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Different Types of Compact Global Attractors for Cocycles with a Noncompact Phase Space of Driving System and the Relationship Between Them

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Abstract. In this paper we study different types of compact global attractors for non-autonomous (cocycle) dynamical systems with noncompact phase space of driving system. We establish the relations between them.

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1 Introduction

The global attractors play a very important role in the study of asymptotic behavior of dynamical systems (both autonomous and non-autonomous). During the last 20-25 years many works dedicated to the study of the global attractors of dynamical systems (including infinite-dimensional systems) were published. See, for example, A. V. Babin and M. I. Vishik [2], I. D. Chueshov [12], J. K. Hale [17], O. A. Ladyzhenskaya [23], J. C. Robinson [24], R. Temam [26] (for autonomous systems), A. N. Carvalho, J. A. Langa and J. C. Robinson[4], D. N. Cheban [8, 9], V. V. Chepyzhov and M. I. Vishik [11], A. Haraux [18], P. E. Kloeden and M. Rasmussen [20] (for non-autonomous systems) and the bibliography therein.

The aim of this paper is studying different types of compact global attractors for non-autonomous (cocycle φ) dynamical systems over dynamical system $(Y, \mathbb{T}, \sigma)$ (driving system) with the fiber W in the case when the phase space Y of driving system is not compact. We establish the relations between global, pullback and forward attractors for cocycle dynamical systems. For cocyle dynamical systems $\langle W, \varphi, (Y, \mathbb{T}, \sigma)\rangle$ with compact phase space Y of driving system $(Y, \mathbb{T}, \sigma)$ this problem was studied in the work of D. Cheban, P. Kloeden and B. Schmalfuss [10] (see also [8, Ch.II]).

2 Some Notions of Non-autonomous Dynamical Systems

In this section we collect some notions from the autonomous and non-autonomous dynamical systems [6] (see also [9, Ch.IX]) which we will use below.

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Let Y be a complete metric space, let $\mathbb{R} := (-\infty, +\infty)$, $\mathbb{Z} := \{0, \pm 1, \pm 2, \dots\}$, $\mathbb{S} = \mathbb{R}$ or $\mathbb{Z}$, $\mathbb{T}_+ = \{t \in \mathbb{T} | \quad t \geq 0\}$, $\mathbb{T}_- = \{t \in \mathbb{T} | \quad t \leq 0\}$ and $\mathbb{T}$ ($\mathbb{T}_+ \subseteq \mathbb{T}$) be a sub-semigroup of group $\mathbb{S}$. Let $(Y, \mathbb{T}, \sigma)$ be an autonomous dynamical system on Y and E be a real or complex Banach space with the norm $|\cdot|$.

Definition 1. (Cocycle on the state space E with the base $(Y, \mathbb{T}, \sigma)$.) The triplet $\langle E, \phi, (Y, \mathbb{T}, \sigma)\rangle$(or briefly ϕ) is said to be a cocycle (see, for example,[9] and [25]) on the state space E with the base $(Y, \mathbb{T}, \sigma)$ if the mapping $\phi : \mathbb{T}_+ \times Y \times E \to E$ satisfies the following conditions:

1. $\phi(0, y, u) = u$ for all $u \in E$ and $y \in Y$;
2. $\phi(t + \tau, y, u) = \phi(t, \phi(\tau, u, y), \sigma(\tau, y))$ for all $t, \tau \in \mathbb{T}_+, u \in E$ and $y \in Y$;
3. the mapping ϕ is continuous.

Definition 2. (Skew-product dynamical system). Let $\langle E, \phi, (Y, \mathbb{T}, \sigma)\rangle$ be a cocycle on $E, X := E \times Y$ and π be a mapping from $\mathbb{T}_+ \times X$ to X defined by equality $\pi = (\phi, \sigma)$, i.e., $\pi(t, (u, y)) = (\phi(t, \omega, u), \sigma(t, y))$ for all $t \in \mathbb{T}_+$ and $(u, y) \in E \times Y$. The triplet $(X, \mathbb{T}_+, \pi)$ is an autonomous dynamical system and it is called [25] a skew-product dynamical system.

Definition 3. (Non-autonomous dynamical system.) Let $\mathbb{T}_1 \subseteq \mathbb{T}_2$ be two sub-semigroups of the group $\mathbb{T}, (X, \mathbb{T}_1, \pi)$ and $(Y, \mathbb{T}_2, \sigma)$ be two autonomous dynamical systems and $h : X \to Y$ be a homomorphism from $(X, \mathbb{T}_1, \pi)$ to $(Y, \mathbb{T}_2, \sigma)$ (i.e., $h(\pi(t, x)) = \sigma(t, h(x))$ for all $t \in \mathbb{T}_1$, $x \in X$ and h is continuous), then the triplet $\langle (X, \mathbb{T}_1, \pi),\ (Y,\ \mathbb{T}_2,\ \sigma), h\rangle$ is called (see [3] and [9]) a non-autonomous dynamical system.

Example 1. (The non-autonomous dynamical system generated by cocycle ϕ.) Let $\langle E, \phi, (Y, \mathbb{T}, \sigma)\rangle$ be a cocycle, $(X, \mathbb{T}_+, \pi)$ be a skew-product dynamical system ($X = E \times Y, \pi = (\phi, \sigma)$) and $h = pr_2 : X \to Y$, then the triplet $\langle (X, \mathbb{T}_+, \pi),\ (Y, \mathbb{T}, \sigma), h\rangle$ is a non-autonomous dynamical system.

3 Global Attractors of Non-autonomous Dynamical Systems

3.1 Non-autonomous sets

Let W and Y be two metric spaces.

Definition 4. A family $\{A_y |\ y \in Y\}$ of subsets A_y of W indexed by $y \in Y$ is called a non-autonomous set.

Let $\{A_y |\ y \in Y\}$ be a non-autonomous set. Denote by $\mathbb{A}$ the subset of $X := W \times Y$ defined by equality

$$\mathbb{A} := \bigcup \{A_p \times \{y\} |\ y \in Y\}\} = \{(w, y) \in X |\ w \in A_y,\ y \in Y\}.$$

Remark 1. 1. Let $\mathcal{A}$ be a subset of $X = W \times Y$, $\mathcal{A}_y := \mathbb{A} \bigcap pr_2^{-1}(y)$ and $A_y := pr_1(\mathcal{A}_y)$, then $\{A_y|\ y \in Y\}$ is a non-autonomous set.

2. Denote by $\mathfrak{A} = \bigcup\{A_y \times \{y\}|\ y \in Y\}$, then $\mathcal{A} \subseteq \mathfrak{A}$.

Definition 5. A non-autonomous set $\{A_y|\ y \in Y\}$is said to be

1. pre-compact (respectively, uniformly pre-compact) if for every $y \in Y$ the set A_y (respectively, $\bigcup\{A_y|\ y \in Y\}$) is a pre-compact subset of W;

2. bounded (respectively, uniformly bounded) if for every $y \in Y$ the set A_y (respectively, $\bigcup\{A_y|\ y \in Y\}$) is a bounded subset of W.

Definition 6. A non-autonomous set $\{A_y|\ y \in Y\}$ is said to be

1. positively (respectively, negatively) invariant (with respect to cocycle φ) if $\varphi(t, A_y, y) \subseteq A_{\sigma(t,y)}$ (respectively, $\varphi(t, A_y, y) \supseteq A_{\sigma(t,y)}$) for any $y \in Y$ and $t \geq 0$;

2. invariant if it is positively and negatively invariant.

Lemma 1. *Assume that the set Y is invariant, that is, $\sigma(t, Y) = Y$ for any $t \in \mathbb{T}$. The non-autonomous set $\{A_y|\ y \in Y\}$ is positively invariant (respectively, negatively invariant or invariant) if and only if the set $\mathbb{A}$ is a positively invariant (respectively, negatively invariant or invariant) subset of skew-product dynamical system $(X, \mathbb{T}, \pi)$.*

Proof. Note that $\mathfrak{A} = \bigcup\{\mathbb{A}_y|\ y \in Y\}$, where $\mathbb{A}_y := A_y \times \{y\}$. Let $\{A_y|\ y \in Y\}$ be a positively invariant (respectively, negatively invariant or invariant) set, then

$$\pi(t, \mathbb{A}) = \bigcup\{\pi(t, \mathbb{A}_y)|\ y \in Y\} = \bigcup\{(\varphi(t, A_y, y), \sigma(t, y))|\ y \in Y\} \subseteq$$
$$\bigcup\{(A_{\sigma(t,y)}, \sigma(t, y))|\ y \in Y\} \subseteq \bigcup\{(A_q, q)|\ q \in Y\} = \bigcup\{\mathbb{A}_q|\ q \in Y\} = \mathbb{A}.$$

Conversely. Let $\mathbb{A}$ be an invariant set of skew-product dynamical system $(X, \mathbb{T}, \pi)$, then

$$\pi(t, \mathbb{A}) = \sqcup\{\pi(t, \mathbb{A}_y)|\ y \in Y\} = \sqcup\{(\varphi(t, A_y, y).\sigma(t, y)\}. \tag{1}$$

On the other hand we have

$$\mathbb{A} = \sqcup\{\mathbb{A}_y|\ y \in Y\} = \sqcup\{\mathbb{A}_{\sigma(t,y)}|\ y \in Y\} = \sqcup\{(A_{\sigma(t,y)}, \sigma(t, y))|\ y \in Y\}. \tag{2}$$

Since $\pi(t, \mathbb{A}) \subseteq \mathbb{A}$ for any $t \in \mathbb{T}$, then from (1)-(2) it follows that $\varphi(t, A_y, y) \subseteq A_{\sigma(t,y)}$ for any $(t, y) \in \mathbb{T} \times Y$.

Analogously one can be consider the case $\mathbb{A} \subseteq \pi(t, \mathbb{A})$ (respectively, $\mathbb{A} = \pi(t, \mathbb{A})$) for any $t \in \mathbb{T}$. $\square$

Lemma 2. *The following statements are equivalent:*

1. *for any compact subset $K \subseteq W$ the set $\bigcup\{A_y|\ y \in K\}$ is pre-compact in W;*

2. *the set* $\mathbb{A} \subseteq X$ *is conditionally pre-compact in* (X, h, Y) *(*$X = W \times Y$ *and* $h := pr_2 : X \to Y$*).*

Proof. Let K be an arbitrary compact subset of Y. Then taking into account the relation

$$h^{-1}(K) \bigcap \mathbb{A} = \bigcup \{A_y \times \{y\}|\ y \in K\} \subseteq \big(\bigcup \{A_y | y \in K\}\big) \times K$$

we conclude that the set $h^{-1}(K) \bigcap \mathbb{A}$ is pre-compact in X.

Conversely, assume that the set $h^{-1}(K) \bigcap \mathbb{A}$ is pre-compact for any compact subset K from Y. Then the set $\bigcup \{A_y|\ y \in K\}$ is pre-compact in W because

$$pr_1(h^{-1}(K) \bigcap \mathbb{A}) = \bigcup \{A_y|\ y \in K\},$$

$h^{-1}(K) \bigcap \mathbb{A}$ is a pre-compact subset of X and $pr_1 : X \to W$ is a continuous mapping. Lemma is proved. □

Corollary 1. *Let* $\{A_y|\ y \in Y\}$ *be a uniformly pre-compact non-autonomous set, then the set* $\mathbb{A}$ *is a conditionally compact subset of* X *with respect to* (X, h, Y)*, where* $h = pr_2$*.*

Proof. This statement follows from Lemma 2 because for any compact subset $K \subseteq Y$ we have

$$h^{-1}(K) \bigcap \mathbb{A} = \bigcup \{A_y \times \{y\}|\ y \in K\} \subseteq \big(\bigcup \{A_y | y \in K\}\big) \times K.$$

On the other hand by condition of Lemma the set $\bigcup \{A_y|\ y \in Y\}$ is pre-compact and, consequently, the set $\bigcup \{A_y|\ y \in K\}$ is so. Now to finish the proof it is sufficient to apply Lemma 2. □

3.2 Maximal compact invariant sets

Definition 7. A non-autonomous compact set $\{A_y|\ y \in Y\}$ ($A_y \subseteq W$) is called a maximal compact invariant set of cocycle φ if the following conditions are fulfilled:

1. $\{A_y|\ y \in Y\}$ is invariant;
2. $A = \bigcup \{A_y|\ y \in Y\}$ is pre-compact;
3. the non-autonomous set $\{A_y|\ y \in Y\}$ is maximal with the properties 1. and 2., i.e., if a non-autonomous set $\{A'_y|\ y \in Y\}$ is invariant and $A' = \bigcup \{A'_y|\ y \in Y\}$ is pre-compact, then $A'_y \subseteq A_y$ for every $y \in Y$.

Lemma 3. *Let* $\{I_y|\ y \in Y\}$ *be a non-autonomous set. Assume that the set* $\mathbb{J} = \bigcup \{J_y = I_y \times \{y\}|\ y \in Y\}$ *is conditionally pre-compact, then the following statements are equivalent:*

1. *the mapping* $y \to I_y$ *is upper semi-continuous;*

2. the set $\mathbb{J}$ is closed in X.

Proof. Assume that the mapping $y \to I_y$ be upper semi-continuous and $(\bar{w}, \bar{y}) \in \overline{\mathbb{J}}$. Then there exists a sequence $\{(w_n, y_n) \in \mathbb{J}\}$ such that $(w_n, y_n) \to (\bar{w}, \bar{y})$. Since $w_n \in I_{y_n}$ and $y \to I_y$ is upper semi-continuous, then $w_0 \in I_{y_0}$ and, consequently, $(w_0, y_0) \in J_{y_0} \subseteq \mathbb{J}$. Thus the set $\mathbb{J}$ is closed.

Let now $\mathbb{J}$ be a closed subset of X. We will show that the mapping $y \to I_y$ is upper semi-continuous. If we suppose that the family $\{I_y |\ y \in Y\}$ is not upper semi-continuous, then there are $\varepsilon_0 > 0$, $y_0 \in Y$ and sequences $\{y_n\} \subset Y$ and $\{w_n\} \subset W$ such that $y_n \to y_0$ as $n \to \infty$, $w_n \in I_{y_n}$ and

$$\rho(w_n, I_{y_0}) \geq \varepsilon_0. \tag{3}$$

Since $\bigcup\{I_{y_n} |\ n \in \mathbb{N}\}$ is pre-compact, then without loss of generality we can suppose that the sequence $\{w_n\}$ is convergent. Denote by $w_0 := \lim\limits_{n\to\infty} w_n$ and passing to the limit in (3) as $n \to \infty$ we obtain $w_0 \notin I_{y_0}$. On the other hand we have $(w_n, y_n) \in \mathbb{J}_{y_n} \subseteq \mathbb{J}$ for any $n \in \mathbb{N}$ and since the set $\mathbb{J}$ is closed and $(w_n, y_n) \to (w_0, y_0)$ as $n \to \infty$, then $(w_0, y_0) \in \mathbb{J}$ and, consequently, $w_0 \in I_{y_0}$. The obtained contradiction proves our statement. □

Let Y be a complete metric space, $(Y, \mathbb{T}, \sigma)$ be a dynamical system and $\langle W, \varphi, (Y, \mathbb{T}, \sigma)\rangle$ be a cocycle over $(Y, \mathbb{T}, \sigma)$ with the fiber W. Below we suppose that the set Y is invariant, i.e., $\sigma(t, Y) = Y$ for any $t \in \mathbb{T}$.

Definition 8. A non-autonomous set $\{I_y |\ y \in Y\}$ ($I_y \subseteq W$) of nonempty compact subsets of W is called a compact pullback (respectively, uniform pullback) attractor [1], [9, Ch.II], [13]) of the cocycle φ if the following conditions are fulfilled:

a. $\mathcal{I} := \bigcup\{I_y | y \in Y\}$ is relatively compact;

b. $\{I_y |\ y \in Y\}$ is invariant w.r.t. cocycle φ, i.e., $\varphi(t, I_y, y) = I_{\sigma(t,y)}$ for any $t \in \mathbb{T}_+$ and $y \in Y$;

c. for every $y \in Y$ and any pre-compact non-autonomous set $\{K_y |\ y \in Y\}$

$$\lim_{t\to+\infty} \beta(\varphi(t, K_y, \sigma(-t, y)), I_y) = 0$$

(respectively,

$$\lim_{t\to+\infty} \sup\{\beta(\varphi(t, K_y, \sigma(-t, y)), I_y) |\ y \in Y\} = 0),$$

where $\beta(A, B) = \sup\{\rho(a, B) : a \in A\}$ is a semi-distance of Hausdorff.

Remark 2. If $\mathbb{T} = \mathbb{T}_+$ and Y is invariant, then:

1. $\sigma(-t, y) := \{\tilde{y} \in Y |\ \sigma(t, \tilde{y}) = y\} = (\sigma^t)^{-1}(y)$, where $\sigma^t(y) := \sigma(t, y)$ for any $(t, y) \in \mathbb{T}_+ \times Y$ and f^{-1} denotes the inverse mapping of f;

2. $I_{\sigma(-t,y)} := \bigcup\{I_{\tilde{y}} |\ \tilde{y} \in \sigma(-t,y)\}$ (see, for example, [19]).

Lemma 4. *Let $\{I_y |\ y \in Y\}$ be a compact pullback attractor of the cocycle $\langle W, \varphi, (Y, \mathbb{T}, \sigma)\rangle$. Then $\{I_y |\ y \in Y\}$ is a maximal compact invariant set of the cocycle φ.*

Proof. Let $\{I'_y |\ y \in Y\}$ be a compact invariant subset of the cocycle φ, then there exists a nonempty compact subset $K \subseteq W$ such that

$$\bigcup\{I'_y | y \in Y\} \subseteq K \tag{4}$$

and

$$\varphi(t, I'_y, y) = I'_{\sigma(t,y)} \tag{5}$$

for any $y \in Y$ and $t \geq 0$. Then taking into consideration (4)-(5) we obtain

$$\beta(I'_y, I_y) = \beta(\varphi(t, I'_{\sigma(-t,y)}, \sigma(-t,y)), I_y) \leq \\ \beta(\varphi(t, K, \sigma(-t,y)), I_y) \to 0 \quad \text{as } t \to \infty$$

and, consequently, $I'_y \subseteq I_y$ for any $y \in Y$. Lemma is proved. □

Corollary 2. *Let $\{I_y |\ y \in Y\}$ be a compact pullback attractor of the cocycle $\langle W, \varphi, (Y, \mathbb{T}, \sigma)\rangle$. Then $\mathbb{J} = \bigcup\{J_y = I_y \times \{y\} |\ y \in Y\}$ is the maximal conditional pre-compact invariant set of the skew-product dynamical system $(X, \mathbb{T}_+, \pi)$ with the property that $pr_1(\mathbb{J})$ is a pre-compact set.*

Proof. Let $\mathbb{J}'$ be an invariant set of the skew-product dynamical system $(X, \mathbb{T}_+, \pi)$ with the property that $pr_1(\mathbb{J}')$ is a pre-compact set. Consider the non-autonomous set $I' = \{I'_y |\ y \in Y\}$, where $I'_y := pr_1(\mathbb{J}'_y)$. By conditions of Corollary 2 there exists a compact subset $K \subseteq W$ such that $I'_y \subseteq K$ for any $y \in Y$. Since $I' = \{I'_y |\ y \in Y\}$ is an invariant subset of the cocycle φ, then by Lemma 4 we have $I'_y \subseteq I_y$ for any $y \in Y$ and, consequently, $\mathbb{J}' \subseteq \mathbb{J}$. □

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Corollary 3. *There exists at most one compact pullback attractor of the cocycle $\langle W, \varphi, (Y, \mathbb{T}, \sigma)\rangle$.*

Proof. Let $\{I^i_y |\ y \in Y\}$ $(i = 1, 2)$ be two compact pullback attractors of the cocycle φ, then by Lemma 4 $I^i_y \subseteq I^j_y$ for any $y \in Y$ and $i, j = 1, 2$ $(i \neq j)$. From the last inclusion it follows that $I^1_y = I^2_y$ for any $y \in Y$. □

Theorem 1. *Let $\{I_y |\ y \in Y\}$ be a compact pullback attractor of the cocycle $\langle W, \varphi, (Y, \mathbb{T}, \sigma)\rangle$.*

Then the following statements hold:

1. *$\mathbb{J} = \bigcup\{J_y = I_y \times \{y\} |\ y \in Y\}$ is the maximal conditional pre-compact set of the skew-product dynamical system $(X, \mathbb{T}_+\pi)$;*

2. *If the metric space Y is compact, then $\mathbb{J}$ is the maximal compact invariant set of $(X, \mathbb{T}_+, \pi)$.*

Proof. By Corollary 2 $\mathbb{J}$ is the maximal conditionally pre-compact invariant set of $(X, \mathbb{T}_+, \pi)$.

Let now Y be a compact metric space. To prove the second statement of Theorem it is sufficient to show that the set $\mathbb{J}$ is compact, i.e., that $\mathbb{J}$ is pre-compact and closed. Note that

$$\mathbb{J} = \bigcup\{\mathbb{J}_y |\ y \in Y\} = \bigcup\{I_y \times \{y\}|\ y \in Y\} \subseteq \left(\bigcup\{I_y|\ y \in Y\}\right) \times Y$$

and, consequently, $\mathbb{J}$ is pre-compact because the sets $\bigcup\{I_y|\ y \in Y\}$ and Y are so.

Finally, we will show that the set $\mathbb{J}$ is closed. Denote by $M := \overline{\mathbb{J}}$ the closure in X of the set $\mathbb{J}$, then it is a compact subset of X. Since the set $\mathbb{J}$ is invariant, then its closure M is also invariant because $\mathbb{J}$ is invariant and pre-compact. Since $pr_1(M)$ is a compact subset of W, then by Corollary 2 we have $\mathbb{J} = M$. Theorem is completely proved. □

Corollary 4. *Let $\{I_y|\ y \in Y\}$ be a compact pullback attractor of the cocycle $\langle W, \varphi, (Y, \mathbb{T}, \sigma)\rangle$. If the metric space Y is compact, then the map $y \to I_y$ is upper semi-continuous.*

Proof. This statement follows from Theorem 1 (item 2.) and Lemma 3. □

Corollary 5. *Under the condition of Corollary 4 there exists a residual set $Y_0 \subseteq Y$ such that the mapping $y \to I_y$ is continuous at every point $y \in Y_0$.*

Proof. This statement follows from Corollary 4 and the fact that the set of points of continuity Y_0 of the semi-continuous function $y \to I_y$ is residual (see, for example,[14] and [22, Ch.I]). □

Remark 3. In the general case Corollary 5 is unimprovable. In the paper [16] (see also [15] and the references therein) an example is given of a non-autonomous (quasi-periodic) dynamical system with discrete time and a compact global attractor $\{I_y|\ y \in Y\}$ in which the mapping $y \to I_y$ is upper semi-continuous, but not continuous.

Theorem 2. *Let $\langle W, \varphi, (Y, \mathbb{T}, \sigma)\rangle$ be a cocycle over dynamical system $(Y, \mathbb{T}, \sigma)$ with the fiber W and $\{I_y|\ y \in Y\}$ be a non-autonomous set. Assume that the following conditions are fulfilled:*

1. *the set Y is invariant;*
2. *the set $\bigcup\{I_y|\ y \in Y\}$ is pre-compact.*

Then the following statements are equivalent:

1. *$\{I_y|\ y \in Y\}$ is a maximal pre-compact and invariant (with respect to cocycle φ) set;*

2. *the set* $\mathbb{J} = \bigcup\{J_y|\ y \in Y\}$, *where* $J_y := I_y \times \{y\}$ *for any* $y \in Y$, *is a maximal conditionally compact invariant set of skew-product dynamical system* $(X, \mathbb{T}_+, \pi)$ *with the property that* $pr_1(\mathbb{J})$ *is precompact.*

Proof. Let $\{I_y|\ y \in Y\}$ be a maximal pre-compact and φ invariant set. By Lemma 2 the set $\mathbb{J}$ is conditionally pre-compact. Let $\mathbb{J}^{'}$ be an invariant set of the skew-product dynamical system $(X, \mathbb{T}_+, \pi)$ with the property that $pr_1(\mathbb{J}^{'})$ is pre-compact. Consider the non-autonomous set $I^{'} = \{I^{'}_y|\ y \in Y\}$, where $I^{'}_y := pr_1(\mathbb{J}^{'}_y)$. By Lemma 1 the non-autonomous set $\{I'_y|\ y \in Y\}$ is invariant and taking into consideration $I'_y = pr_1(J_y)$ for any $y \in Y$, then $\{I'_y|\ y \in Y\}$ is pre-compact. Since $\{I_y|\ y \in Y\}$ is a maximal pre-compact φ invariant set, then $I'_y \subseteq I_y$ for any $y \in Y$ and, consequently, $\mathbb{J}' \subseteq \mathbb{J}$.

Conversely, if $\mathbb{J} \subset X$ is an invariant set of $(X, \mathbb{T}_+, \pi)$, then by Lemma 1 the non-autonomous set $\{I_y|\ y \in Y\}$ $(I_y = pr_1 J_y)$ is φ invariant. Since $\bigcup\{I_y|\ y \in Y\} = pr_1\mathbb{J}$, then the set $\bigcup\{I_y|\ y \in Y\}$ is pre-compact. Now we will establish that the φ invariant non-autonomous set $\{I_y|\ y \in Y\}$ is maximal. In fact, l et $\{I'_y|\ y \in Y\}$ be a non-autonomous set possessing the following properties:

a. the set $\bigcup\{I'_y|\ y \in Y\}$ is pre-compact;

b. the non-autonomous set $\{I'_y|\ y \in Y\}$ is φ invariant;

c. $I_y \subseteq I'_y$ for any $y \in Y$.

According to Lemma 1 the set $\mathbb{J}' = \bigcup\{I'_y \times \{y\}|\ y \in Y\} \subseteq X$ is invariant with respect to skew-product dynamical system $(X, \mathbb{T}_+, \pi)$. By Lemma 2 the set $\mathbb{J}'$ is conditionally pre-compact. From condition *c.* we have $\mathbb{J} \subseteq \mathbb{J}'$. Taking into consideration the fact that $\mathbb{J}$ is a maximal conditionally pre-compact invariant set of $(X, \mathbb{T}_+, \pi)$ we conclude that $\mathbb{J}' \subseteq \mathbb{J}$ and, consequently, $I'_y \subseteq I_y$ for any $y \in Y$, i.e., $I'_y = I_y$ for any $y \in Y$. Theorem is proved. ☐

4 Pullback Attractors of Cocycles

Let $M = \{M_y|\ y \in Y\}$ be a non-autonomous set and

$$\omega_y(M) := \bigcap_{t \geq 0} \overline{\bigcup_{\tau \geq t} \varphi(\tau, M_{\sigma(-\tau,y)}, \sigma(-\tau, y))}$$

for any $y \in Y$.

Lemma 5. *Let* Y *be a complete metric space,* $(Y, \mathbb{T}, \sigma)$ *be a two-sided dynamical system and* $\langle W, \varphi, (Y, \mathbb{T}, \sigma)\rangle$ *be a cocycle over dynamical system* $(Y, \mathbb{T}, \sigma)$ *with the fiber* W*. Then the following statements hold:*

1. *the point* $p \in \omega_y(M)$ *if and only if there exit* $t_k \to +\infty$ *and* $u_k \in M_{\sigma(-t_k,y)}$ *such that* $p = \lim\limits_{k \to +\infty} \varphi(t_k, u_k, \sigma(-t_k, y))$;

2. *$U(t,y)\omega_y(M) \subseteq \omega_{\sigma(t,y)}(M)$ for any $y \in Y$ and $t \in \mathbb{T}_+$, where $U(t,y) := \varphi(t,\cdot,y)$;*

3. *if $M = \{M_y |\ y \in Y\}$ is a compact non-autonomous set such that $\bigcup\{M_y |\ y \in Y\}$ is pre-compact, then for any point $w \in \omega_y(M)$ the motion $\varphi(t,w,y)$ is defined on $\mathbb{S}$;*

4. *if there exits a compact non-autonomous set $K = \{K_y |\ y \in Y\}$ such that $K_y \neq \emptyset$ for any $y \in Y$ and*

$$\lim_{t\to+\infty} \beta(\varphi(t, M_{\sigma(-t,y)}, \sigma(-t,y)), K_y) = 0, \tag{6}$$

then $\omega_y(M) \neq \emptyset$, is compact,

$$\lim_{t\to+\infty} \beta(\varphi(t, M_{\sigma(-t,y)}, \sigma(-t,y)), \omega_y(M)) = 0 \tag{7}$$

and

$$U(t,y)\omega_y(M) = \omega_{\sigma(t,y)}(M) \tag{8}$$

for any $y \in Y$ and $t \in \mathbb{S}_+$.

Proof. The first statement of Lemma follows directly from equality (6).

Let $w \in \omega_y(M)$, then there exit $t_k \to +\infty$ and $u_k \in M_{\sigma(-t_k,y)}$ such that

$$w = \lim_{k\to+\infty} \varphi(t_k, u_k, \sigma(-t_k,y))$$

and, hence,

$$\varphi(t,w,y) = \lim_{k\to+\infty} \varphi(t, \varphi(t_k, u_k, \sigma(-t_k,y), y) = \lim_{k\to+\infty} \varphi(t + t_k, u_k, \sigma(-t_k - t, \sigma(t,y))). \tag{9}$$

Since $u_k \in M_{\sigma(-t_k,y)} = M_{\sigma(-t_k-t,\sigma(t,y))}$, then $\varphi(t,w,y) \in \omega_{\sigma(t,y)}(M)$, that is $U(t,y)\omega_y(M) \subseteq \omega_{\sigma(t,y)}(M)$ for any $y \in Y$ and $t \in \mathbb{T}_+$.

From the equality (9), it follows that the motion $\varphi(t,w,y)$ is defined on $\mathbb{S}$ like $\varphi(t + t_k, u_k, \sigma(-t_k,y))$ is defined on $[-t_k, +\infty)$ and $t_k \to +\infty$.

Let us show that from (6) it follows that $\omega_y(M) \neq \emptyset$ if $\emptyset \neq K_y \in C(W)$ for any $y \in Y$. Since $M_y \in C(W)$ for any $y \in Y$, $u_k \in M_{\sigma(-t_k,y)}$ and $t_k \to +\infty$, then according to (6) the sequence $\{\varphi(t_k, u_k, \sigma(-t_k,y))\}$ can be considered convergent. Assume

$$\overline{u} = \lim_{k\to+\infty} \varphi(t_k, u_k, \sigma(-t_k,y)),$$

then $\overline{u} \in \omega_y(M)$ and consequently $\omega_y(M) \neq \emptyset$.

Let us show $\omega_y(M)$ is compact. Let $\varepsilon_k \downarrow 0$ and $\{v_k\} \subseteq \omega_y(M)$, then there exist $t_k \to +\infty$ as $k \to \infty$ and $u_k \in M$ such that

$$\rho(\varphi(t_k, u_k, \sigma(-t_k,y)), v_k) < \varepsilon_k.$$

According to the condition (6), the sequence $\{\varphi(t_k, u_k, \sigma(-t_k, y)\}$ is relatively compact and since $\varepsilon_k \downarrow 0$, $\{v_k\}$ also is relatively compact. By definition of $\omega_y(M)$ it is closed and, consequently, $\omega_y(M)$ is a compact subset of W.

Now we will establish equality (7). If we assume that it is not true, then there are $\varepsilon_0 > 0$, $t_k \to +\infty$ and $u_k \in M$ such that

$$\rho(\varphi(t_k, u_k, \sigma(-t_k, y)), \omega_y(M)) \geq \varepsilon_0$$

for any $k \in \mathbb{N}$. Taking into account (6) without loss of generality we can suppose that the sequence $\{\varphi(t_k, u_k, \sigma(-t_k, y))\}$ is convergent. Denote its limit by $\bar{u}$, then

$$\bar{u} \in \omega_y(M). \tag{10}$$

On the other hand passing to the limit in (7) as $k \to \infty$ we obtain

$$\rho(\bar{u}, \omega_y(M)) \geq \varepsilon_0. \tag{11}$$

Relations (10) and (11) are contradictory. The obtained contradiction proves our statement.

To establish equality (8) it is sufficient to show that $\omega_{\sigma(t,y)}(M) \subseteq \varphi(t, \omega_y(M), y)$. Let $u \in \omega_{\sigma(t,y)}(M)$, then there are $t_k \to +\infty$ and $u_k \in M_{\sigma(-t_k, \sigma(t,y))} = M_{\sigma(-t_k+t,y)}$ such that

$$\begin{aligned} u = \lim_{k\to\infty} \varphi(t_k, u_k, \sigma(-t_k, \sigma(t, y))) = \\ \lim_{k\to\infty} \varphi(t_k - t + t, u_k, \sigma(-t_k + t, y)) = \\ \lim_{k\to\infty} \varphi(t, \varphi(t_k - t, u_k, \sigma(-t_k, \sigma(t, y)), y). \end{aligned} \tag{12}$$

By (6) without loss of generality we can assume that the sequence

$$\{\varphi(t_k - t, u_k, \sigma(-t_k, \sigma(t, y))\} = \{\varphi(t_k - t, u_k, \sigma(-t_k + t, y)\}$$

converges. Denote its limit by u_t, then $u_t \in \omega_y(M)$ because $u_k \in M_{\sigma(-t_k+t,y)}$ for any $k \in \mathbb{N}$ and $t_k - t \to +\infty$ as $k \to \infty$. From (12) we obtain $u = \varphi(t, u_t, y) \in \varphi(t, \omega_y(M), y)$, i.e., $\omega_{\sigma(t,y)} \subseteq \varphi(t, \omega_y(M), y)$. $\square$

Remark 4. Lemma 5 remains true if instead of two-sided dynamical system $(Y, \mathbb{T}, \sigma)$ we consider a one-sided dynamical system $(Y, \mathbb{T}_+, \sigma)$ and the phase space Y is compact and invariant (i.e., $\sigma(t, Y) = Y$ for any $t \in \mathbb{T}_+$).

To establish this fact it is sufficient to note if $(Y, \mathbb{T}_+, \sigma)$ is a one-sided dynamical system and the phase space Y is compact and invariant, then on Y a two-sided set-valued dynamical system $(Y, \mathbb{T}, \tilde{\sigma})$ is defined such that $\tilde{\sigma}(t, y) = \sigma(t, y)$ for any $(t, y) \in \mathbb{T}_+ \times Y$. Now to obtain the proof of Remark 4 it is sufficient to apply Lemma 3.8.1 from [7, Ch.III] (see also [19]).

Definition 9. A cocycle $\langle W, \varphi, (Y, \mathbb{S}, \sigma)\rangle$ is said to be compact (respectively, uniformly compact) pullback dissipative, if there exists a uniformly compact non-autonomous set $K = \{K_y |\ y \in Y\}$ possessing the following properties:

1. $K_y \neq \emptyset$ for any $y \in Y$;

2. for any compact non-autonomous set $M = \{M_y|\ y \in Y\}$ we have

$$\lim_{t\to+\infty} \beta(\varphi(t, M_{\sigma(-t,y)}, \sigma(-t,y)), K_y) = 0 \tag{13}$$

for any $y \in Y$ (respectively, equality (13) takes place uniformly with respect to $y \in Y$).

Denote by $\mathcal{B}(W)$ (respectively, by $\mathbb{B}(W)$) the family of all bounded (respectively, bounded and closed) subsets of W.

Recall that for any nonempty subsets $A, B \subseteq \mathbb{B}(W)$ we define $\beta(A,B) := \sup\{\rho(a,B)|\ a \in A\}$, where $\rho(a,B); = \inf\{\rho(a,b)|\ b \in B\}$.

Lemma 6. *Let $B_1 \subseteq B_2$ be two arbitrary nonempty closed bounded subsets of W, then for any $A \in W$ we have $\beta(A,B_2) \leq \beta(A,B_1)$.*

Proof. Let $a \in A$, then

$$\rho(a,B_2) = \inf\{\rho(a,B_2)|\ b \in B_2\} \leq \inf\{\rho(a,B_1)|\ b \in B_1\} = \rho(a,B_1) \tag{14}$$

and, consequently, from (14) we obtain

$$\beta(A,B_2) = \sup\{\rho(a,B_2)|\ a \in A\} \leq \sup\{\rho(a,B_1)|\ a \in A\} = \beta(A,B_1).$$

Lemma is proved. $\square$

Theorem 3. *[4, Part I], [5], [21] Let $\langle W, \varphi, (Y, \mathbb{T}, \sigma)\rangle$ be a compactly pullback dissipative cocycle and K be the nonempty compact non-autonomous set appearing in (13), then:*

1. *$I_y := \omega_y(K) \neq \emptyset$, is compact, $I_y \subseteq K_y$ and*
$$\lim_{t\to+\infty} \beta(U(t,\sigma(-t,y)))K_{\sigma(-t,y)}, I_y) = 0$$
for every $y \in Y$;

2. *$U(t,y)I_y = I_{\sigma(t,y)}$ for any $y \in Y$ and $t \in \mathbb{T}_+$;*

3. $$\lim_{t\to+\infty} \beta(U(t,\sigma(-t,y))M_{\sigma(-t,y)}, I_y) = 0$$
for any compact non-autonomous set $M = \{M_y|\ y \in Y\}$, i.e., $I = \{I_y\}$ is a compact pullback attractor of the cocycle φ;

4. *if the set $\bigcup\{K_y|\ y \in Y\}$ is pre-compact, then*

   (a) the set
   $$\mathcal{I} := \bigcup\{I_y|\ y \in Y\}$$
   is a pre-compact subset of W;

(b)

$$\lim_{t\to+\infty}\beta(U(t,\sigma(-t,y))M_{\sigma(-t,y)},\mathcal{I})=0 \tag{15}$$

for any compact non-autonomous set $M=\{M_y|\ y\in Y\}$ *and* $y\in Y$*;*

5. *if the space* Y *is compact and the cocycle* φ *is uniform compactly pullback dissipative then*

   (a) *the set* $\mathcal{I}$ *is closed and, consequently,* $\mathcal{I}=\mathfrak{I}$*, where*

$$\mathfrak{I}:=\overline{\bigcup\{I_y|\ y\in Y\}};$$

   (b)

$$\lim_{t\to+\infty}\sup\{\beta(U(t,\sigma(-t,y))M_{\sigma(-t,y)},\mathfrak{I})\ |\ y\in Y\}=0 \tag{16}$$

for any compact non-autonomous set $M=\{M_y|\ y\in Y\}$*.*

Proof. The first three assertions of the theorem follow from Lemma 5.

To establish the fourth statement we note that $I_y=\omega_y(M)\subseteq K_y\subseteq\bigcup\{K_y|\ y\in Y\}$ and, consequently, $\bigcup\{I_y|\ y\in Y\}$ is a pre-compact subset of W.

Let us prove now equality (15). Since $I_y\subseteq K_y\subseteq\mathfrak{I}$ for any $y\in Y$, then by Lemma 6 we have

$$\beta(U(t,\sigma(-t,y))M_{\sigma(-t,y)},\mathfrak{I})\le\beta(U(t,\sigma(-t,y))M_{\sigma(-t,y)},I_y). \tag{17}$$

Passing to the limit in (17) as $t\to+\infty$ we obtain (15).

Since the space Y is compact, then by Theorem 1 the set $\mathbb{J}=\bigcup\{I_y\times\{y\}|\ y\in Y\}$ is a maximal, compact invariant set of the skew-product dynamical system $(X,\mathbb{T},\pi)$. Since $\mathcal{I}=pr_1(\mathbb{J})$ and the map $pr_1:X\to W$ is continuous, then the set $\mathcal{I}$ is compact and, consequently, it is closed. Thus we have

$$\bigcup\{I_y|\ y\in Y\}=\mathcal{I}=\overline{\mathcal{I}}=\overline{\bigcup\{I_y|\ y\in Y\}}=\mathfrak{I}.$$

Now we will prove equality (16). Assuming that it is false we will have a positive number ε_0, non-autonomous set $M_0=\{M_y^0|\ y\in Y\}$, $y_k\in Y$, $u_k\in M^0_{y_k}$ and $t_k\to+\infty$ such that

$$\rho(U(t_k,y_k^{-t_k})u_k,\mathfrak{I})\ge\varepsilon_0 \tag{18}$$

and, consequently,

$$\rho(U(t_k,\sigma(-t_k,y_k))u_k,I_y)\ge\varepsilon_0$$

for any $y\in Y$, because $I_y\subseteq\mathfrak{I}$ and $\rho(u,\mathfrak{I})\le\rho(u,I_y)$ $(u\in W$ and $y\in Y)$.

Since Y is compact, then we can assume that the sequences $\{y_k\}$ and $\{y_kt_k\}$ are convergent. Suppose $y_0:=\lim\limits_{k\to+\infty}y_k$ and $\overline{y}:=\lim\limits_{k\to+\infty}y_kt_k$. According to (6), for given $\varepsilon_0>0$ and $y_0\in Y$, there exists $t_0=t_0(\varepsilon_0,y_0)$ such that

$$\beta(U(t,\sigma(t,y_0))M_{\sigma(-t,y_0},I_{y_0})<\frac{\varepsilon_0}{2} \tag{19}$$

for all $t \geq t_0(\varepsilon_0, y_0)$. Let us note that

$$U(t_k, \sigma(-t_k, y_k))u_k = U(t_0, \sigma(-t_0, y_k))U(t_k - t_0, \sigma(-t_k, y_k))u_k. \tag{20}$$

As $\langle W, \varphi, (Y, \mathbb{T}, \sigma)\rangle$ is uniformly compactly dissipative, then the sequence $\{U(t_k, \sigma(-t_k, y_k))u_k\}$ may be considered a convergent one. Suppose $u' = \lim\limits_{k\to+\infty} \varphi(t_k - t_0, u_k, y_k^{t_k})$ and let us notice that according to (13) $u' \in \mathcal{K} := \bigcup\{K_y |\ y \in Y\}$. From the equality (20), it follows that $U(t_k, \sigma(-t_k, y_k))u_k \to U(t_0, \sigma(-t_0, y_0))u'$ and, hence, from (18) we have

$$U(t_0, \sigma(-t_0, y_0))u' \notin B(I_{y_0}, \varepsilon_0).$$

On the other hand, from (19) and from $u' \in \mathcal{K}$, it follows that

$$U(t_0, \sigma(-t_0, y_0))u' \in B(I_{y_0}, \frac{\varepsilon_0}{2}).$$

The last inclusion contradicts (19), and this finishes the proof of the fourth assertion. The theorem is completely proved. $\square$

Remark 5. Theorem 3 remains true if instead of two-sided dynamical system $(Y, \mathbb{T}, \sigma)$ we consider a one-sided dynamical system $(Y, \mathbb{T}_+, \sigma)$ and the phase space Y is compact and invariant (i.e., $\sigma(t, Y) = Y$ for any $t \in \mathbb{T}_+$).

This fact can be proved with the slight modifications of the proof of Lemma 3 and using Remark 4.

5 Global Attractors of Cocycles

Definition 10. A cocycle φ over $(Y, \mathbb{T}, \sigma)$ with the fiber W is said to be compactly dissipative (respectively, uniformly compact dissipative) if there exits a nonempty compact $K \subseteq W$ such that

$$\lim_{t\to+\infty} \beta(U(t, y)M, K) = 0 \tag{21}$$

for any $M \in C(W)$ and $y \in Y$ (respectively, uniformly with respect to $y \in Y$).

Remark 6. Let K be a pre-compact subset of W such that (21) holds then

$$\lim_{t\to+\infty} \beta(U(t, y)M, \overline{K}) = 0$$

for any $M \in C(W)$ and $y \in Y$ (respectively, uniformly with respect to $y \in Y$), where by $\overline{K}$ the closure of K is denoted.

In fact, since

$$\beta(U(t, y)M, \overline{K}) \leq \beta(U(t, y)M, K) + \beta(K, \overline{K}) = \beta(U(t, y)M, K) \to 0$$

as $t \to +\infty$ for any $M \in C(W)$ and $y \in Y$ (respectively, uniformly with respect to $y \in Y$), because $K \subseteq \overline{K}$ and hence $\beta(K, \overline{K}) = 0$.

Denote by ρ_W (respectively, ρ_Y) the distance on the metric space W (respectively, Y), $X := W \times Y$, $pr_1 : X \to W$ (respectively, $pr_2 : X \to Y$) is the first (respectively, second) projection and $\rho_X := \rho_W + \rho_Y$ (i.e., $\rho_X((u_1, y_1), (u_2, y_2)) := \rho_W(u_1, u_2) + \rho_Y(y_1, y_2)$ for any $(u_i, y_i) \in W \times Y$ and $i = 1, 2$).

Remark 7. When there is no risk of misunderstanding, we will omit the index in the notation of the metric ρ_X (respectively, ρ_Y and ρ_W).

Theorem 4. *[5], [9, Ch.II] Let Y be a compact metric space, then the following statements are equivalent:*

1. *the cocycle $\langle W, \varphi, (Y, \mathbb{T}, \sigma)\rangle$ is uniformly compactly dissipative;*
2. *the skew-product dynamical system $(X, \mathbb{T}, \pi)$ $(X := W \times Y, \pi = (\varphi, \sigma))$ is compact dissipative.*

Proof. Let K be a nonempty compact subset of W figuring in (21) and M be an arbitrary compact subset of W. Denote by $\mathcal{K} := K \times Y$ and $\mathcal{M} := M \times Y$. Since $\beta(\pi(t, (u, y)), \mathcal{K}) \leq \beta(\varphi(t, u, y), K)$ then we have

$$\beta(\pi(t, \mathcal{M}), \mathcal{K}) \leq \sup_{(u,y)\in M\times Y} \beta(\varphi(t, u, y), K) \leq \sup_{(y\in Y} \beta(\varphi(t, M, y), K). \tag{22}$$

From (21) and (22) we obtain

$$\lim_{t\to+\infty} \beta(\pi(t, \mathcal{M}), \mathcal{K}) = 0 \tag{23}$$

for any $\mathcal{M}$.

Let now P be an arbitrary compact subset of W. Since Y is a compact metric space, then $M := pr_1(P)$ is a compact subset of W and $P \subseteq \mathcal{M} = M \times Y$ and, consequently,

$$\beta(\pi(t, P), \mathcal{K}) \leq \beta(\pi(t, \mathcal{M}), \mathcal{K}) \tag{24}$$

for any $P \in C(X)$. From (23) and (24) we obtain

$$\lim_{t\to+\infty} \beta(\pi(t, P), \mathcal{K}) = 0$$

for any $P \in C(X)$, that is, the skew-product dynamical system $(X, \mathbb{T}, \pi)$ is compactly dissipative.

Conversely, suppose that the skew-product dynamical system $(X, \mathbb{T}, \pi)$ is compactly dissipative, then there exists a non-empty compact subset $A_0 \in C(X)$ such that

$$\lim_{t\to+\infty} \beta(\pi(t, A), A_0) = 0 \tag{25}$$

for any $A \in C(X)$. Denote by $K := pr_1(A_0)$, then K is a nonempty compact subset of W because $X = W \times Y$ and Y is compact. Let $M \in C(W)$ be an arbitrary compact subset of W and $(u, y) \in \mathcal{M} = M \times Y$. Since $\rho_W(\varphi(t, u, y), K) \le \rho_X(\pi(t, (u, y)), A_0)$, then we obtain

$$\sup_{u \in M} \rho_W(\varphi(t, u, y), K) \le \sup_{u \in M} \rho_X(\pi(t, (u, y)), A_0)$$

and

$$\sup_{y \in Y} \sup_{u \in M} \rho_W(\varphi(t, u, y), K) \le \sup_{y \in Y} \sup_{u \in M} \rho_X(\pi(t, (u, y)), A_0) \le \beta(\pi(t, A), A_0). \quad (26)$$

From (25) and (26) we receive

$$\lim_{t \to +\infty} \sup_{y \in Y} \beta(\varphi(t, M, y), K) = \lim_{t \to +\infty} \sup_{y \in Y} \sup_{u \in M} \rho_W(\varphi(t, u, y), K) = 0 \quad (27)$$

for any $M \in C(W)$ which means that the cocycle φ is uniform compact dissipative. Theorem is proved. □

Definition 11. A non-autonomous set $I = \{I_y |\ y \in Y\}$ is said to be a compact global attractor for the cocycle $\langle W, \varphi, (Y, \mathbb{T}, \sigma) \rangle$ if it possesses the following properties:

1. the set $\mathcal{I} := \bigcup \{I_y |\ y \in Y\}$ is pre-compact;
2. $\{I_y |\ y \in Y\}$ is invariant, i.e., $\varphi(t, I_y, y) = I_{\sigma(t,y)}$ for any $(t, y) \in \mathbb{T} \times Y$;
3. $$\lim_{t \to +\infty} \sup_{y \in Y} \beta(\varphi(t, M, y), \mathcal{I}) = 0$$
   for any $M \in C(W)$, where $\mathcal{I} = \bigcup \{I_y |\ y \in Y\}$.

Theorem 5. *Let Y be a compact metric space, Y be invariant (i.e., $\sigma(t, Y) = Y$ for any $t \in \mathbb{T}$) and φ be a cocycle over $(Y, \mathbb{T}, \sigma)$ with the fiber W. If the cocycle φ is uniformly compactly dissipative, then it has a compact global attractor.*

Proof. Let Y be a compact metric space and φ be uniformly compactly dissipative. Then there exists a nonempty compact subset $K \subseteq W$ such that (21) holds for every $M \in C(W)$. Reasoning as in the proof of Theorem 4 we conclude that the compact $\mathcal{K} := K \times Y \in C(X)$ attracts every compact subset $P \in C(X)$. In particular, $\mathcal{K}$ attracts itself and, consequently, $\omega(\mathcal{K}) \subseteq \mathcal{K}$. Since $(X, \mathbb{T}, \pi)$ is compactly dissipative and $\mathcal{K}$ attracts every compact subset from X, then $J := \omega(\mathcal{K})$ is its Levinson center. For any $y \in Y$ we denote by $I_y := pr_1(J_y)$, where $J_y := pr_2^{-1}(y) \bigcap J$. Note that the non-autonomous set $I = \{I_y |\ y \in Y\}$ possesses the following properties:

1. for any $y \in Y$ the set I_y is a non-empty compact subset of W;
2. the set $\mathcal{I}$ is closed and, consequently, $\mathcal{I} = \mathfrak{I}$;
3. $\{I_y |\ y \in Y\}$ is φ invariant, i.e., $\varphi(t, I_y, y) = I_{\sigma(t,y)}$ for any $(t, y) \in \mathbb{T} \times Y$;

4.
$$\lim_{t\to+\infty}\sup_{y\in Y}\beta(\varphi(t,M,y),\mathcal{I})=0$$

for any $M\in C(W)$.

The properties (i)-(iii) listed above of the non-autonomous set $I=\{I_y|\ y\in Y\}$ follow from the properties of the Levinson center. Namely, J is a nonempty, compact and invariant set of the skew-product dynamical system. Besides $pr_2(J)=Y$, $pr_1(J)=\mathcal{I}$ and $\mathcal{I}$ is a compact subset of W. Finally, the property (iv) follows from the equality (27). Theorem is completely proved. □

6 Forward Attractors for Cocycles

Lemma 7. *Suppose that the cocycle $\langle W,\varphi,(Y,\mathbb{T},\sigma)\rangle$ is uniformly compactly dissipative, then it is uniformly pullback compactly dissipative.*

Proof. Assume that the cocycle $\langle W,\varphi,(Y,\mathbb{T},\sigma)\rangle$ is uniformly compactly dissipative, then there exists a non-empty compact subset $K\in C(W)$ such that

$$\lim_{t\to+\infty}\sup_{y\in Y}\beta(\varphi(t,M,y),K)=0 \tag{28}$$

for any $M\in C(W)$. Since $\sigma(t,Y)=Y$ for any $t\in\mathbb{T}$, then $\sigma(-t,Y)=Y$ (if T is a semi-group, then $\sigma(-t,y):=\sigma(t,\cdot)^{-1}(y)$, where f^{-1} denotes the inverse mapping for the map $f:Y\to Y$) and, consequently,

$$\sup_{q\in Y}\beta(\varphi(t,M,q),K)=\sup_{y\in Y}\beta(\varphi(t,M,\sigma(-t,y)),K) \tag{29}$$

for any $M\in C(W)$. From (28) and (29) we obtain

$$\lim_{t\to+\infty}\sup_{y\in Y}\beta(\varphi(t,M,\sigma(-t,y)),K)=0$$

for any $M\in C(W)$. Lemma is proved. □

Theorem 6. *Let $\langle W,\varphi,(Y,\mathbb{S},\sigma)\rangle$ be uniformly compactly dissipative and K be the nonempty compact set appearing in the equality (21), then:*

1. *$I_y:=\omega_y(K)\neq\emptyset$, is compact, $I_y\subseteq K_y$ and*
$$\lim_{t\to+\infty}\beta(U(t,\sigma(-t,y)))K_{\sigma(-t,y)},I_y)=0$$
for every $y\in Y$;

2. *$U(t,y)I_y=I_{\sigma(t,y)}$ for any $y\in Y$ and $t\in\mathbb{S}_+$;*

3.

$$\lim_{t\to+\infty}\beta(U(t,\sigma(-t,y))M_{\sigma(-t,y)},I_y)=0$$

for any compact non-autonomous set $M=\{M_y|\ y\in Y\}$, i.e., $I=\{I_y\}$ is a compact pullback attractor of the cocycle φ;

4.

$$\lim_{t\to+\infty}\beta(U(t,\sigma(-t,y))M_{\sigma(-t,y)},\mathfrak{I})=0$$

for any compact non-autonomous set $M=\{M_y|\ y\in Y\}$), where

$$\mathfrak{I}:=\overline{\bigcup\{I_y\ |\ y\in Y\}};$$

5. if the cocycle φ is uniformly compactly pullback dissipative then

$$\lim_{t\to+\infty}\sup\{\beta(U(t,\sigma(-t,y))M_{\sigma(-t,y)},\mathfrak{I})\ |\ y\in Y\}=0$$

for any compact non-autonomous set $M=\{M_y|\ y\in Y\}$);

6. if Y is compact and the cocycle φ is compactly (respectively, uniformly compactly) pullback dissipative then

$$\lim_{t\to+\infty}\beta(U(t,\sigma(-t,y))M_{\sigma(-t,y)},\mathcal{I})=0$$

(respectively,

$$\lim_{t\to+\infty}\sup\{\beta(U(t,\sigma(-t,y))M_{\sigma(-t,y)},\mathcal{I})\ |\ y\in Y\}=0\ )$$

for any compact non-autonomous set $M=\{M_y|\ y\in Y\}$), where

$$\mathcal{I}:=\bigcup\{I_y\ |\ y\in Y\}.$$

Proof. This statement follows from Theorem 4 and Lemma 7. □

Remark 8. Theorem 6 remains true if instead of two-sided dynamical system $(Y,\mathbb{T},\sigma)$ we consider a one-sided dynamical system $(Y,\mathbb{T}_+,\sigma)$ and the phase space Y is compact and invariant (i.e., $\sigma(t,Y)=Y$ for any $t\in\mathbb{T}_+$).

This fact can be proved with the slight modifications of the proof of Theorem 6 and using Remark 5.

Definition 12. Let $\langle W,\varphi,(Y,\mathbb{T},\sigma)\rangle$ be compactly dissipative, K be the nonempty compact subset of W appearing in the equality (21) and $I_y:=\omega_y(K)$ for any $y\in Y$. The family of compact subsets $\{I_y|\ y\in Y\}$ is said to be a Levinson center (compact global attractor) of non-autonomous (cocycle) dynamical system $\langle W,\varphi,(Y,\mathbb{T},\sigma)\rangle$.

Remark 9. According to Theorem 6 by Definition 12 the notion of Levinson center (compact global attractor) for non-autonomous (cocycle) dynamical system $\langle W,\varphi,(Y,\mathbb{T},\sigma)\rangle$ is well defined.

Corollary 6. *Let $\langle W,\varphi,(Y,\mathbb{T},\sigma)\rangle$ be compactly dissipative non-autonomous dynamical system, $\{I_y|\ y\in Y\}$ be its Levinson center and $\gamma:\mathbb{T}\mapsto W$ be a pre-compact full trajectory of φ (i.e., $\gamma(\mathbb{T})$ is pre-compact subset and there exists a point $y_0\in Y$ such that $\gamma(t+s)=\varphi(t,\gamma(s),\sigma(s,y_0))$ for any $t\geq 0$ and $s\in\mathbb{T}$), then $\gamma(0)\in I_{y_0}$.*

Definition 13. A non-autonomous set $I=\{I_y|\ y\in Y\}$ is said to be a forward (respectively, a uniformly forward) compact global attractor for the cocycle $\langle W,\varphi,(Y,\mathbb{T},\sigma)\rangle$ if it possesses the following properties:

1. the set $\mathcal{I}:=\bigcup\{I_y|\ y\in Y\}$ is pre-compact;
2. $\{I_y|\ y\in Y\}$ is invariant;
3. $$\lim_{t\to+\infty}\sup_{y\in Y}\beta(\varphi(t,M,y),I_{\sigma(t,y)})=0 \tag{30}$$

   (respectively, equality (30) takes place uniformly w.r.t $y\in Y$)) for any $M\in C(W)$.

Lemma 8. *Assume that the cocycle $\langle W,\varphi,(Y,\mathbb{T},\sigma)\rangle$ has a forward (respectively, a uniformly forward) compact global attractor $I=\{I_y|\ y\in Y\}$, then it is compactly (respectively, uniformly compactly) dissipative.*

Proof. Denote by $\mathfrak{I}:=\overline{\bigcup\{I_y|\ y\in Y\}}$, then by condition of Lemma the set $\mathfrak{I}$ is a non-empty compact from W. Since $I_{\sigma(t,y)}\subseteq\mathfrak{I}$ for any $(t,y)\in\mathbb{T}\times Y$, then by Lemma 6 we have

$$\beta(\varphi(t,M,y),\mathfrak{I})\leq\beta(\varphi(t,M,y),I_{\sigma(t,y)}) \tag{31}$$

for any $(t,y)\in\mathbb{T}\times Y$ and $M\in C(W)$. From equality (31) we obtain

$$\lim_{t\to+\infty}\beta(\varphi(t,M,y),\mathfrak{I})=0$$

for any $y\in Y$ (respectively, uniformly w.r.t. $y\in Y$) and $M\in C(W)$. Lemma is proved. □

Theorem 7. *Suppose that the cocycle $\langle W,\varphi,(Y,\mathbb{T},\sigma)\rangle$ is compactly (respectively, uniformly compactly) dissipative and $\{I_y|\ y\in Y\}$ is its Levinson center. If the metric space Y is compact and the mapping $y\to I_y$ is lower semi-continuous. Then $\{I_y|\ y\in Y\}$ is a compact global forward (respectively, uniformly forward) attractor of cocycle φ.*

Proof. Assume that the cocycle $\langle W, \varphi, (Y, \mathbb{T}, \sigma)\rangle$ is compactly (respectively, uniformly compactly) dissipative and $\{I_y |\ y \in Y\}$ is its Levinson center. Let M be an arbitrary compact subset of W. We will show that

$$\lim_{t\to+\infty} \beta(\varphi(t, M, y), I_{\sigma(t,y)}) = 0 \tag{32}$$

(respectively,

$$\lim_{t\to+\infty} \sup_{y\in Y} \beta(\varphi(t, M, y), I_{\sigma(t,y)}) = 0\ ) \tag{33}$$

for any $y \in Y$.

Firstly we establish equality (32). If we suppose that (32) is not true, then there are $\varepsilon_0 > 0$, a sequence $t_k \to +\infty$, $y_0 \in Y$, $M_0 \in C(W)$ and a sequence $\{u_k\} \subseteq M_0$ such that

$$\rho(\varphi(t_k, u_k, y_0), I_{\sigma(t_k,y_0)}) \geq \varepsilon_0 \tag{34}$$

for any $k \in \mathbb{N}$. Since the space Y is compact, then without loss of generality we can suppose that the sequence $\{\sigma(t_k, y_0)\}$ converges. Denote its limit by $\bar{y}$. According to Theorem 4 the skew-product dynamical system $(X, \mathbb{T}, \pi)$ is compact dissipative and, consequently the set $\Sigma^+_{M_0} := \bigcup\{\pi(t, (M_0, Y)|\ t \geq 0\}$ is pre-compact in the space $X = W \times Y$. From this fact it follows that the sequence $\{\varphi(t_k, u_k, y_0)\}$ is pre-compact and, consequently we can suppose that it converges. Denote by $\bar{u} = \lim\limits_{k\to\infty} \varphi(t_k, u_k, y_0)$, then $(\bar{u}, \bar{y}) = \bar{x} \in J$ and, consequently,

$$\bar{u} \in I_{\bar{y}} \tag{35}$$

because $I_{\bar{y}} = pr_1(J_{\bar{y}})$, where $J_y := J \bigcap pr_2^{-1}(y)$ for any $y \in Y$. On the other hand from (34) we obtain

$$\varepsilon_0 \leq \rho(\varphi(t_k,_k, y_0), I_{\sigma(t_k,y_0)}) \leq \rho(\varphi(t_k, u_k, y_0), I_{\bar{y}}) + \beta(I_{\bar{y}}, I_{\sigma(t_k,y_0)}). \tag{36}$$

Passing to the limit in (36) as $k \to \infty$ and taking into consideration that the map $y \to I_y$ is lower semi-continuous we will have

$$\varepsilon_0 \leq \limsup_{k\to\infty} \rho(\varphi(t_k, u_k, y_0), I_{\bar{y}}).$$

This means, in particular, that there exists a subsequence $\{t_{k_m}\} \subseteq \{t_k\}$ such that

$$\rho(\varphi(t_{k_m}, u_{k_m}, y_0), I_{\bar{y}}) \geq \varepsilon_0/2 \tag{37}$$

for any $m \in \mathbb{N}$. The relations (35) and (37) are contradictory. The obtained contradiction proves our statement.

To finish the proof of Theorem we will prove equality (33). Assuming that it is false we will have a positive number ε_0, $M_0 \in C(W)$, sequences $\{y_k\} \subseteq Y, \{u_k\} \subseteq W, \{t_k\}) \subseteq \mathbb{T}$ such that $t_k \to +\infty$ as $k \to +\infty$ and

$$\rho(\varphi(t_k, u_k, y_k), I_{\sigma(t_k,y_k)}) \geq \varepsilon_0 \tag{38}$$

for any $k \in \mathbb{N}$. Without loss of generality we can suppose that the sequence $\{\sigma(t_k, y_k)\}$ converges because the space Y is compact. Denote by $\bar{y} = \lim_{k\to\infty} \sigma(t_k, y_k)$. Since the skew-product dynamical system $(X, \mathbb{T}, \pi)$ is compactly dissipative (see Theorem 4) then the set $\Sigma^+_{M_0} := \bigcup\{\pi(t, (M_0, Y)|\ t \geq 0\}$ is pre-compact and, consequently, the sequence $\{\varphi(t_k, u_k, y_k)\}$ is pre-compact. Without loss of generality we can suppose that it is convergent. Let $\bar{u} = \lim_{k\to\infty} \varphi(t_k, u_k, y_k)$, then $(\bar{u}, \bar{y}) = \bar{x} \in J$ and, consequently,

$$\bar{u} \in I_{\bar{y}}. \tag{39}$$

On the other hand from (38) we obtain

$$\varepsilon_0 \leq \rho(\varphi(t_{k,k}, y_k), I_{\sigma(t_k, y_k)}) \leq \rho(\varphi(t_k, u_k, y_k), I_{\bar{y}}) + \beta(I_{\bar{y}}, I_{\sigma(t_k, y_k)}). \tag{40}$$

Since the map $y \to I_y$ is lower semi-continuous, then passing to the limit in (40) as $k \to \infty$ we receive

$$\varepsilon_0 \leq \limsup_{k\to\infty} \rho(\varphi(t_k, u_k, y_k), I_{\bar{y}}). \tag{41}$$

From (41) it follows that there exists a subsequence $\{t_{k_m}\} \subseteq \{t_k\}$ such that

$$\rho(\varphi(t_{k_m}, u_{k_m}, y_{k_m}, I_{\bar{y}}) \geq \varepsilon_0/2 \tag{42}$$

for any $m \in \mathbb{N}$. The relations (39) and (42) are contradictory. The obtained contradiction proves our statement. Theorem is completely proved. □

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References

[1] Arnold L., *Random Dynamical Systems.* Springer-Verlag, 1998.

[2] Babin A. V. and Vishik M. I., Regular Attractor of Semi-Groups and Evolutional Equations. *Journal Math. Pures et Appl.*, **62**, 1983, pp.172-189.

[3] Bronsteyn I. U., *Extensions of Minimal Transformation Group. Kishinev,* Shtiintsa, 1974 (in Russian). [English translation: Extensions of Minimal Transformation Group, Sijthoff & Noordhoff, Alphen aan den Rijn. The Netherlands Germantown, Maryland USA, 1979]

[4] Carvalho A. N., Langa J. A. and Robinson J. C., Attractors for Infinite-Dimensional Nonautonomous Dynamical Systems. Applied Mathematical Sciences, Vol. 182. *Springer,* New York, 2013, xxxvi+409 pp.

[5] Cheban D. N., Global Attractors of Infinite-Dimensional Nonautonomous Dynamical Systems, I. *Bulletin of Academy of Sciences of Republic of Moldova.* Mathematics. 1997, N3 (25), pp. 42-55 (in Russian)

[6] Cheban D. N., Global Pullback Atttactors of C-Analytic Nonautonomous Dynamical Systems. *Stochastics and Dynamics.* 2001, v.1 , No.4, pp.511-535.

[7] Cheban D. N., *Global Attractors of Set-Valued Dynamical and Control Systems.* Nova Science Publishers Inc, New York, 2010, xvii+269 pp.

[8] Cheban D. N., *Global Attractors of Non-Autonomous Dissipstive Dynamical Systems.* Interdisciplinary Mathematical Sciences 1. River Edge, NJ: World Scientific, 2004, xxiii+502 pp.

[9] Cheban D. N., *Global Attractors of Nonautonomous Dynamical and Control Systems. 2nd Edition.* Interdisciplinary Mathematical Sciences, vol.18, River Edge, NJ: World Scientific, 2015, xxv+589 pp.

[10] Cheban D. N., Kloeden P. E. and Schmalfuss B., Relation Between Pullback and Global Attractors of Nonautonomous Dynamical Systems. *Nonlinear Dynamics and Systems Theory*, v. **2**, No.2 (2002), pp.9-28.

[11] Chepyzhov V. V. and Vishik M. I., *Attractors for Equations of Mathematical Physics.* Amer. Math. Soc., Providence, RI, 2002, xi+363 pp.

[12] Chueshov I. D., *Introduction to the Theory of Infinite-Dimensional Dissipative Systems. University Lectures in Contemporary Mathematics.* AKTA, Kharkiv, 1999. 436 pp. (in Russian)

[13] Flandoli F. and Schmalfuss B., Random Attractors for the Stochastic 3D Navier–Stokes Equation with Multiplicative White Noise, *Stochastics and Stochastics Reports*, **59** (1996), 21–45.

[14] Fort M. K., Points of continuity of semicontinuous functions. *Publ. Math.,* Debrecen, 2, 1951, pp.100-102.

[15] Gledininning Paul, Global attractors of pinched skew products. *Dynamical Systems,* Vol. 17, No. 3, 2002, pp. 287-294.

[16] Grebogi C., Ott E., Pelikan S. and Yorke J. A., Strange attractors that are not chaotic. *Physica D,* 13, 1984, pp.261-268.

[17] Hale J. K., *Asymptotic Behaviour of Dissipative Systems.* Amer. Math. Soc., Providence, RI, 1988, vii+196 pp.

[18] Haraux A., *Systèmes Dynamiques Dissipativs et Applications.* Masson, Paris–Milan–Barselona–Rome, 1991, x+130 pp.

[19] Kloeden P. E., Pullback attractors of non-autonomous semidynamical systems. *Stochstics and Dynamics,* Vol.3, No.1, 2003, pp.101-112.

[20] Peter E. Kloeden and Martin Rasmussen, *Nonautonomous Dynamical Systems* American Mathematical Society, Providence, Rhode Island, 2011, vii+264 pp.

[21] Kloeden P. E. and Schmalfuss B., Cocycle Attractors of Variable Time Step Discretizations of Lorenzian Systems. *J. Difference Eqns. Applns.*, **3** (1997), pp.125–145.

[22] Kuratovskii K., *Topology.* Mir, Moscow, 1966. 2 volumes, Vol. 1.

[23] Ladyzhenskaya O. A., *Attractors for Semigroups and Evolution Equations.* Lizioni Lincei, Cambridge Univ. Press, Cambridge, New-York, 1991, xi+73 p.

[24] Robinson J. C., *Infinite-Dimensional Dynamical Systems.* Cambridge University Press, 2001, xvii+461 pp.

[25] Sell G. R., *Lectures on Topological Dynamics and Differential Equations.* Van Nostrand–Reinbold, London, 1971, ix+199 pp.

[26] Temam T., *Infinite–Dimensional Dynamical Systems in Mechanics and Physics.* 2nd edition, Springer Verlag, New York, 1997.

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