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# Methods of construction of Hausdorff extensions

Laurenţiu Calmuţchi

**Abstract.** In this paper we study the extensions of Hausdorff spaces generated by discrete families of open sets.

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## 1 Introduction

Any space is considered to be a Hausdorff space. We use the terminology from [3]. For any completely regular space $X$ denote by $\beta X$ the Stone-Čech compactification of the space $X$.

Fix a space $X$. A space $eX$ is an extension of the space $X$ if $X$ is a dense subspace of $eX$. If $eX$ is a compact space, then $eX$ is called a compactification of the space $X$. The subspace $eX \setminus X$ is called a remainder of the extension $eX$.

Denote by $Ext(X)$ the family of all extensions of the space $X$. If $X$ is a completely regular space, then by $Ext_\rho(X)$ we denote the family of all completely regular extensions of the space $X$. Obviously, $Ext_\rho(X) \subset Ext(X)$. Let $Y, Z \in Ext(X)$ be two extensions of the space $X$. We consider that $Z \leq Y$ if there exists a continuous mapping $f : Y \longrightarrow Z$ such that $f(x) = x$ for each $x \in X$. If $Z \leq Y$ and $Y \leq Z$, then we say that extensions $Y$ and $Z$ are equivalent and there exists a unique homeomorphism $f : Y \longrightarrow Z$ of $Y$ onto $Z$ such that $f(x) = x$ for each $x \in X$. We identify the equivalent extensions. In this case $Ext(X)$ and $Ext_\rho$ are partially ordered sets.

Let $\tau$ be an infinite cardinal. Denote by $O(\tau)$ the set of all ordinal numbers of cardinality $< \tau$. We consider that $\tau$ is the first ordinal number of the cardinality $\tau$. For any $\alpha \in O(\tau)$ we put $O(\alpha) = \{\beta \in O(\tau) : \beta < \alpha\}$. In this case $O(\tau)$ is well ordered set such that $|O(\tau)| = \tau$ and $|O(\alpha)| < \tau$ for every $\alpha \in O(\tau)$.

A point $x \in X$ is called a $P(\tau)$-point of the space $X$ if for any non-empty family $\gamma$ of open subsets of $X$ for which $x \in \cap\gamma$ and $|\gamma| < \tau$ there exists an open subset $U$ of $X$ such that $x \in U \subset \cap\gamma$. If any point of $X$ is a $P(\tau)$-point, then we say that $X$ is a $P(\tau)$-space.

Any point is an $\aleph_0$-point. If $\tau = \aleph_1$, then the $P(\tau)$-point is called the $P$-point.

## 2 Hausdorff extensions of discrete spaces

Let $\tau$ be an infinite cardinal. Let $E$ be a discrete space of the cardinality $\geq \tau$.

A family $\eta$ of subsets of $E$ is called $\tau$-centered if the family $\eta$ is non-empty, $\cap\eta = \emptyset$, $\emptyset \notin \eta$ and for any subfamily $\zeta \subset \eta$, with cardinality $|\zeta| < \tau$, there exists $l \in \eta$ such that $L \subset \cap\zeta$.

Two families $\eta$ and $\zeta$ of subsets of the space $E$ are almost disjoint if there exist $L \in \eta$ and $Z \in \zeta$ such that $L \cap Z = \emptyset$.

Any family of subsets is ordered by the following order: $L \preceq H$ if and only if $H \subset L$. Relative to this oder some families of sets are well-ordered.

**Proposition 1.** *Let $k = |E| \geq \tau$ and $\Sigma\{k^m : m < \tau\} = k$. Then on $E$ there exists a set $\Omega$ of well-ordered almost disjoint $\tau$-centered families such that $|\Omega| = k^\tau$ and $|\eta| = \tau$ for each $\eta \in \Omega$.*

*Proof.* We fix an element $0 \in E$. For every $\alpha \in O(\tau)$ we put $E_\alpha = E$ and $0_\alpha = 0$. Then $E^\tau = \Pi\{E_\alpha : \alpha \in O(\tau)\}$. For each $x = (x_\alpha : \alpha \in O(\tau)) \in E^\tau$ we put $\phi(x) = sup\{0, \alpha : x_\alpha \neq 0_\alpha\}$. Obviously, $0 \leq \phi(x) \leq \tau$. Let $D = \{x = (x_\alpha : \alpha \in O(\tau)) \in E^\tau : \phi(x) < \tau\}$. By construction, $|D| = \Sigma\{k^m : m < \tau\} = k$ and $|E^\tau| = k^\tau$. Since $|E| = |D|$, we can fix a one-to-one mapping $f : E \longrightarrow D$. Fix a point $x = (x_\alpha : \alpha \in O(\tau)) \in E^\tau$. For any $\beta \in O(\tau)$ we put $V(x,\beta) = \{y = (y_\alpha : \alpha \in O(\tau)) \in E^\tau : y_\alpha = x_\alpha$ for every $\alpha \leq \beta\}$ and $\eta_x = \{L(x,\beta) = f^{-1}(D \cap V(x,\beta) : \beta \in 0(\tau)\}$. Then $\Omega = \{\eta_x : x \in E^\tau\}$ is the desired set of $\tau$-centered families. □

*Remark* 1. Let $|E| = k \geq \tau$. Since on $E$ there exists $k$ mutually disjoint subsets of cardinality $\tau$, on $E$ there exists a set $\Phi$ of well-ordered almost disjoint $\tau$-centered families such that $|\Phi| \geq k$ and $|\eta| = \tau$ for each $\eta \in \Phi$.

Fix a set $\Phi$ of almost disjoint $\tau$-centered families of subsets of the set $E$. We put $e_\Phi E = E \cup \Phi$. On $e_\Phi E$ we construct two topologies.

**Topology** $T^s(\Phi)$. The basis of the topology $T^s(\Phi)$ is the family $\mathcal{B}^s(\Phi) = \{U_L = L \cup \{\eta \in \Phi : H \subset L$ for some $H \in \eta\} : L \subset E\}$.

**Topology** $T_m(\Phi)$. For each $x \in E$ we put $B_m(x) = \{\{x\}\}$. For every $\eta \in \Phi$ we put $B_m(\eta) = \{V_{(\eta,L)} = \{\eta\} \cup L : L \in \eta\}$. The basis of the topology $T_m(\Phi)$ is the family $\mathcal{B}_m(\Phi) = \cup\{B_m(x) : x \in e_\Phi E\}$.

**Theorem 1.** *The spaces $(e_\Phi E, T^s(\Phi))$ and $(e_\Phi E, T_m(\Phi))$ are Hausdorff zero-dimensional extensions of the discrete space $E$, and $T^s(\Phi) \subset T_m(\Phi))$. In particular, $(e_\Phi E, T^s(\Phi)) \leq (e_\Phi E, T_m(\Phi))$.*

*Proof.* The inclusion $T^s(\Phi) \subset T_m(\Phi))$ follows from the constructions of the topologies $T^s(\Phi)$ and $T_m(\Phi))$. If $L \in \eta \in \Phi$, then $\eta \in clL$. Hence the set $E$ is dense in the spaces $(e_\Phi E, T^s(\Phi))$ and $(e_\Phi E, T_m(\Phi))$. If the families $\eta, \zeta \in \Phi$ are distinct, then there exist $L \in \eta$ and $Z \in \zeta$ such that $L \cap Z = \emptyset$. Then $U_L \cap U_Z = \emptyset$. If $L \subset E$ and $|L| < \tau$, then $L$ is an open-and-closed subset of the spaces $(e_\Phi E, T^s(\Phi))$ and $(e_\Phi E, T_m(\Phi))$. Hence the topologies $T^s(\Phi)$ and $T_m(\Phi)$ are discrete on $E$ and the spaces $(e_\Phi E, T^s(\Phi))$ and $(e_\Phi E, T_m(\Phi))$ are Hausdorff extensions of the discrete space $E$. Since the sets $U_L$ and $V_{(\eta,L)}$ are open-and-closed in the topologies $T^s(\Phi)$ and $T_m(\Phi))$, respectively, the spaces $(e_\Phi E, T^s(\Phi))$ and $(e_\Phi E, T_m(\Phi))$ are zero-dimensional. □

**Theorem 2.** *The spaces* $(e_\Phi E, T^s(\Phi))$ *and* $(e_\Phi E, T_m(\Phi))$ *are* $P(\tau)$*-spaces.*

*Proof.* Fix $\eta \in \Phi$. If $\zeta \subset \eta$ and $|\zeta| < \tau$, then there exists $L(\zeta) \in \eta$ such that $L(\zeta) \subset \cap\zeta$. From this fact immediately follows that $(e_\Phi E, T_m(\Phi))$ is a $P(\tau)$-space. Assume that $\{L_\mu : \mu \in M\}$ is a family of subsets of $E$, $|M| < \tau$, $\eta \in \Phi$ and $\eta \in \cap\{L_\mu : \mu \in M\}$. Then there exists $L \in \eta$ such that $L \subset \cap\{L_\mu : \mu \in M\}$. Thus $\eta \in U_L \in \cap\{U_{L_\mu} : \mu \in M\}$. From this fact immediately follows that $(e_\Phi E, T^s(\Phi))$ is a $P(\tau)$-space. □

**Corollary 1.** *If* $T^s(\Phi) \subset \mathcal{T} \subset T_m(\Phi))$*, then* $(e_\Phi E, \mathcal{T})$ *is a Hausdorff extension of the discrete space* $E$*, and* $(e_\Phi E, T^s(\Phi)) \leq (e_\Phi E, \mathcal{T}) \leq (e_\Phi E, T_m(\Phi))$*.*

**Theorem 3.** *The space* $(e_\Omega E, T^s(\Omega))$*, where* $\Omega$ *is the set of well-ordered almost disjoint* $\tau$*-centered families from Proposition 1, is a zero-dimensional paracompact space with character* $\chi(e_\Omega E, T^s(\Omega)) = \tau$ *and weight* $\Sigma\{|E|^m : m < \tau\}$*.*

*Proof.* We consider that $E = D$. The family $\mathcal{B} = \{\{x\} : x \in D\} \cup \{V(x, \beta) : x \in E^\tau, \beta \in O(\tau)\}$ is a base of the topology $T^s(\Omega)$. If $U, V \in \mathcal{B}$, then either $U \subset V$, or $V \subset U$, or $U \cap V = \emptyset$. From the A. V. Arhangel'skii Theorem [1] it follows that $(e_\Omega E, T^s(\Omega))$ is a zero-dimensional paracompact space. □

## 3 Construction of Hausdorff extensions

Let $\tau$ be an infinite cardinal. Fix a $P(\tau)$-space $X$. Let $\gamma = \{H_\mu : \mu \in M\}$ be a discrete family of non-empty open subsets of the space $X$ and $\tau \leq |M|$. For any $\mu \in M$ we fix a point $e_\mu \in U_\mu$ and a family $\xi_\mu = \{H_{(\mu,\alpha)} : \alpha \in O(\tau)\}$ of open subsets of $X$ such that $e_\mu \in \cap\xi_\mu$ and $H_{(\mu,\beta)} \subset H_{(\mu,\alpha)} \subset H_\mu$ for all $\alpha \in O(\tau)$ and $\beta \in O(\alpha)$. Then $E = \{e_\mu : \mu \in M\}$ is a discrete closed subspace of the space $X$.

Consider the Hausdorff extension $rE$ of the space $E$. We put $e_{rE}X = X \cup (rE \backslash E)$. In $e_{rE}X$ we construct the topology $\mathcal{T} = T(\gamma, E, \xi_\mu, \tau)$ as follows:

– we consider $X$ as an open subspace of $e_{(E,Y)}X$;

– let $T_X$ be the topology of $X$ and $T_{rE}$ be the topology of the space $rE$;

– if $V \in T_{rE}$, then $e_\alpha V = V \cup \{H_{(\mu,\alpha)} : e_\mu \in V\}$;

– $\mathcal{B} = T_X \cup \{e_\alpha V : V \in T_{rE}\}$ is an open base of the topology $\mathcal{T} = T(\gamma, E, \xi_\mu, \tau)$.

**Theorem 4.** *The space* $(e_{(E,Y)}X, T(\gamma, E, \xi_\mu, \tau))$ *is a Hausdorff extension of the space* $X$*.*

*Proof.* If $V, W \in T_{rE}$, then:

– $e_\alpha W \subset e_\alpha V$ if and only if $W \subset V$;

– $e_\alpha W \cap e_\alpha V = \emptyset$ if and only if $W \cap tV = \emptyset$;

– $e_\alpha V \cap rE = V$.

These facts and Theorem 1 complete the proof. □

**Theorem 5.** *If* $rE$ *is a* $P(\tau)$*-space, then* $(e_{(E,Y)}X, T(\gamma, E, \xi_\mu, \tau))$ *is a* $P(\tau)$*-space too. Moreover,* $\chi(e_{(E,Y)}X, T(\gamma, E, \xi_\mu, \tau)) = \chi(X) + \chi(rE)$ *and*

$$w(e_{(E,Y)}X, T(\gamma, E, \xi_\mu, \tau)) = w(X) + w(rE).$$

*Proof.* Follows immediately from the construction of the sets $e_\alpha V$. □

**Theorem 6.** *Assume that the spaces $rE$ and $X$ are zero-dimensional, and the sets $H_{(\mu,\alpha)}$ are open-and-closed in $X$. Then:*

*1. $(e_{(E,Y)}X, T(\gamma, E, \xi_\mu, \tau))$ is a zero-dimensional space.*

*2. The space $(e_{(E,Y)}X, T(\gamma, E, \xi_\mu, \tau))$ is paracompact if and only if the spaces $rE$ and $X$ are paracompact.*

*Proof.* If the set $V$ is open-and-closed in $rE$ and the sets $H_{(\mu,\alpha)}$ are open-and-closed in $X$, then the sets $e_\alpha V$ are open-and-closed in $(e_{(E,Y)}X, T(\gamma, E, \xi_\mu, \tau))$. If $\{V_\lambda : \lambda \in L\}$ is a discrete cover of $rE$, and $\alpha(\lambda) \in O(\tau)$, then $\{e_{\alpha(\lambda)}V_\lambda : \lambda \in L\}$ is a discrete family of open-and-closed sets. This fact completes the proof. □

## References

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Laurenţiu Calmuţchi
Tiraspol State University,
str. Gh. Iablocikin 5, Chişinău
Republic of Moldova
E-mail: *lcalmutchi@gmail.com*

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