_{H^V(Y)}\big)\big] + N \subseteq (C_r)^*_Y(N)$, i.e. $(C_{r^*})_Y(N) \subseteq (C_r)^*_Y(N)$ for every $N \subseteq Y$. This means that $C_{r^*} \leq (C_r)^*$ for every $r \in \mathbb{PR}(R)$.

Theorem 4.1 is proved. $\square$

2. The last case of our investigation consists in the study of the square with the following mappings:

$$\begin{tikzcd}
\mathbb{PR}(R) \arrow[d, "\Psi_2^R"] & \mathbb{PR}(S) \arrow[l, "(-)^*"'] \arrow[d, "\Psi_2^S"] \\
\mathbb{CO}(R) & \mathbb{CO}(S)\,. \arrow[l, "(-)^*"']
\end{tikzcd}$$

Theorem 4.2. *For every preradical $s \in \mathbb{PR}(S)$ the relation $C_{s^*} \leq (C_s)^*$ is true.*

Proof. a) Let $s \in \mathbb{PR}(S)$. Considering the way: $s \overset{(-)^*}{\rightsquigarrow} s^* \overset{\Psi_2^R}{\rightsquigarrow} C_{s^*}$ we obtain:

$$(C_{s^*})_X(M) = \text{Ker}\,[\varphi_X \cdot H^V(\pi^s_{H^U(X)})] + M \tag{4.3}$$

for every $M \subseteq X$ of R-Mod, since by the definition $(C_{s^*})_X(M) \overset{\text{def}}{=\!=} s^*(X) + M$ and $s^*(X) \overset{\text{def}}{=\!=} \text{Ker}\,[\varphi_X \cdot H^V(\pi^s_{H^U(X)})]$.

b) For $s \in \mathbb{PR}(S)$ consider now the way: $s \overset{\Psi_2^S}{\rightsquigarrow} C_s \overset{(-)^*}{\rightsquigarrow} (C_s)^*$. The operator C_s is determined by the rule: $(C_s)_Y(N) \overset{\text{def}}{=\!=} s(Y) + N$ for every $N \subseteq Y$ of S-Mod. It remains to define the transition $C_s \overset{(-)^*}{\rightsquigarrow} (C_s)^*$. For every inclusion $m : M \xrightarrow{\subseteq} X$ of R-Mod by H^U and C_s we obtain in R-Mod the situation:

$$\begin{tikzcd}
s\big(H^U(X)\big) \arrow[dr, "i^s_{H^U(X)}", "\subseteq"'] \arrow[dd, dashed, "l", "\cap|"'] & & H^U(X)/s\big(H^U(X)\big) \arrow[dd, dashed, "\pi^{(l)}"] \\
& H^U(X) \arrow[ur, "\pi^s_{H^U(X)}", "nat"'] \arrow[dr, "\pi^m_{C_s}", "nat"'] & \\
(C_s)_{H^U(X)}\big(Im\, H^U(m)\big) \arrow[ur, "i^m_{C_s}", "\subseteq"'] & & H^U(X)/\big(C_s)_{H^U(X)}\big(Im\, H^U(m)\big),
\end{tikzcd}$$

where $(C_s)_{H^U(X)}\big(Im\, H^U(m)\big) \overset{\text{def}}{=\!=} s\big(H^U(X)\big) + Im\, H^U(m)$.

By the definition of $(C_s)^*$ we have:

$$(C_s)^*_X(M) \overset{\text{def}}{=\!=} \text{Ker}\,[\varphi_X \cdot H^V(\pi^m_{C_s})]. \tag{4.4}$$

c) To compare the modules $(C_{s^*})_X(M)$ and $(C_s)^*_X(M)$ defined by (4.3) and (4.4) we use the diagram:

$$\begin{tikzcd}
(C_{s^*})_X(M) \arrow[dr, "\subseteq"] \arrow[dd, dashed, "\cap|"] & & & H^V\big[H^U(X)/s\big(H^U(X)\big)\big] \arrow[dd, dashed, "H^V(\pi^{(l)})"] \\
& X \arrow[r, "\varphi_X"] & H^VH^U(X) \arrow[ur, "H^V\left(\pi^s_{H^U(X)}\right)"] \arrow[dr, "H^V(\pi^m_{C_s})"'] & \\
(C_s)^*_X(M) \arrow[ur, "\subseteq"] & & & H^V\big[H^U(X)/\big(C_s\big)_{H^U(X)}\big(Im\, H^U(m)\big)\big],
\end{tikzcd}$$

from which we obtain:

$$\text{Ker}\,[\varphi_X \cdot H^V\big(\pi^s_{H^U(X)}\big)] \subseteq \text{Ker}\,[\varphi_X \cdot H^V\big(\pi^s_{H^U(X)}\big) \cdot H^V(\pi^{(l)})] =$$

$$= \text{Ker}\,[\varphi_X \cdot H^V(\pi^m_{C_s})] = (C_s)^*{}_X(M).$$

Therefore $\text{Ker}\,[\varphi_X \cdot H^V\big(\pi^s_{H^U(X)}\big)] + M \subseteq (C_{s^*})_X(M)$, i.e. $(C_{s^*})_X(M) \subseteq (C_s)^*{}_X(M)$ for every $M \subseteq X$. This means that $C_{s^*} \leq (C_s)^*$ for every $s \in \mathbb{PR}(S)$.

Theorem 4.2 is proved. ☐

Totalizing the exposed above facts we can affirm that for a Morita context the studied ten mappings are well concordant between them in the sense of commutativity of suitable diagrams. Indeed, in the four cases the commutativity is proved (Theorems 2.1, 2.2, 3.1, 3.2), while in other cases the inclusion relations are shown.

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