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Subordination and superordination for certain analytic functions associated with Ruscheweyh derivative and a new generalised multiplier transformation

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Abstract. In the present paper, we study the operator defined by using Ruscheweyh derivative $\mathcal{R}^m$ and new generalized multiplier transformation

$$\mathcal{D}^m_{\lambda_1,\lambda_2,\ell,d}f(z) = z + \sum_{k=n+1}^{\infty}\left[\frac{\ell(1+(\lambda_1+\lambda_2)(k-1))+d}{\ell(1+\lambda_2(k-1))+d}\right]^m a_k z^k$$

denoted by $\mathcal{RD}^{m,\alpha}_{\lambda_1,\lambda_2,\ell,d} : \mathcal{A}_n \to \mathcal{A}_n$, $\mathcal{RD}^{m,\alpha}_{\lambda_1,\lambda_2,\ell,d}f(z) = (1-\alpha)\mathcal{R}^m f(z) + \alpha\mathcal{D}^m_{\lambda_1,\lambda_2,\ell,d}f(z)$, where $\mathcal{A}_n = \left\{f \in \mathcal{H}(\mathbb{U}), f(z) = z + a_{n+1}z^{n+1} + a_{n+2}z^{n+2} + ..., z \in \mathbb{U}\right\}$ is the class of normalized analytic functions with $\mathcal{A}_1 = \mathcal{A}$. We obtain several differential subordinations associated with the operator $\mathcal{RD}^{m,\alpha}_{\lambda_1,\lambda_2,\ell,d}f(z)$. Further, sandwich-type results for this operator are considered.

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1 Introduction

Let $\mathcal{A}$ denote the class of functions of the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k, \tag{1}$$

which are analytic in the unit disk $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$. Let $\mathcal{H}(\mathbb{U})$ be the space of holomorphic functions in $\mathbb{U}$. For $a \in \mathbb{C}$ and $n \in \mathbb{N}$ we denoted by

$$\mathcal{H}[a,n] = \left\{f \in \mathcal{H}(\mathbb{U}), f(z) = a + a_n z^n + a_{n+1}z^{n+1} + ..., z \in \mathbb{U}\right\},$$

and

$$\mathcal{A}_n = \left\{f \in \mathcal{H}(\mathbb{U}), f(z) = z + a_{n+1}z^{n+1} + a_{n+2}z^{n+2} + ..., z \in \mathbb{U}\right\},$$

with $\mathcal{A}_1 = \mathcal{A}$. A function $f \in \mathcal{H}(\mathbb{U})$ is said to be starlike in $\mathbb{U}$ if and only if $f'(0) \neq 0$ and $\Re e\left(\frac{zf'(z)}{f(z)}\right) > 0$. Further, a function $f \in \mathcal{H}(\mathbb{U})$ is said to be convex in $\mathbb{U}$ if and only if $f'(0) \neq 0$ and $\Re e\{1 + \frac{zf''(z)}{f'(z)}\} > 0, z \in \mathbb{U}$.

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If f and g are analytic functions in $\mathbb{U}$, we say that f is subordinate to g, (or g is superordinate to f), and write $f(z) \prec g(z)$ $(z \in \mathbb{U})$. If there exists a Schwarz function $w(z)$, analytic in $\mathbb{U}$, with $w(0) = 0$ and $|w(z)| < 1$, then $f(z) = g(w(z))$ $(z \in \mathbb{U})$. In particular if g is univalent in $\mathbb{U}$, then $f(z) \prec g(z)$ is equivalent to $f(0) = g(0)$ and $f(\mathbb{U}) \subset g(\mathbb{U})$.

The method of differential subordinations (also known as the admissible functions method) was perhaps first introduced by Miller and Mocanu in 1978 [20] and the theory started to develop in 1981 [17]. All the details can be found in a book written by Miller and Mocanu [18].

For our work, we may need the following definitions and lemmas. First, we state the following generalized derivative operator: Let $m, d \in \mathbb{N}_0 = \{0, 1, 2, ...\}$, $\lambda_2 \geq \lambda_1 \geq 0, \ell \geq 0$, and $\ell + d > 0$. Then, for $f \in \mathcal{A}_n$, the operator $\mathcal{D}^m_{\lambda_1,\lambda_2,\ell,d}$ is defined by $\mathcal{D}^m_{\lambda_1,\lambda_2,\ell,d} : \mathcal{A}_n \to \mathcal{A}_n$,

$$\mathcal{D}^0_{\lambda_1,\lambda_2,\ell,d} f(z) = f(z),$$

$$\mathcal{D}^1_{\lambda_1,\lambda_2,\ell,d} f(z) = \frac{[\ell(1 + \lambda_2(k-1) - \lambda_1) + d]\, \mathcal{D}^0_{\lambda_1,\lambda_2,\ell,d} f(z) + z\ell\lambda_1 (\mathcal{D}^0_{\lambda_1,\lambda_2,\ell,d} f(z))'}{\ell(1 + \lambda_2(k-1)) + d},$$

$$\mathcal{D}^2_{\lambda_1,\lambda_2,\ell,d} f(z) = \frac{[\ell(1 + \lambda_2(k-1) - \lambda_1) + d]\, \mathcal{D}^1_{\lambda_1,\lambda_2,\ell,d} f(z) + z\ell\lambda_1 (\mathcal{D}^1_{\lambda_1,\lambda_2,\ell,d} f(z))'}{\ell(1 + \lambda_2(k-1)) + d},$$

$$\vdots$$

$$\mathcal{D}^m_{\lambda_1,\lambda_2,\ell,d} f(z) = \mathcal{D}_{\lambda_1,\lambda_2,\ell,d}(\mathcal{D}^{m-1}_{\lambda_1,\lambda_2,\ell,d}) f(z)).$$

Remark 1. If $f(z) \in \mathcal{A}_n$ and $f(z) = z + \sum\limits_{k=n+1}^{\infty} a_k z^k$, then the linear operator

$$\mathcal{D}^m_{\lambda_1,\lambda_2,\ell,d} f(z) = z + \sum_{k=n+1}^{\infty} \left[\frac{\ell(1 + (\lambda_1 + \lambda_2)(k-1)) + d}{\ell(1 + \lambda_2(k-1)) + d}\right]^m a_k z^k. \tag{2}$$

It can be easily shown that

$$\begin{aligned}\big[\ell(1 + \lambda_2(k-1)) + d\big]\mathcal{D}^{m+1}_{\lambda_1,\lambda_2,\ell,d} f(z) = \\ \big[\ell(1 + \lambda_2(k-1) - \lambda_1) + d\big]\mathcal{D}^m_{\lambda_1,\lambda_2,\ell,d} f(z) + \ell\lambda_1 z\big(\mathcal{D}^m_{\lambda_1,\lambda_2,\ell,d} f(z)\big)'.\end{aligned} \tag{3}$$

We note that

- $\mathcal{D}^m_{1,0,1,d} f(z) = I^m_\lambda f(z)$ (see Cho and Srivastava [4]).
- $\mathcal{D}^m_{1,0,\ell,d} f(z) = I^m_{\alpha,\beta} f(z)$ (see Swamy [24]).
- $\mathcal{D}^m_{\lambda_1,0,1,d} f(z) = I^m(\lambda, \ell) f(z)$ (see Cătaş [3]).
- $\mathcal{D}^m_{\lambda_1,0,1,0} f(z) = D^m_\lambda f(z)$ (see Al-Oboudi [1]).

Definition 1. (Ruscheweyh [23]) For $f \in \mathcal{A}_n, m \in \mathbb{N}$, the operator $\mathcal{R}^m$ is defined by $\mathcal{R}^m : \mathcal{A}_n \to \mathcal{A}_n$,

$$\begin{aligned} \mathcal{R}^0 f(z) &= f(z), \\ \mathcal{R}^1 f(z) &= z f'(z), \\ &\vdots \\ (m+1)\mathcal{R}^{m+1} f(z) &= z(\mathcal{R}^m f(z))' + m\mathcal{R}^m f(z), \quad z \in \mathbb{U}. \end{aligned} \tag{4}$$

Remark 2. If $f(z) \in \mathcal{A}_n$ and $f(z) = z + \sum\limits_{k=n+1}^{\infty} a_k z^k$, then the linear operator $\mathcal{R}^m f(z) = z + \sum\limits_{k=n+1}^{\infty} C_{m+k-1}^{m} a_k z^k, z \in \mathbb{U}$.

Definition 2. Let $m, d \in \mathbb{N}_0 = \{0, 1, 2, ...\}, \lambda_2 \geq \lambda_1 \geq 0, \ell \geq 0, \ell + d > 0$ and $\alpha \geq 0$. Denote by $\mathcal{RD}_{\lambda_1,\lambda_2,\ell,d}^{m,\alpha}$ the operator given by $\mathcal{RD}_{\lambda_1,\lambda_2,\ell,d}^{m,\alpha} : \mathcal{A}_n \to \mathcal{A}_n$,

$$\mathcal{RD}_{\lambda_1,\lambda_2,\ell,d}^{m,\alpha} f(z) = (1-\alpha)\mathcal{R}^m f(z) + \alpha \mathcal{D}_{\lambda_1,\lambda_2,\ell,d}^{m} f(z), z \in \mathbb{U}.$$

Remark 3. If $f(z) \in \mathcal{A}_n$ and $f(z) = z + \sum\limits_{k=n+1}^{\infty} a_k z^k$, then

$$\begin{aligned} &\mathcal{RD}_{\lambda_1,\lambda_2,\ell,d}^{m,\alpha} f(z) \\ &= z + \sum_{k=n+1}^{\infty} \left\{ (1-\alpha) C_{m+k-1}^{m} + \alpha \left[\frac{\ell(1 + (\lambda_1 + \lambda_2)(k-1)) + d}{\ell(1 + \lambda_2(k-1)) + d} \right]^m \right\} a_k z^k. \end{aligned}$$

Remark 4. The operator $\mathcal{RD}_{1,0,\ell,d}^{m,\alpha} f(z) = RI_{\alpha,\beta,\lambda}^{m} f(z)$ was studied in [25, 26]. The operator $\mathcal{RD}_{\lambda_1,0,1,0}^{m,\alpha} f(z) = RD_{\lambda,\alpha}^{m} f(z)$ was studied in [7–10]. The operator $\mathcal{RD}_{\lambda_1,0,1,d}^{m,\alpha} f(z) = RI_{m,\lambda,\ell}^{\alpha}$ was studied in [11] whereas operator $\mathcal{RD}_{1,0,1,0}^{m,\alpha} f(z) = L_{\alpha}^{m} f(z)$ was studied in [12–15].

Also we note that:
For $\alpha = 0, \mathcal{RD}_{\lambda_1,\lambda_2,\ell,d}^{m,0} f(z) = \mathcal{R}^m f(z)$, where $z \in \mathbb{U}$.
For $\alpha = 1, \mathcal{RD}_{\lambda_1,\lambda_2,\ell,d}^{m,0} f(z) = \mathcal{D}_{\lambda_1,\lambda_2,\ell,d}^{m} f(z)$, where $z \in \mathbb{U}$.
For $m = 0$,
$\mathcal{RD}_{\lambda_1,\lambda_2,\ell,d}^{0,\alpha} f(z) = (1-\alpha)\mathcal{R}^0 f(z) + \alpha \mathcal{D}_{\lambda_1,\lambda_2,\ell,d}^{0} f(z) = f(z) = \mathcal{R}^0 f(z) = \mathcal{D}_{\lambda_1,\lambda_2,\ell,d}^{0} f(z)$, where $z \in \mathbb{U}$.

Definition 3. [19] Denote by Q the set of functions f that are analytic and injective on $\overline{\mathbb{U}} \setminus \mathbf{E}(f)$, where

$$\mathbf{E}(f) = \left\{ \eta \in \partial\mathbb{U} : \lim_{z \to \eta} f(z) = \infty \right\},$$

and are such that $f'(\eta) \neq 0, \eta \in \partial\mathbb{U} \setminus \mathbf{E}(f)$.

Lemma 1. [18] *Let q be univalent function in $\mathbb{U}$ and let θ and ϕ be analytic functions in a domain D containing $q(\mathbb{U})$, with $\phi(w) \neq 0$ when $w \in q(\mathbb{U})$.*

Set

$$Q(z) = zq'(z)\phi\big[q(z)\big], \quad h(z) = \theta\,[q(z)] + Q(z).$$

Suppose that
(i) $Q(z)$ is starlike univalent in $\mathbb{U}$,
(ii) $\Re e\left\{\dfrac{zh'(z)}{Q(z)}\right\} > 0$. If p is analytic in $\mathbb{U}$, with $p(0) = q(0), p(\mathbb{U}) \subset D$ and

$$\theta\big[p(z)\big] + zp'(z)\phi\big[p(z)\big] \prec \theta\big[q(z)\big] + zq'(z)\phi\big[q(z)\big], \tag{5}$$

then $p \prec q$ and q is the best dominant of (5).

Lemma 2. [2] *Let q be convex univalent in the unit disc $\mathbb{U}$ and ϑ and φ be analytic in a domain D containing $q(\mathbb{U})$. Suppose that*
(i) $\Re e\left\{\dfrac{\vartheta'[q(z)])}{\varphi[q(z)]}\right\} > 0$,
(ii) $zq'(z)\varphi(q(z))$ is starlike univalent in $\mathbb{U}$.

If $p(z) \in \mathcal{H}[q(0), 1] \cap Q$, with $p(\mathbb{U}) \subseteq D$, and $\vartheta[p(z)] + zp'(z)\varphi[p(z)]$ is univalent in $\mathbb{U}$, and

$$\vartheta[q(z)] + zq'(z)\varphi[q(z)] \prec \vartheta[p(z)] + zp'(z)\varphi[p(z)]$$

then

$$q(z) \prec p(z), \quad (z \in \mathbb{U})$$

and $q(z)$ is the best subordinant.

2 Main results

Theorem 1. *Let $f \in \mathcal{A}_n, m, d \in \mathbb{N}_0, \lambda_2 \geq \lambda_1 \geq 0, \ell \geq 0, \ell + d > 0, \delta > 0$ and $\alpha \geq 0$. Let the function q be univalent in $\mathbb{U}$ and suppose that it satisfies the conditions*

$$\Re e\{q(z)\} > 0, \quad z \in \mathbb{U}, \tag{6}$$

and

$$\Re e\left[\frac{zq''(z)}{q'(z)} - \frac{zq'(z)}{q(z)} + 1\right] > 0, z \in \mathbb{U}. \tag{7}$$

Let

$$\Psi(\delta, m, \alpha, \lambda_1, \lambda_2, \ell, d) = \left(\frac{\mathcal{RD}^{m,\alpha}_{\lambda_1,\lambda_2,\ell,d}f(z)}{z}\right)^{\delta} + \delta(m+1)\frac{\mathcal{RD}^{m+1,\alpha}_{\lambda_1,\lambda_2,\ell,d}f(z)}{\mathcal{RD}^{m,\alpha}_{\lambda_1,\lambda_2,\ell,d}f(z)}$$
$$+ \delta\alpha\Bigg[\left(\frac{\big[\ell(1+\lambda_2(k-1)) + d\big]}{\ell\lambda_1} - (m+1)\right)\frac{\mathcal{D}^{m+1}_{\lambda_1,\lambda_2,\ell,d}f(z)}{\mathcal{RD}^{m,\alpha}_{\lambda_1,\lambda_2,\ell,d}f(z)}$$

$$-\left(\frac{\left[\ell(1+\lambda_2(k-1)-\lambda_1)+d\right]}{\ell\lambda_1}-m\right)\frac{\mathcal{D}^{m}_{\lambda_1,\lambda_2,\ell,d}f(z)}{\mathcal{RD}^{m,\alpha}_{\lambda_1,\lambda_2,\ell,d}f(z)}\Bigg]-\delta(m+1). \quad (8)$$

If

$$\Psi(\delta,m,\alpha,\lambda_1,\lambda_2,\ell,d)\prec q(z)+\frac{zq'(z)}{q(z)},\quad z\in\mathbb{U}, \quad (9)$$

then $\left(\frac{\mathcal{RD}^{m,\alpha}_{\lambda_1,\lambda_2,\ell,d}f(z)}{z}\right)^{\delta}\prec q(z)$ *and* q *is the best dominant.*

Proof. Define the function $p(z)$ by

$$p(z)=\left(\frac{\mathcal{RD}^{m,\alpha}_{\lambda_1,\lambda_2,\ell,d}f(z)}{z}\right)^{\delta},z\in\mathbb{U}. \quad (10)$$

By logarithmically differentiating both sides of (10) with respect to z, and multiplying the resulting equation by z, we obtain

$$\frac{zp'(z)}{p(z)}=\delta\left[\frac{z(\mathcal{RD}^{m,\alpha}_{\lambda_1,\lambda_2,\ell,d}f(z))'}{\mathcal{RD}^{m,\alpha}_{\lambda_1,\lambda_2,\ell,d}f(z)}-1\right]. \quad (11)$$

Using (3) and (1), (11) becomes

$$p(z)+\frac{zp'(z)}{p(z)}=\left(\frac{\mathcal{RD}^{m,\alpha}_{\lambda_1,\lambda_2,\ell,d}f(z)}{z}\right)^{\delta}+\delta(m+1)\frac{\mathcal{RD}^{m+1,\alpha}_{\lambda_1,\lambda_2,\ell,d}f(z)}{\mathcal{RD}^{m,\alpha}_{\lambda_1,\lambda_2,\ell,d}f(z)}$$
$$+\delta\alpha\Bigg[\left(\frac{\left[\ell(1+\lambda_2(k-1))+d\right]}{\ell\lambda_1}-(m+1)\right)\frac{\mathcal{D}^{m+1}_{\lambda_1,\lambda_2,\ell,d}f(z)}{\mathcal{RD}^{m,\alpha}_{\lambda_1,\lambda_2,\ell,d}f(z)}$$
$$-\left(\frac{\left[\ell(1+\lambda_2(k-1)-\lambda_1)+d\right]}{\ell\lambda_1}-m\right)\frac{\mathcal{D}^{m}_{\lambda_1,\lambda_2,\ell,d}f(z)}{\mathcal{RD}^{m,\alpha}_{\lambda_1,\lambda_2,\ell,d}f(z)}\Bigg]-\delta(m+1). \quad (12)$$

From (8), (9) and (12), we get

$$p(z)+\frac{zp'(z)}{p(z)}\prec q(z)+\frac{zq'(z)}{q(z)},z\in\mathbb{U}.$$

By setting

$$\theta(w)=w,\quad \text{and}\quad \phi(w)=\frac{1}{w},$$

it can be easily observed that $\theta(w)$ is analytic function in $\mathbb{C}$ and $\phi(w)$ is analytic function in $\mathbb{C}\setminus\{0\}$ and that $\phi(w)\neq 0$. Also we see that

$$Q(z)=zq'(z)\phi\left[q(z)\right]=\frac{zq'(z)}{q(z)},$$

and

$$h(z) = \theta\big[q(z)\big] + Q(z) = q(z) + \frac{zq'(z)}{q(z)}.$$

We can calculate

$$\Re e\left\{\frac{zQ'(z)}{Q(z)}\right\} = \Re e\left[\frac{zq''(z)}{q'(z)} - \frac{zq'(z)}{q(z)} + 1\right], z \in \mathbb{U},$$

hence by (7) $\Re e\left\{\frac{zQ'(z)}{Q(z)}\right\} > 0$, then Q is starlike in $\mathbb{U}$, and by using (7)

$$\Re e\left\{\frac{zh'(z)}{Q(z)}\right\} = \Re e\left\{q(z) + \frac{zq''(z)}{q'(z)} - \frac{zq'(z)}{q(z)} + 1\right\} > 0.$$

Since Q is starlike, and $\Re e\left\{\frac{zh'(z)}{Q(z)}\right\} > 0,\ z \in \mathbb{U}.$ and then, by using Lemma 1 we deduce that the subordination (9) implies $p(z) \prec q(z)$, and the function q is the best dominant of (9).

Theorem 2. *Let q be convex univalent in $\mathbb{U}$, and let us assume that it satisfies the equation* (7) *and*

$$\Re e\big\{q(z)q'(z)\big\} > 0, \quad z \in \mathbb{U}. \tag{13}$$

If $f \in \mathcal{A}_n, m, d \in \mathbb{N}_0, \lambda_2 \geq \lambda_1 \geq 0, \ell \geq 0, \ell + d > 0, \delta > 0, \alpha \geq 0$ and

$$\left(\frac{\mathcal{RD}^{m,\alpha}_{\lambda_1,\lambda_2,\ell,d} f(z)}{z}\right)^{\delta} \in \mathcal{H}[q(0), 1] \cap Q. \tag{14}$$

If $\Psi(\delta, m, \alpha, \lambda_1, \lambda_2, \ell, d)$ given by (8) *is univalent in $\mathbb{U}$ and satisfies the following superordination*

$$q(z) + \frac{zq'(z)}{q(z)} \prec \Psi(\delta, m, \alpha, \lambda_1, \lambda_2, \ell, d), \tag{15}$$

then

$$q(z) \prec \left(\frac{\mathcal{RD}^{m,\alpha}_{\lambda_1,\lambda_2,\ell,d} f(z)}{z}\right)^{\delta}, \text{ and } q \text{ is the best subordinant.}$$

Proof. It can be proved easily by using the same method of Theorem 1 and by an application of Lemma 2.

Combining Theorems 1 and 2, we state the following sandwich theorem.

Theorem 3. *Let q_i be univalent in $\mathbb{U}\,(i = 1, 2)$, and q_1 be convex. Suppose that $q_1(z)$ satisfies* (7) *and* (13)*, and $q_2(z)$ satisfies* (6) *and* (7)*.*
Let $f \in \mathcal{A}_n, m, d \in \mathbb{N}_0, \lambda_2 \geq \lambda_1 \geq 0, \ell \geq 0, \ell + d > 0, \delta > 0, \alpha \geq 0$ and

$$\left(\frac{\mathcal{RD}^{m,\alpha}_{\lambda_1,\lambda_2,\ell,d} f(z)}{z}\right)^{\delta} \in \mathcal{H}[q(0), 1] \cap Q. \tag{16}$$

Let $\Psi(\delta, m, \alpha, \lambda_1, \lambda_2, \ell, d)$ *given by* (8) *be univalent in* $\mathbb{U}$.
If

$$q_1(z) + \frac{zq_1'(z)}{q_1(z)} \prec \Psi(\delta, m, \alpha, \lambda_1, \lambda_2, \ell, d) \prec q_2(z) + \frac{zq_2'(z)}{q_2(z)},$$

then

$$q_1(z) \prec \left(\frac{\mathcal{RD}^{m,\alpha}_{\lambda_1,\lambda_2,\ell,d} f(z)}{z}\right)^{\delta} \prec q_2(z).$$

and q_1, q_2, *respectively, are the best subordinant and the best dominant.*

Theorem 4. *Let* $f \in \mathcal{A}_n, m, d \in \mathbb{N}_0, \lambda_2 \geq \lambda_1 \geq 0, \ell \geq 0, \ell + d > 0, \delta > 0$ *and* $\alpha \geq 0$. *Let the function* q *be univalent in* $\mathbb{U}$ *and suppose that it satisfies the conditions* (6) *and* (7).
Let

$$\begin{aligned}
\Phi(\delta, m, \alpha, \lambda_1, \lambda_2, \ell, d) &= \left(\frac{\mathcal{RD}^{m+1,\alpha}_{\lambda_1,\lambda_2,\ell,d} f(z)}{z}\right)\left(\frac{z}{\mathcal{RD}^{m,\alpha}_{\lambda_1,\lambda_2,\ell,d} f(z)}\right)^{\delta} \\
&+ (m+2)\left[\frac{\mathcal{RD}^{m+2,\alpha}_{\lambda_1,\lambda_2,\ell,d} f(z)}{\mathcal{RD}^{m+1,\alpha}_{\lambda_1,\lambda_2,\ell,d} f(z)} - 1\right] - \delta(m+1)\left[\frac{\mathcal{RD}^{m+1,\alpha}_{\lambda_1,\lambda_2,\ell,d} f(z)}{\mathcal{RD}^{m,\alpha}_{\lambda_1,\lambda_2,\ell,d} f(z)} - 1\right] \\
&+ \alpha\left(\frac{\left[\ell(1+\lambda_2(k-1)) + d\right]}{\ell\lambda_1} - (m+2)\right)\frac{\mathcal{D}^{m+2}_{\lambda_1,\lambda_2,\ell,d} f(z)}{\mathcal{RD}^{m+1,\alpha}_{\lambda_1,\lambda_2,\ell,d} f(z)} \\
&- \alpha\delta\left(\frac{\left[\ell(1+\lambda_2(k-1)) + d\right]}{\ell\lambda_1} - (m+1)\right)\frac{\mathcal{D}^{m+1}_{\lambda_1,\lambda_2,\ell,d} f(z)}{\mathcal{RD}^{m,\alpha}_{\lambda_1,\lambda_2,\ell,d} f(z)} \\
&- \alpha\left(\frac{\left[\ell(1+\lambda_2(k-1) - \lambda_1) + d\right]}{\ell\lambda_1} - (m+1)\right)\frac{\mathcal{D}^{m+1}_{\lambda_1,\lambda_2,\ell,d} f(z)}{\mathcal{RD}^{m+1,\alpha}_{\lambda_1,\lambda_2,\ell,d} f(z)} \\
&+ \alpha\delta\left(\frac{\left[\ell(1+\lambda_2(k-1) - \lambda_1) + d\right]}{\ell\lambda_1} - m\right)\frac{\mathcal{D}^{m}_{\lambda_1,\lambda_2,\ell,d} f(z)}{\mathcal{RD}^{m+1,\alpha}_{\lambda_1,\lambda_2,\ell,d} f(z)}. \qquad (17)
\end{aligned}$$

If

$$\Phi(\delta, m, \alpha, \lambda_1, \lambda_2, \ell, d) \prec q(z) + \frac{zq'(z)}{q(z)}, \qquad (18)$$

then

$$\left(\frac{\mathcal{RD}^{m+1,\alpha}_{\lambda_1,\lambda_2,\ell,d} f(z)}{z}\right)\left(\frac{z}{\mathcal{RD}^{m,\alpha}_{\lambda_1,\lambda_2,\ell,d} f(z)}\right)^{\delta} \prec q(z), z \in \mathbb{U},$$

and q *is the best dominant.*

Proof. Define the function $p(z)$ by

$$p(z) = \left(\frac{\mathcal{RD}^{m,\alpha}_{\lambda_1,\lambda_2,\ell,d} f(z)}{z}\right)^{\delta}, z \in \mathbb{U}. \qquad (19)$$

Then the function $p(z)$ is analytic in $\mathbb{U}$ and $p(0) = 1$. By logarithmically differentiating both sides of (19) with respect to z, and multiplying the resulting equation by z, and using (3) and (1), we obtain

$$\begin{aligned} p(z) + \frac{zp'(z)}{p(z)} &= \left(\frac{\mathcal{RD}^{m+1,\alpha}_{\lambda_1,\lambda_2,\ell,d}f(z)}{z}\right)\left(\frac{z}{\mathcal{RD}^{m,\alpha}_{\lambda_1,\lambda_2,\ell,d}f(z)}\right)^{\delta} \\ &+ (m+2)\left[\frac{\mathcal{RD}^{m+2,\alpha}_{\lambda_1,\lambda_2,\ell,d}f(z)}{\mathcal{RD}^{m+1,\alpha}_{\lambda_1,\lambda_2,\ell,d}f(z)} - 1\right] - \delta(m+1)\left[\frac{\mathcal{RD}^{m+1,\alpha}_{\lambda_1,\lambda_2,\ell,d}f(z)}{\mathcal{RD}^{m,\alpha}_{\lambda_1,\lambda_2,\ell,d}f(z)} - 1\right] \\ &+ \alpha\left(\frac{[\ell(1+\lambda_2(k-1))+d]}{\ell\lambda_1} - (m+2)\right)\frac{\mathcal{D}^{m+2}_{\lambda_1,\lambda_2,\ell,d}f(z)}{\mathcal{RD}^{m+1,\alpha}_{\lambda_1,\lambda_2,\ell,d}f(z)} \\ &- \alpha\delta\left(\frac{[\ell(1+\lambda_2(k-1))+d]}{\ell\lambda_1} - (m+1)\right)\frac{\mathcal{D}^{m+1}_{\lambda_1,\lambda_2,\ell,d}f(z)}{\mathcal{RD}^{m,\alpha}_{\lambda_1,\lambda_2,\ell,d}f(z)} \\ &- \alpha\left(\frac{[\ell(1+\lambda_2(k-1)-\lambda_1)+d]}{\ell\lambda_1} - (m+1)\right)\frac{\mathcal{D}^{m+1}_{\lambda_1,\lambda_2,\ell,d}f(z)}{\mathcal{RD}^{m+1,\alpha}_{\lambda_1,\lambda_2,\ell,d}f(z)} \\ &+ \alpha\delta\left(\frac{[\ell(1+\lambda_2(k-1)-\lambda_1)+d]}{\ell\lambda_1} - m\right)\frac{\mathcal{D}^{m}_{\lambda_1,\lambda_2,\ell,d}f(z)}{\mathcal{RD}^{m+1,\alpha}_{\lambda_1,\lambda_2,\ell,d}f(z)}. \end{aligned} \tag{20}$$

From (17), (18) and (20), we get

$$p(z) + \frac{zp'(z)}{p(z)} \prec q(z) + \frac{zq'(z)}{q(z)}, z \in \mathbb{U}.$$

Taking $\theta(w) = w$, and $\phi(w) = \frac{1}{w}$, and applying Lemma 1, we obtain the conclusion of Theorem 4.

Theorem 5. *Let* $f \in \mathcal{A}_n, m, d \in \mathbb{N}_0, \lambda_2 \geq \lambda_1 \geq 0, \ell \geq 0, \ell + d > 0,\ \delta > 0,\ \alpha \geq 0,$ *and* $\left(\frac{\mathcal{RD}^{m+1,\alpha}_{\lambda_1,\lambda_2,\ell,d}f(z)}{z}\right)\left(\frac{z}{\mathcal{RD}^{m,\alpha}_{\lambda_1,\lambda_2,\ell,d}f(z)}\right)^{\delta} \in \mathcal{H}[q(0),1] \cap Q$. *Let* $\Phi(\delta, m, \alpha, \lambda_1, \lambda_2, \ell, d)$ *defined by* (17)*be univalent in* $\mathbb{U}$. *Let* q *be convex univalent in* $\mathbb{U}$, *and let us assume that it satisfies the equations* (7) *and* (13).
If

$$q(z) + \frac{zq'(z)}{q(z)} \prec \Phi(\delta, m, \alpha, \lambda_1, \lambda_2, \ell, d), \tag{21}$$

then

$$q(z) \prec \left(\frac{\mathcal{RD}^{m+1,\alpha}_{\lambda_1,\lambda_2,\ell,d}f(z)}{z}\right)\left(\frac{z}{\mathcal{RD}^{m,\alpha}_{\lambda_1,\lambda_2,\ell,d}f(z)}\right)^{\delta},$$

and q *is the best subordinant.*

Proof. Theorem 5 follows by using the same technique of proof of Theorem 4 and by an application of Lemma 2.

Combining Theorems 4 and 5, we state the following sandwich theorem.

Theorem 6. *Let q_i be univalent in $\mathbb{U}\,(i=1,2)$, and q_1 be convex. Suppose that $q_1(z)$ satisfies (7) and (13), and $q_2(z)$ satisfies (6) and (7).*
Let $f \in \mathcal{A}_n, m, d \in \mathbb{N}_0, \lambda_2 \geq \lambda_1 \geq 0, \ell \geq 0, \ell + d > 0, \delta > 0, \alpha \geq 0$ and

$$\left(\frac{\mathcal{RD}^{m+1,\alpha}_{\lambda_1,\lambda_2,\ell,d}f(z)}{z}\right)\left(\frac{z}{\mathcal{RD}^{m,\alpha}_{\lambda_1,\lambda_2,\ell,d}f(z)}\right)^{\delta} \in \mathcal{H}[q(0),1] \cap Q. \tag{22}$$

Let $\Phi(\delta, m, \alpha, \lambda_1, \lambda_2, \ell, d)$ given by (17) be univalent in $\mathbb{U}$.
If

$$q_1(z) + \frac{zq_1'(z)}{q_1(z)} \prec \Phi(\delta, m, \alpha, \lambda_1, \lambda_2, \ell, d) \prec q_2(z) + \frac{zq_2'(z)}{q_2(z)},$$

then

$$q_1(z) \prec \left(\frac{\mathcal{RD}^{m+1,\alpha}_{\lambda_1,\lambda_2,\ell,d}f(z)}{z}\right)\left(\frac{z}{\mathcal{RD}^{m,\alpha}_{\lambda_1,\lambda_2,\ell,d}f(z)}\right)^{\delta} \prec q_2(z).$$

and q_1, q_2, respectively, are the best subordinant and the best dominant.

Taking $\alpha = 1$ in Theorem 6, we get the following result

Corollary 1. *Let q_i be univalent in $\mathbb{U}\,(i=1,2)$, and q_1 be convex. Suppose that $q_1(z)$ satisfies (7) and (13), and $q_2(z)$ satisfies (6) and (7).*
Let $f \in \mathcal{A}_n, m, d \in \mathbb{N}_0, \lambda_2 \geq \lambda_1 \geq 0, \ell \geq 0, \ell + d > 0, \delta > 0, \alpha \geq 0$ and

$$\left(\frac{\mathcal{D}^{m+1}_{\lambda_1,\lambda_2,\ell,d}f(z)}{z}\right)\left(\frac{z}{\mathcal{D}^{m}_{\lambda_1,\lambda_2,\ell,d}f(z)}\right)^{\delta} \in \mathcal{H}[q(0),1] \cap Q. \tag{23}$$

Let

$$\begin{aligned}\tau(\delta, m, \lambda_1, \lambda_2, \ell, d) &= \left(\frac{\mathcal{D}^{m+1}_{\lambda_1,\lambda_2,\ell,d}f(z)}{z}\right)\left(\frac{z}{\mathcal{D}^{m}_{\lambda_1,\lambda_2,\ell,d}f(z)}\right)^{\delta} \\ &+ \frac{[\ell(1+\lambda_2(k-1))+d]}{\ell\lambda_1}\left[\left(\frac{\mathcal{D}^{m+2}_{\lambda_1,\lambda_2,\ell,d}f(z)}{\mathcal{D}^{m+1}_{\lambda_1,\lambda_2,\ell,d}f(z)} - 1\right) + \delta\left(1 - \frac{\mathcal{D}^{m+1}_{\lambda_1,\lambda_2,\ell,d}f(z)}{\mathcal{D}^{m}_{\lambda_1,\lambda_2,\ell,d}f(z)}\right)\right]\end{aligned} \tag{24}$$

If

$$q_1(z) + \frac{zq_1'(z)}{q_1(z)} \prec \tau(\delta, m, \lambda_1, \lambda_2, \ell, d) \prec q_2(z) + \frac{zq_2'(z)}{q_2(z)},$$

then

$$q_1(z) \prec \left(\frac{\mathcal{D}^{m+1}_{\lambda_1,\lambda_2,\ell,d}f(z)}{z}\right)\left(\frac{z}{\mathcal{D}^{m}_{\lambda_1,\lambda_2,\ell,d}f(z)}\right)^{\delta} \prec q_2(z).$$

and q_1, q_2, respectively, are the best subordinant and the best dominant.

Theorem 7. *Let $f \in \mathcal{A}_n, m, d \in \mathbb{N}_0, \lambda_2 \geq \lambda_1 \geq 0, \ell \geq 0, \ell + d > 0, \delta > 0$ and $\alpha \geq 0$. Let the function q be univalent in $\mathbb{U}$ and suppose that it satisfies the conditions* (6) *and* (7).
Let

$$\begin{aligned}\Theta(\delta, m, \alpha, \lambda_1, \lambda_2, \ell, d) = (m+2)\frac{\mathcal{RD}^{m+2,\alpha}_{\lambda_1,\lambda_2,\ell,d}f(z)}{\mathcal{RD}^{m+1,\alpha}_{\lambda_1,\lambda_2,\ell,d}f(z)} - m\frac{\mathcal{RD}^{m+1,\alpha}_{\lambda_1,\lambda_2,\ell,d}f(z)}{\mathcal{RD}^{m,\alpha}_{\lambda_1,\lambda_2,\ell,d}f(z)} \\ + \alpha\left(\frac{\left[\ell(1+\lambda_2(k-1))+d\right]}{\ell\lambda_1} - (m+2)\right)\frac{\mathcal{D}^{m+2,\alpha}_{\lambda_1,\lambda_2,\ell,d}f(z)}{\mathcal{RD}^{m+1,\alpha}_{\lambda_1,\lambda_2,\ell,d}f(z)} \\ - \alpha\left(\frac{\left[\ell(1+\lambda_2(k-1))+d\right]}{\ell\lambda_1} - (m+1)\right)\frac{\mathcal{D}^{m+1}_{\lambda_1,\lambda_2,\ell,d}f(z)}{\mathcal{RD}^{m,\alpha}_{\lambda_1,\lambda_2,\ell,d}f(z)} \\ - \alpha\left(\frac{\left[\ell(1+\lambda_2(k-1)-\lambda_1)+d\right]}{\ell\lambda_1} - (m+1)\right)\frac{\mathcal{D}^{m+1}_{\lambda_1,\lambda_2,\ell,d}f(z)}{\mathcal{RD}^{m+1,\alpha}_{\lambda_1,\lambda_2,\ell,d}f(z)} \\ + \alpha\left(\frac{\left[\ell(1+\lambda_2(k-1)-\lambda_1)+d\right]}{\ell\lambda_1} - m\right)\frac{\mathcal{D}^{m}_{\lambda_1,\lambda_2,\ell,d}f(z)}{\mathcal{RD}^{m,\alpha}_{\lambda_1,\lambda_2,\ell,d}f(z)} - 1. \quad (25)\end{aligned}$$

If

$$\Theta(\delta, m, \alpha, \lambda_1, \lambda_2, \ell, d) \prec q(z) + \frac{zq'(z)}{q(z)}, \quad (26)$$

then

$$\frac{\mathcal{RD}^{m+1,\alpha}_{\lambda_1,\lambda_2,\ell,d}f(z)}{\mathcal{RD}^{m,\alpha}_{\lambda_1,\lambda_2,\ell,d}f(z)} \prec q(z), z \in \mathbb{U},$$

and q is the best dominant.

Proof. Define the function $p(z)$ by

$$p(z) = \frac{\mathcal{RD}^{m+1,\alpha}_{\lambda_1,\lambda_2,\ell,d}f(z)}{\mathcal{RD}^{m,\alpha}_{\lambda_1,\lambda_2,\ell,d}f(z)}.$$

Then

$$p(z) + \frac{zp'(z)}{p(z)} = \Theta(\delta, m, \alpha, \lambda_1, \lambda_2, \ell, d),$$

which, in light of hypothesis (26) of Theorem 7, yields the following subordination

$$p(z) + \frac{zp'(z)}{p(z)} \prec q(z) + \frac{zq'(z)}{q(z)}.$$

The assertion of Theorem 7 now follows by an application of Lemma 1 with $\theta(w) = w, \phi(w) = \dfrac{1}{w}$.

Theorem 8. *Let q be convex univalent in $\mathbb{U}$, and let us assume that it satisfies the equations* (7) *and* (13). *Let $f \in \mathcal{A}_n, m, d \in \mathbb{N}_0, \lambda_2 \geq \lambda_1 \geq 0, \ell \geq 0, \ell + d > 0$, $\delta > 0, \alpha \geq 0$ and $\left(\dfrac{\mathcal{RD}^{m+1,\alpha}_{\lambda_1,\lambda_2,\ell,d} f(z)}{\mathcal{RD}^{m,\alpha}_{\lambda_1,\lambda_2,\ell,d} f(z)}\right) \in \mathcal{H}[q(0), 1] \cap Q$. Let $\Theta(\delta, m, \alpha, \lambda_1, \lambda_2, \ell, d)$ given by* (25) *be univalent in $\mathbb{U}$ and satisfies the following superordination*

$$q(z) + \frac{zq'(z)}{q(z)} \prec \Theta(\delta, m, \alpha, \lambda_1, \lambda_2, \ell, d), \tag{27}$$

then

$$q(z) \prec \left(\frac{\mathcal{RD}^{m+1,\alpha}_{\lambda_1,\lambda_2,\ell,d} f(z)}{\mathcal{RD}^{m,\alpha}_{\lambda_1,\lambda_2,\ell,d} f(z)}\right), \textit{ and } q \textit{ is the best subordinant.}$$

Proof. By using the same method of Theorem 7 and by an application of Lemma 2.

Combining Theorems 1 and 2, we state the following sandwich theorem.

Theorem 9. *Let q_i be univalent in $\mathbb{U}\,(i = 1, 2)$, and q_1 be convex. Suppose that $q_1(z)$ satisfy* (7) *and* (13), *and $q_2(z)$ satisfy* (6) *and* (7).
Let $f \in \mathcal{A}_n, m, d \in \mathbb{N}_0, \lambda_2 \geq \lambda_1 \geq 0, \ell \geq 0, \ell + d > 0, \delta > 0, \alpha \geq 0$ and

$$\frac{\mathcal{RD}^{m+1,\alpha}_{\lambda_1,\lambda_2,\ell,d} f(z)}{\mathcal{RD}^{m,\alpha}_{\lambda_1,\lambda_2,\ell,d} f(z)} \in \mathcal{H}[q(0), 1] \cap Q. \tag{28}$$

Let $\Theta(\delta, m, \alpha, \lambda_1, \lambda_2, \ell, d)$ given by (25) *be univalent in $\mathbb{U}$ and satisfy the following superordination*

$$q_1(z) + \frac{zq_1'(z)}{q_1(z)} \prec \Theta(\delta, m, \alpha, \lambda_1, \lambda_2, \ell, d) \prec q_2(z) + \frac{zq_2'(z)}{q_2(z)},$$

then

$$q_1(z) \prec \frac{\mathcal{RD}^{m+1,\alpha}_{\lambda_1,\lambda_2,\ell,d} f(z)}{\mathcal{RD}^{m,\alpha}_{\lambda_1,\lambda_2,\ell,d} f(z)} \prec q_2(z),$$

and q_1, q_2, respectively, are the best subordinant and the best dominant.

Taking $\alpha = 1$ in Theorem 9, we get the following result

Corollary 2. *Let q_i be univalent in $\mathbb{U}\,(i = 1, 2)$, and q_1 be convex. Suppose that $q_1(z)$ satisfies* (7) *and* (13), *and $q_2(z)$ satisfies* (6) *and* (7). *Let $f \in \mathcal{A}_n, m$, $d \in \mathbb{N}_0, \lambda_2 \geq \lambda_1 \geq 0, \ell \geq 0, \ell + d > 0, \delta > 0, \alpha \geq 0$ and*

$$\frac{\mathcal{D}^{m+1}_{\lambda_1,\lambda_2,\ell,d} f(z)}{\mathcal{D}^{m}_{\lambda_1,\lambda_2,\ell,d} f(z)} \in \mathcal{H}[q(0), 1] \cap Q. \tag{29}$$

Let

$$\Upsilon(\delta,m,\lambda_1,\lambda_2,\ell,d)=\left(\frac{\big[\ell(1+\lambda_2(k-1))+d\big]}{\ell\lambda_1}\right)\frac{\mathcal{D}^{m+2}_{\lambda_1,\lambda_2,\ell,d}f(z)}{\mathcal{D}^{m+1}_{\lambda_1,\lambda_2,\ell,d}f(z)} - \left(\frac{\big[\ell(1+\lambda_2(k-1))+d\big]}{\ell\lambda_1}-1\right)\frac{\mathcal{D}^{m+1}_{\lambda_1,\lambda_2,\ell,d}f(z)}{\mathcal{D}^{m}_{\lambda_1,\lambda_2,\ell,d}f(z)} \quad (30)$$

If

$$q_1(z)+\frac{zq_1'(z)}{q_1(z)}\prec\Upsilon(\delta,m,\lambda_1,\lambda_2,\ell,d)\prec q_2(z)+\frac{zq_2'(z)}{q_2(z)},$$

then

$$q_1(z)\prec\frac{\mathcal{D}^{m+1}_{\lambda_1,\lambda_2,\ell,d}f(z)}{\mathcal{D}^{m}_{\lambda_1,\lambda_2,\ell,d}f(z)}\prec q_2(z).$$

and q_1, q_2, respectively, are the best subordinant and the best dominant.

Remark 5. Some other works related to differential subordination can be found in [5–7, 21, 22].

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