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# Upper Bounds for the Number of Limit Cycles for a Class of Polynomial Differential Systems Via The Averaging Method

S. Benadouane, A. Berbache, A. Bendjeddou

**Abstract.** In this paper, we study the number of limit cycles of polynomial differential systems of the form

$$\begin{cases} \dot{x} = y \\ \dot{y} = -x - \varepsilon(h_1(x)\,y^{2\alpha} + g_1(x)\,y^{2\alpha+1} + f_1(x)\,y^{2\alpha+2}) \\ \qquad - \varepsilon^2(h_2(x)\,y^{2\alpha} + g_2(x)\,y^{2\alpha+1} + f_2(x)\,y^{2\alpha+2}) \end{cases}$$

where $m, n, k$ and $\alpha$ are positive integers, $h_i$, $g_i$ and $f_i$ have degree $n, m$ and $k$, respectively for each $i = 1, 2$, and $\varepsilon$ is a small parameter. We use the averaging theory of first and second order to provide an accurate upper bound of the number of limit cycles that bifurcate from the periodic orbits of the linear center $\dot{x} = y, \dot{y} = -x$. We give an example for which this bound is reached.

**Mathematics subject classification:** 34C07, 34C23, 37G15.
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## 1 Introduction and statement of the main results

One of the main problems in the theory of ordinary differential equations is the study of the existence of limit cycles, their number and stability. A limit cycle of a differential equation is a periodic orbit in the set of all isolated periodic orbits of the differential equation. The second part of the 16th Hilbert's problem (see [8]) is related to the least upper bound on the number of limit cycles of polynomial vector fields having a fixed degree.

Many of the results on the limit cycles of polynomial differential systems have been obtained by considering limit cycles that bifurcate from a single degenerate singular point (i.e. from a Hopf bifurcation), which are called small amplitude limit cycles, see Lloyd [14]. There are partial results concerning the maximum number of small-amplitude limit cycles for Liénard polynomial differential systems. The number of small-amplitude limit cycles gives a lower bound for the maximum number of limit cycles that a polynomial differential system can have. There are many results concerning the existence of small-amplitude limit cycles for the following generalization of the classical Liénard polynomial differential system

$$\dot{x} = y \qquad \text{and} \qquad \dot{y} = -g(x) - f(x)y \tag{1}$$

where $f(x)$ and $g(x)$ are polynomials in the variable $x$ of degrees $n$ and $m$, respectively. We denote by $H(m,n)$ and $\hat{H}(m,n)$ the maximum number of limit cycles that system (1) can have and the maximum number of small-amplitude limit cycles that system(1) can have, respectively. The first number is usually called Hilbert number for system (1). Since the work of Liénard [10] to the present time several authors have found particular values of these numbers $H$ and $\hat{H}$, to find a survey about these values see [13]. The authors of [12] computed the maximum number of limit cycles $\hat{H}_k(m,n)$ of system(1) that bifurcate from the periodic orbits of the linear center $\dot{x}=y, \dot{y}=-x$, using the averaging theory of order $k$. More specifically it was found that $\hat{H}_1(m,n) = [(n+m-1)/2]$. In order to find the maximum number of limit cycles it is interesting to know what families of system (1) have a center. This is because we can perturb these centers and control the number of small-amplitude limit cycles or the number of limit cycles that bifurcate from the periodic orbits of these centers, (see [5, 6]). We recall that a singular point is a center if there is an open neighborhood consisting, besides the singularity, of periodic orbits. The center problem consists in determining what families of a given system have a center. For more information about the Hilbert's 16th problem and related topics see [9]. Now we are citing some results about the limit cycles on Liénard differential systems (see [12]) In 1928, Liénard proved that if $m=1$ and $F(x)=\int_0^x f(s)ds$ is a continuous odd function, which has a unique root at $x=a$ and is monotone increasing for $x \geq a$, then equations (1.2) have a unique limit cycle. In 1977 Lins, de Melo and Pugh [11] stated the conjecture that if $f(x)$ has degree $n \geq 1$ and $g(x)=x$ then system (1) has at most $[n/2]$ limit cycles. They prove this conjecture for $n=1,2$. In 1998 Gasull and Torregrosa [4] obtained upper bounds for $\hat{H}(7,6), \hat{H}(6,7)$, $\hat{H}(7,7)$ and $\hat{H}(4,20)$. In 2010, Llibre et al, computed the maximum number of limit cycles $\hat{H}_k(m,n)$ of system (1) that bifurcate from the periodic orbits of the linear centre $\dot{x}=y, \dot{y}=-x$, using the averaging theory of order $k$, for $k=1,2,3$. In 2014 B. Garca, J. Llibre, and J. S. Pérez del Rio 1001[3] using the averaging theory of first and second order, they studied the maximum number of medium amplitude limit cycles bifurcating from the linear center $\dot{x}=y, \dot{y}=-x$ of the more generalized polynomial Liénard differential systems of the form

$$\begin{cases} \dot{x}=y \\ \dot{y}=-x-\varepsilon(h_1(x)+p_1(x)y+q_1(x)y^2) \\ \qquad -\varepsilon^2(h_2(x)+p_2(x)+q_2(x)y^2) \end{cases}$$

where $h_1, h_2, p_1, q_1, p_2$ and $q_2$ have degree $n$.

In this work using the averaging theory, we study the maximum number of limit cycles which can bifurcate from the periodic orbits of a linear center perturbed inside the class of generalized polynomial Liénard differential equations

$$\begin{cases} \dot{x}=y \\ \dot{y}=-x-\varepsilon(h_1(x)y^{2\alpha}+g_1(x)y^{2\alpha+1}+f_1(x)y^{2\alpha+2}) \\ \qquad -\varepsilon^2(h_2(x)y^{2\alpha}+g_2(x)y^{2\alpha+1}+f_2(x)y^{2\alpha+2}) \end{cases} \tag{2}$$

where $m,n,k$ and $\alpha$ are positive integers, $h_i$, $g_i$ and $f_i$ have degree $n,m$ and $k$, respectively for each $i = 1, 2$, and $\varepsilon$ is a small parameter.

Let $[\cdot]$ denote the integer part function. Our main result is the following one.

**Theorem 1.** *For $|\varepsilon|$ sufficiently small, the maximum number of limit cycles of the polynomial differential systems* (2) *bifurcating from the periodic orbits of the linear center $\dot{x} = y, \dot{y} = -x$, using the averaging theory*
*(a) of first order is*

$$\lambda_1 = \left[\frac{m}{2}\right],$$

*(b) of second order is*

$$\lambda = \max\left\{\left[\frac{m}{2}\right]; \left[\frac{m-1}{2}\right] + \left[\frac{n}{2}\right] + \alpha; \left[\frac{m-1}{2}\right] + \left[\frac{k}{2}\right] + 1 + \alpha\right\}.$$

The proof of the above theorem is given in Section 3.

## 2 The averaging theory of first and second order

In this section we present the basic results from the averaging theory that we shall need for proving the main results of this paper. The averaging theory up to second order for studying specifically periodic orbits was developed in [1, 2]. It is summarized as follows.

Consider the differential system

$$\dot{x}(t) = \varepsilon F_1(t, x) + \varepsilon^2 F_2(t, x) + \varepsilon^3 R(t, x, \varepsilon),$$

where $F_1, F_2 : \mathbb{R} \times D \to \mathbb{R}$, $R : \mathbb{R} \times D \times (-\varepsilon_f, \varepsilon_f) \to \mathbb{R}$ are continuous functions, $T-$periodic in the first variable, and $D$ is an open subset of $\mathbb{R}^n$. Assume that the following hypotheses hold.
(i) $F_1(t,\cdot) \in C^2(D), F_2(t,\cdot) \in C^1(D)$ for all $t \in \mathbb{R}, F_1, F_2, R$ are locally Lipschitz with respect to $x$, and $R$ is twice differentiable with respect to $\varepsilon$.
We define $F_{k0} : D \to \mathbb{R}$ for $k = 1, 2$ as

$$\begin{aligned} F_{10}(x) &= \frac{1}{T}\int_0^T F_1(s, x)ds, \\ F_{20}(x) &= \frac{1}{T}\int_0^T (D_x F_1(s, x))\, y_1(s, x) + F_2(s, x)ds, \end{aligned}$$

where

$$y_1(s, x) = \int_0^s F_1(t, x)dt.$$

(ii) For an open and bounded set $V \subset D$ and for each $\varepsilon \in (-\varepsilon f, \varepsilon f)\backslash\{0\}$, there exists $a_\varepsilon \in V$ such that $F_{10}(a_\varepsilon) + \varepsilon F_{20}(a_\varepsilon) = 0$ and $d_B(F_{10} + \varepsilon F_{20}, V, a_\varepsilon) \neq 0$.

Then, for $|\varepsilon| > 0$ sufficiently small there exists a $T-$periodic solution $x(.,\varepsilon)$ of the system such that $x(0,\varepsilon) \to a_\varepsilon$ as $\varepsilon \to 0$.

The expression $d_B(F_{10} + \varepsilon F_{20}, V, a_\varepsilon) \neq 0$ means that the Brouwer degree of the function $F_{10} + \varepsilon F_{20} : V \to \mathbb{R}^n$ at the fixed point $a_\varepsilon$ is not zero. A sufficient condition of this inequality holding is that the Jacobian of the function $F_{10} + \varepsilon F_{20}$ at $a_\varepsilon$ is not zero.

If $F_{10}$ is not identically zero, then the zeros of $F_{10} + \varepsilon F_{20}$ are mainly the zeros of $F_{10}$ for $\varepsilon$ sufficiently small. In this case the previous result provides the *averaging theory of first order.*

If $F_{10}$ is identically zero and $F_{20}$ is not identically zero, then the zeros of $F_{10} + \varepsilon F_{20}$ are mainly the zeros of $F_{20}$ for $\varepsilon$ sufficiently small. In this case the previous result provides the *averaging theory of second order.*

## 3 Proof of Theorem 1

For the proof we shall use the first order averaging theory as it was stated in Section 2. We write system (2) in polar coordinates $(r,\theta)$ given by $x = r\cos\theta$ and $y = r\sin\theta$. In this way, system (2) will become written in the standard form for applying the averaging theory. If we write

$$\begin{aligned} h_1(x) &= \sum_{i=0}^{n} a_i x^i, \; g_1(x) = \sum_{i=0}^{m} c_i x^i, \; f_1(x) = \sum_{i=0}^{k} d_i x^i, \\ h_2(x) &= \sum_{i=0}^{n} A_i x^i, \; g_2(x) = \sum_{i=0}^{m} C_i x^i, \; f_2(x) = \sum_{i=0}^{k} D_i x^i \end{aligned}$$

then, system (2) becomes

$$\begin{cases} \dot{r} = -\varepsilon E_1(r,\theta) - \varepsilon^2 H_1(r,\theta), \\ \dot{\theta} = -1 - \dfrac{\varepsilon}{r} E_2(r,\theta) - \dfrac{\varepsilon^2}{r} H_2(r,\theta), \end{cases}$$

where

$$E_1(r,\theta) = \sum_{i=0}^{n} a_i h_{i,2\alpha+1}(\theta) r^{2\alpha+i} + \sum_{i=0}^{k} d_i h_{i,2\alpha+3}(\theta) r^{2\alpha+i+2} + $$
$$+ \sum_{i=0}^{m} c_i h_{i,2\alpha+2}(\theta) r^{2\alpha+i+1},$$

$$H_1(r,\theta) = \sum_{i=0}^{n} A_i h_{i,2\alpha+1}(\theta) r^{2\alpha+i} + \sum_{i=0}^{k} D_i h_{i,2\alpha+3}(\theta) r^{2\alpha+i+2} + $$
$$+ \sum_{i=0}^{m} C_i h_{i,2\alpha+2}(\theta) r^{2\alpha+i+1},$$

$$E_2(r,\theta)=\sum_{i=0}^{n} a_i h_{i+1,2\alpha}(\theta)\, r^{2\alpha+i}+\sum_{i=0}^{k} d_i h_{i+1,2\alpha+2}(\theta)\, r^{2\alpha+i+2}+$$
$$+\sum_{i=0}^{m} c_i h_{i+1,2\alpha+1}(\theta)\, r^{2\alpha+i+1},$$
$$H_2(r,\theta)=\sum_{i=0}^{n} A_i h_{i+1,2\alpha}(\theta)\, r^{2\alpha+i}+r^2\sum_{i=0}^{k} D_i h_{i+1,2\alpha+2}(\theta)\, r^{2\alpha+i+2}+$$
$$+\sum_{i=0}^{m} C_i h_{i+1,2\alpha+1}(\theta)\, r^{2\alpha+i+1},$$

where $h_{i,\alpha}(\theta)=\cos^i\theta\sin^i\theta$ Taking $\theta$ as the new independent variable, system (2) becomes

$$\frac{dr}{d\theta}=\varepsilon F_1(r,\theta)+\varepsilon^2F_2(r,\theta)+O\left(\varepsilon^3\right), \tag{3}$$

where

$$\begin{aligned} F_1(r,\theta) &= E_1(r,\theta), \\ F_2(r,\theta) &= H_1(r,\theta)-\frac{1}{r}E_1(r,\theta)\,E_2(r,\theta). \end{aligned} \tag{4}$$

First we shall study the limit cycles of the differential equation (3) using the averaging theory of first order. Therefore, by Section 2 we must study the simple positive zeros of the function

$$F_{10}(r)=\frac{1}{2\pi}\int_0^{2\pi} F_1(r,\theta)\,d\theta.$$

For every one of these zeros we will have a limit cycle of the polynomial differential system (2). If $F_{10}(r)$ is identically zero, applying the theory of averaging of second order (see again Section 2) every simple positive zero of the function

$$F_{20}(r)=\frac{1}{2\pi}\int_0^{2\pi}\left(\frac{d}{dr}F_1(r,\theta)\left(\int_0^{\theta}F_1(r,s)ds\right)+F_2(r,\theta)\right)d\theta,$$

will provide a limit cycle of the polynomial differential system (2).

## 3.1 Proof of statement (a) of Theorem 1

Taking into account the expression of (4), in order to obtain $F_{10}$ is necessary to evaluate the integrals of the form

$$\int_0^{\pi}\cos^i\theta\sin^j\theta d\theta$$

In the following lemma we compute these integrals.

**Lemma 1.** *Let* $h_{i,j}(\theta) = \cos^i\theta \sin^j\theta$ *and* $\delta_{i,j}(\theta) = \int_0^\theta h_{i,j}(s)\,ds$ *Then*

$$\delta_{i,j}(2\pi) = \begin{cases} 0 & \text{if } i \text{ is odd or } j \text{ is odd,} \\ \frac{(j-1)(j-3)\dots 1}{(j+i)(j+i-2)\dots(i+2)} \frac{1}{2^{i-1}} \binom{i}{\frac{i}{2}} \pi & \text{if } i \text{ and } j \text{ are even,} \end{cases} \tag{5}$$

*where* $\binom{i}{\frac{i}{2}} = \dfrac{i!}{\left(\frac{i}{2}!\right)^2}$

*Proof.* Using the integrals 12 and 13 given at the appendix with $\theta = 2\pi$ and taking into account that $h_{i,j}(2\pi) = 0$ if $j \neq 0$ we have that

$$\delta_{i,2j}(2\pi) = \frac{(2j-1)(2j-3)\dots 1}{(2j+i)(2j+i-2)(i+2)} \delta_{i,0}(2\pi), \delta_{i,2j+1}(2\pi) = 0. \tag{6}$$

Again, using the integrals 10 and 11 given in the appendix, with $\theta = 2\pi$, we have that $\delta_{2i,0}(2\pi) = \frac{(2i-1)(2i-3)}{2^i i!} 2\pi$ and $\delta_{2i+1,0}(2\pi) = 0$, Substituting $\delta_{2i,0}(2\pi)$ and $\delta_{2i+1,0}(2\pi)$ given as above into (6) we obtain (5). Using this lemma we shall obtain in the next proposition the function $F_{10}(r)$: □

**Proposition 1.** *We have*

$$F_{10}(r) = \frac{r^{2\alpha+1}}{2\pi} \sum_{i=0}^{\left[\frac{m}{2}\right]} c_{2i}\ \delta_{2i,2\alpha+2}(2\pi)\, r^{2i}. \tag{7a}$$

*Proof.* The function $F_{10}(r)$ is given by

$$\begin{aligned} F_{10}(r) &= \frac{1}{2\pi} \int_0^{2\pi} \sum_{i=0}^{n} a_i h_{i,2\alpha+1}(\theta)\, r^{2\alpha+i} d\theta + \frac{1}{2\pi} \int_0^{2\pi} \sum_{i=0}^{k} d_i h_{i,2\alpha+3}(\theta)\, r^{2\alpha+i+2} d\theta \\ &\quad + \frac{1}{2\pi} \int_0^{2\pi} \sum_{i=0}^{m} c_i h_{i,2\alpha+2}(\theta)\, r^{2\alpha+i+1} d\theta. \end{aligned}$$

Using lemma 1, we obtain

$$\int_0^{2\pi} h_{i,2\alpha+1}(\theta)\, d\theta = \int_0^{2\pi} h_{i,2\alpha+3}(\theta)\, d\theta = 0,\ \forall i \in \mathbb{N}.$$

Then

$$F_{10}(r) = \frac{1}{2\pi} \int_0^{2\pi} \sum_{i=0}^{m} c_i h_{i,2\alpha+2}(\theta)\, r^{2\alpha+i+1} d\theta$$

$$\begin{aligned} &= \int_0^{2\pi} \sum_{\substack{i=0\\ i \text{ odd}}}^{m} c_i h_{i,2\alpha+2}(\theta)\, r^{2\alpha+i+1} + \sum_{\substack{i=0\\ i \text{ even}}}^{m} c_i h_{i,2\alpha+2}(\theta)\, r^{2\alpha+i+1} d\theta \\ &= \sum_{i=0}^{\left[\frac{m-1}{2}\right]} c_{2i+1} \int_0^{2\pi} h_{2i+1,2\alpha+2}(\theta)\, r^{2\alpha+2i+2} + \sum_{i=0}^{\left[\frac{m}{2}\right]} c_{2i} \int_0^{2\pi} h_{2i,2\alpha+2}(\theta)\, r^{2\alpha+2i+1} d\theta. \end{aligned}$$

Again, using lemma 1, we conclude that $\int_0^{2\pi} h_{2i+1,2\alpha+2}(\theta)\, d\theta = 0$, then

$$F_{10}(r) = \frac{r^{2\alpha+1}}{2\pi} \sum_{i=0}^{\left[\frac{m}{2}\right]} c_{2i} \delta_{2i,2\alpha+2}(2\pi)\, r^{2i}.$$

□

From Proposition 1, the polynomial $F_{10}(r)$ has at most $\lambda_1 = \left\{\left[\frac{m}{2}\right]\right\}$ positive roots, and we can choose $c_{2i}$ in such a way that $F_{10}(r)$ has exactly $\lambda_1$ simple positive roots, hence the statement (a) of Theorem 1 is proved.

## 3.2 Proof of statement (b) of Theorem 1

Now using the results stated in Section 2 we shall apply the second order averaging theory to the previous differential equation. For this we put $F_{10}(r) \equiv 0$, which is equivalent to

$$c_i = 0, \quad \text{for all } i \text{ even}. \tag{8}$$

We must study the simple positive zeros of the function

$$F_{20}(r) = \frac{1}{2\pi} \int_0^{2\pi} \left( \frac{d}{dr} F_1(r,\theta) \left( \int_0^{\theta} F_1(r,s) ds \right) + F_2(r,\theta) \right) d\theta.$$

We split the computation of the function $F_{20}(r)$ in two pieces, i.e. we define $2\pi F_{20}(r) = \Phi(r) + \Psi(r)$, where

$$\begin{aligned} \Phi(r) &= \int_0^{2\pi} \frac{d}{dr} F_1(r,\theta) \left( \int_0^{\theta} F_1(r,s) ds \right) d\theta, \\ \Psi(r) &= \int_0^{2\pi} F_2(r,\theta)\, d\theta = \int_0^{2\pi} \left( H_1(r,\theta) - \frac{1}{r} E_1(r,\theta) E_2(r,\theta) \right) d\theta. \end{aligned}$$

First we compute the integrals $\int_0^{2\pi} \delta_{i,j}(\theta) h_{p,q}(\theta) d\theta$, in the following lemma.

**Lemma 2.** *Let* $\eta_{i,j}^{p,q}(2\pi) = \int_0^{2\pi} \delta_{i,j}(\theta) h_{p,q}(\theta) d\theta$. *Then the following equalities hold:*

*a) The integral* $\eta_{2i+1,0}^{p,q}(2\pi)$ *is zero if* $p$ *is odd or* $q$ *is even, and is equal to*

$$\frac{1}{2i+1}\left(\sum_{l=0}^{i-1} \frac{2^l j(j-1)\dots(j-l+1)}{(2i-1)(2i-3)\dots(2i-2l-1)}\delta_{2i+p+2l-2;q+1}(2\pi)\right) + \frac{1}{2i+1}\delta_{2i+p;q+1}(2\pi)$$

*if* $p$ *is even and* $q$ *is odd.*

*b) The integral* $\eta_{2i+1,2j+1}^{p,q}(2\pi)$ *is zero if* $p$ *is odd or* $q$ *is odd, and is equal to*

$$-\frac{1}{2j+2i+2}\left(\sum_{l=1}^{j-1} \frac{\left(2^l j(j-1)\dots(j-l+1)\right)\delta_{2i+p+2;2j-2l+q}(2\pi)}{(2j+2i)(2j+2i-2)\dots(2j+2i-2l+2)}\right) - \frac{1}{2j+2i+2}\delta_{2i+p+2,2j+q}(2\pi)$$

*if* $p$ *is even and* $q$ *is even.*

*c) The integral* $\eta_{2i,2j+1}^{p,q}(2\pi)$ *is zero if* $p$ *is even or* $q$ *is odd, and is equal to*

$$-\frac{1}{2j+2i+1}\left(\sum_{l=1}^{j-1} \frac{\left(2^l j(j-1)\dots(j-l+1)\right)\delta_{2i+p+1;2j-2l+q}(2\pi)}{(2j+2i-1)(2j+2i-3)\dots(2j+2i-2l+1)}\right) - \frac{1}{2j+2i+1}\delta_{2i+p+1,2j+q}(2\pi)$$

*if* $p$ *is odd and* $q$ *is even.*

*(d) The integral* $\eta_{2i+1,2j}^{p,q}(2\pi)$ *is zero if* $p$ *is odd or* $q$ *is even, and is equal to*

$$-\frac{1}{2j+2i+1}\left(\sum_{l=1}^{j-1} \frac{\left((2j-1)(2j-3)\dots(2j-2l+1)\right)\delta_{2i+p+2;2j-2l+q-1}(2\pi)}{(2j+2i-1)(2j+2i-3)\dots(2j+2i-2l+1)}\right) - \frac{1}{2j+2i+1}\delta_{2i+p+2;2j+q+1}(2\pi) + \frac{(2j-1)(2j-3)\dots 1}{(2j+2i+1)(2j+2i-1)\dots(2i+3)}\eta_{2i+1,0}^{p,q}(2\pi)$$

*if* $p$ *is even and* $q$ *is odd.*

*Proof.* Using the integral 12 of the appendix and taking into account $h_{i,j}(\theta)\, h_{p,q}(\theta) = h_{i+p,j+q}(\theta)$, we have

$$\begin{aligned}\eta^{p,q}_{2i+1,0}(2\pi) &= \frac{1}{2i+1}\sum_{l=0}^{i-1}\frac{2^l j(j-1)\ldots.(j-l+1)}{(2i-1)(2i-3)\ldots(2i-2l-1)}\int_0^{2\pi} h_{2i+p+2l-2;q+1}(\theta)d\theta \\ &\quad +\frac{1}{2i+1}\int_0^{2\pi} h_{2i+p;q+1}(\theta)d\theta.\end{aligned}$$

By using lemma 2, statement (a) follows. Using the integral 14 of the appendix and taking into account $h_{i,j}(\theta)\, h_{p,q}(\theta) = h_{i+p,j+q}(\theta)$, we have

$$\begin{aligned}\eta^{p,q}_{2i+1,2j+1}(2\pi) &= -\frac{1}{2j+2i+2}\int_0^{2\pi} h_{2i+p+2,2j+q}(\theta)d\theta \\ &\quad -\frac{1}{2j+2i+2}\left(\begin{array}{c}\sum\limits_{l=1}^{j-1}\frac{2^l j(j-1)\ldots.(j-l+1)}{(2j+2i)(2j+2i-2)\ldots(2j+2i-2l+2)} \\ *\int_0^{2\pi} h_{2i+p+2;2j-2l+q}(\theta)d\theta\end{array}\right)\end{aligned}$$

and

$$\begin{aligned}\eta^{p,q}_{2i,2j+1}(2\pi) &= -\frac{1}{2j+2i+1}\int_0^{2\pi} h_{2i+p+1,2j+q}(\theta)d\theta \\ &\quad -\frac{1}{2j+2i+1}\left(\begin{array}{c}\sum\limits_{l=1}^{j-1}\frac{2^l j(j-1)\ldots.(j-l+1)}{(2j+2i-1)(2j+2i-3)\ldots(2j+2i-2l+1)} \\ *\int_0^{2\pi} h_{2i+p+1;2j-2l+q}(\theta)d\theta.\end{array}\right)\end{aligned}$$

Using again lemma 2, statement (b), (c) follows. Using the integral 12 and 13 of the appendix and taking into account $h_{i,j}(\theta)\, h_{p,q}(\theta) = h_{i+p,j+q}(\theta)$ and using lemma 2, we obtain

$$\begin{aligned}\eta^{p,q}_{2i+1,2j}(2\pi) &= \frac{(2j-1)(2j-3)\ldots.1}{(2j+2i+1)(2j+2i-1)\ldots(2i+3)}\eta^{p,q}_{2i+1,0}(2\pi) \\ &\quad -\frac{1}{2j+2i+1} \\ &\quad *\left(\sum_{l=1}^{j-1}\frac{((2j-1)(2j-3)\ldots.(2j-2l+1))\,\delta_{2i+p+2;2j-2l+q-1}(2\pi)}{(2j+2i-1)(2j+2i-3)\ldots(2j+2i-2l+1)}\right) \\ &\quad -\frac{1}{2j+2i+1}\left(\delta_{2i+p+2;2j+q+1}(2\pi)\right).\end{aligned}$$

Hence statement (d) of lemma 2 is proved. □

**Proposition 2.** *The integral $\Phi(r)$ can be expressed by*

$$\Phi(r) = r^{4\alpha+1} P_1\left(r^2\right).$$

*where $P_1\left(r^2\right)$ is a polynomial in the variable $r^2$ of degree*

$$\lambda_2 = \max\left\{\left[\frac{n}{2}\right] + \left[\frac{m-1}{2}\right]; \left[\frac{k}{2}\right] + \left[\frac{m-1}{2}\right] + 1\right\}.$$

*Proof.* First, we have

$$\begin{aligned}
F_1(r,\theta) &= \sum_{\substack{i=0\\ \text{i odd}}}^{n} a_i h_{i,2\alpha+1}(\theta) r^{2\alpha+i} + \sum_{\substack{i=0\\ i \text{ odd}}}^{k} d_i h_{i,2\alpha+3}(\theta) r^{2\alpha+i+2} \\
&\quad + \sum_{\substack{i=0\\ i \text{ odd}}}^{m} c_i h_{i,2\alpha+2}(\theta) r^{2\alpha+i+1} + \sum_{\substack{i=0\\ i \text{ even}}}^{n} a_i h_{i,2\alpha+1}(\theta) r^{2\alpha+i} \\
&\quad + \sum_{\substack{i=0\\ i \text{ even}}}^{k} d_i h_{i,2\alpha+3}(\theta) r^{2\alpha+i+2} + \sum_{\substack{i=0\\ i \text{ even}}}^{m} c_i h_{i,2\alpha+2}(\theta) r^{2\alpha+i+1} \\
&= \sum_{i=0}^{\left[\frac{n-1}{2}\right]} a_{2i+1} h_{2i+1,2\alpha+1}(\theta) r^{2\alpha+2i+1} + \sum_{i=0}^{\left[\frac{n}{2}\right]} a_{2i} h_{2i,2\alpha+1}(\theta) r^{2\alpha+2i} \\
&\quad + \sum_{i=0}^{\left[\frac{k-1}{2}\right]} d_{2i+1} h_{2i+1,2\alpha+3}(\theta) r^{2\alpha+2i+3} + \sum_{i=0}^{\left[\frac{k}{2}\right]} d_{2i} h_{2i,2\alpha+3}(\theta) r^{2\alpha+2i+2} \\
&\quad + \sum_{i=0}^{\left[\frac{m-1}{2}\right]} c_{2i+1} h_{2i+1,2\alpha+2}(\theta) r^{2\alpha+2i+2}.
\end{aligned}$$

Next we calculate the terms of this integral. First we have that

$$\begin{aligned}
\frac{d}{dr} F_1(r,\theta) &= \sum_{i=0}^{\left[\frac{n-1}{2}\right]} (2\alpha+2i+1) a_{2i+1} h_{2i+1,2\alpha+1}(\theta) r^{2\alpha+2i} \\
&\quad + \sum_{i=0}^{\left[\frac{k-1}{2}\right]} (2\alpha+2i+3) d_{2i+1} h_{2i+1,2\alpha+3}(\theta) r^{2\alpha+2i+2} \\
&\quad + \sum_{i=0}^{\left[\frac{m-1}{2}\right]} (2\alpha+2i+2) c_{2i+1} h_{2i+1,2\alpha+2}(\theta) r^{2\alpha+2i+1} \\
&\quad + \sum_{i=0}^{\left[\frac{n}{2}\right]} (2\alpha+2i) a_{2i} h_{2i,2\alpha+1}(\theta) r^{2\alpha+2i-1}
\end{aligned}$$

$$+\sum_{i=0}^{\left[\frac{k}{2}\right]}\left(2\alpha+2i+2\right)d_{2i}h_{2i,2\alpha+3}\left(\theta\right)r^{2\alpha+2i+1}$$

Then

$$\begin{aligned}\int_0^\theta F_1(r,s)ds &= \sum_{i=0}^{\left[\frac{n-1}{2}\right]} a_{2i+1}\delta_{2i+1,2\alpha+1}\left(\theta\right)r^{2\alpha+2i+1}\\ &+\sum_{i=0}^{\left[\frac{k-1}{2}\right]} d_{2i+1}\delta_{2i+1,2\alpha+3}\left(\theta\right)r^{2\alpha+2i+3}\\ &+\sum_{i=0}^{\left[\frac{m-1}{2}\right]} c_{2i+1}\delta_{2i+1,2\alpha+2}\left(\theta\right)r^{2\alpha+2i+2}\\ &+\sum_{i=0}^{\left[\frac{n}{2}\right]} a_{2i}\delta_{2i,2\alpha+1}\left(\theta\right)r^{2\alpha+2i}\\ &+\sum_{i=0}^{\left[\frac{k}{2}\right]} d_{2i}\delta_{2i,2\alpha+3}\left(\theta\right)r^{2\alpha+2i+2}.\end{aligned}$$

By using lemma 2, from the 25 main products of $\Phi\left(r\right)$ only the following 4 are not zero when we integrate them between 0 and $2\pi$. So the terms of $\Phi\left(r\right)$ which will contribute to $F_{20}(r)$ are :

$$\begin{aligned}\Phi\left(r\right) &= \sum_{i=0}^{\left[\frac{n}{2}\right]}\sum_{p=0}^{\left[\frac{m-1}{2}\right]}\left(2\alpha+2i\right)a_{2i}c_{2p+1}\eta_{2p+1,2\alpha+2}^{2i,2\alpha+1}\left(2\pi\right)r^{4\alpha+2i+2p+1}\\ &+\sum_{i=0}^{\left[\frac{k}{2}\right]}\sum_{p=0}^{\left[\frac{m-1}{2}\right]}\left(2\alpha+2i+2\right)d_{2i}c_{2p+1}\eta_{2p+1,2\alpha+2}^{2i,2\alpha+3}\left(2\pi\right)r^{4\alpha+2i+2p+3}\\ &+\sum_{i=0}^{\left[\frac{m-1}{2}\right]}\sum_{p=0}^{\left[\frac{n}{2}\right]}\left(2\alpha+2i+2\right)c_{2i+1}a_{2p}\eta_{2p,2\alpha+1}^{2i+1,2\alpha+2}\left(2\pi\right)r^{4\alpha+2i+2p+1}\\ &+\sum_{i=0}^{\left[\frac{m-1}{2}\right]}\sum_{p=0}^{\left[\frac{k}{2}\right]}\left(2\alpha+2i+2\right)c_{2i+1}d_{2p}\eta_{2p,2\alpha+3}^{2i+1,2\alpha+2}\left(2\pi\right)r^{4\alpha+2i+2p+3}\\ &= r^{4\alpha+1}P_1\left(r^2\right)\end{aligned}$$

where $P_1$ is polynomial in the variable $r^2$ of degree $\lambda_2$,

$$\lambda_2=\max\left\{\left[\frac{m-1}{2}\right]+\left[\frac{n}{2}\right];\left[\frac{m-1}{2}\right]+\left[\frac{k}{2}\right]+1\right\}.$$

Finally, we obtain $\Phi(r)$ is a polynomial in the variable $r^2$ of the form

$$\Phi(r) = r^{4\alpha+1} P_1\left(r^2\right).$$

This completes the proof of the Proposition 2. $\square$

In order to complete the computation of $F_{20}(r)$ we must determine the function $\Psi(r)$.

**Proposition 3.** *The integral $\Psi(r)$ can be expressed by*

$$\Psi(r) = r^{2\alpha+1}\left(P_2\left(r^2\right) + r^{2\alpha} P_3\left(r^2\right)\right)$$

*where $P_2\left(r^2\right)$ is a polynomial in the variable $r^2$ of degree*

$$\lambda_1 = \left[\frac{m}{2}\right],$$

*$P_3\left(r^2\right)$ is a polynomial in the variable $r^2$ of degree*

$$\lambda_3 = \max\left\{\left[\frac{m-1}{2}\right] + \left[\frac{n}{2}\right]; \left[\frac{m-1}{2}\right] + \left[\frac{k}{2}\right] + 1\right\}.$$

*Proof.* Firstly we calculate,

$$\begin{aligned}\int_0^{2\pi} H_1(r,\theta)\, d\theta &= \sum_{i=0}^{n} A_i r^{2\alpha+i} \int_0^{2\pi} h_{i,2\alpha+1}(\theta)\, d\theta + \sum_{i=0}^{k} D_i r^{2\alpha+i+2} \int_0^{2\pi} h_{i,2\alpha+3}(\theta)\, d\theta \\ &\quad + \sum_{i=0}^{m} C_i r^{2\alpha+i+1} \int_0^{2\pi} h_{i,2\alpha+2}(\theta)\, d\theta.\end{aligned}$$

Using lemma 2, we conclude that $\int_0^{2\pi} h_{i,2\alpha+1}(\theta)\, d\theta = \int_0^{2\pi} h_{i,2\alpha+3}(\theta)\, d\theta = 0$, and we have

$$\int_0^{2\pi} H_1(r,\theta)\, d\theta = \sum_{\substack{i=0 \\ \text{i even}}}^{m} C_i r^{2\alpha+i+1} \int_0^{2\pi} h_{i,2\alpha+2}(\theta)\, d\theta = \sum_{i=0}^{\left[\frac{m}{2}\right]} C_i r^{2\alpha+i+1} \int_0^{2\pi} h_{i,2\alpha+2}(\theta)\, d\theta.$$

Then

$$\begin{aligned}\int_0^{2\pi} H_1(r,\theta)\, d\theta &= \pi \sum_{i=0}^{\left[\frac{m}{2}\right]} C_{2i}\delta_{2i,2\alpha+2}(2\pi)\, r^{2\alpha+2i+1} \\ &= r^{2\alpha+1} P_2\left(r^2\right)\end{aligned}$$

where $P_2$ is a polynomial in the variable $r^2$ of degree $\lambda_1$.

Finally, we shall study the contribution of the second part $\int_0^{2\pi} \frac{1}{r} E_1(r,\theta) E_2(r,\theta)\, d\theta$ of $F_2(r,\theta)$ to $F_{20}(r)$. Using the expressions of $E_1(r,\theta)$ and $E_2(r,\theta)$ and taking into account that $c_i = 0$ for all $i$ even, we have :

$$\begin{aligned} E_1(r,\theta) = &\sum_{i=0}^{\left[\frac{n-1}{2}\right]} a_{2i+1} h_{2i+1,2\alpha+1}(\theta)\, r^{2\alpha+2i+1} + \sum_{i=0}^{\left[\frac{k-1}{2}\right]} d_{2i+1} h_{2i+1,2\alpha+3}(\theta)\, r^{2\alpha+2i+3} \\ &+ \sum_{i=0}^{\left[\frac{m-1}{2}\right]} c_{2i+1} h_{2i+1,2\alpha+2}(\theta)\, r^{2\alpha+2i+2} + \sum_{i=0}^{\left[\frac{n}{2}\right]} a_{2i} h_{2i,2\alpha+1}(\theta)\, r^{2\alpha+2i} \\ &+ \sum_{i=0}^{\left[\frac{k}{2}\right]} d_{2i} h_{2i,2\alpha+3}(\theta)\, r^{2\alpha+2i+2} \end{aligned}$$

and

$$\begin{aligned} E_2(r,\theta) = &\sum_{p=0}^{\left[\frac{n-1}{2}\right]} a_{2p+1} h_{2p+2,2\alpha}(\theta)\, r^{2\alpha+2p+1} + \sum_{p=0}^{\left[\frac{k-1}{2}\right]} d_{2p+1} h_{2p+2,2\alpha+2}(\theta)\, r^{2\alpha+2p+3} \\ &+ \sum_{p=0}^{\left[\frac{m-1}{2}\right]} c_{2p+1} h_{2p+2,2\alpha+1}(\theta)\, r^{2\alpha+2p+2} + \sum_{p=0}^{\left[\frac{n}{2}\right]} a_{2p} h_{2p+1,2\alpha}(\theta)\, r^{2\alpha+2p} \\ &+ \sum_{p=0}^{\left[\frac{k}{2}\right]} d_{2p} h_{2p+1,2\alpha+2}(\theta)\, r^{2\alpha+2p+2}. \end{aligned}$$

Using Lemma 2, from the 25 main products of $\int_0^{2\pi} \frac{1}{r} E_1(r,\theta) E_2(r,\theta)\, d\theta$, only the following 4 are not zero when we integrate them between 0 and $2\pi$, So the terms which will contribute to $F_{20}(r)$ are

$$\begin{aligned} \int_0^{2\pi} \frac{1}{r} E_1(r,\theta) E_2(r,\theta)\, d\theta = &\sum_{i=0}^{\left[\frac{n}{2}\right]} \sum_{p=0}^{\left[\frac{m-1}{2}\right]} a_{2i} c_{2p+1} \delta_{2i+2p+2,4\alpha+2}(2\pi)\, r^{4\alpha+2i+2p+1} \\ &+ \sum_{i=0}^{\left[\frac{k}{2}\right]} \sum_{p=0}^{\left[\frac{m-1}{2}\right]} d_{2i} c_{2p+1} \delta_{2i+2p+2,4\alpha+2}(2\pi)\, r^{4\alpha+2i+2p+3} \\ &+ \sum_{i=0}^{\left[\frac{m-1}{2}\right]} \sum_{p=0}^{\left[\frac{n}{2}\right]} c_{2i+1} a_{2p} \delta_{2i+2p+2,4\alpha+2}(2\pi)\, r^{4\alpha+2i+2p+1} \\ &+ \sum_{i=0}^{\left[\frac{m-1}{2}\right]} \sum_{p=0}^{\left[\frac{k}{2}\right]} c_{2i+1} d_{2p} \delta_{2i+2p+2,4\alpha+2}(2\pi)\, r^{4\alpha+2i+2p+3} \\ &= r^{4\alpha+1} P_3\left(r^2\right) \end{aligned}$$

where $P_3$ is a polynomial in the variable $r^2$ of degree

$$\lambda_3 = \max\left\{ \left[\frac{m-1}{2}\right] + \left[\frac{n}{2}\right]; \left[\frac{m-1}{2}\right] + \left[\frac{k}{2}\right] + 1 \right\}.$$

Then, we obtain $\Psi(r)$ is a polynomial in the variable $r^2$

$$\Psi(r) = r^{2\alpha+1}\left(P_2\left(r^2\right) + r^{2\alpha} P_3\left(r^2\right)\right)$$

of degree

$$\lambda_{\Psi(r)} = \max\{\lambda_1, \lambda_3 + \alpha\}.$$

Finally, we obtain $F_{20}(r)$ is a polynomial in the variable $r^2$ of the form

$$F_{20}(r) = \frac{r^{2\alpha+1}}{2\pi}\left(r^{2\alpha}P_1\left(r^2\right) + P_2\left(r^2\right) + r^{2\alpha}P_3\left(r^2\right)\right).$$

To find the real positive roots of $F_{20}$ we must find the zeros of a polynomial in $r^2$ of degree $\lambda = \max\{\lambda_1, \lambda_2 + \alpha, \lambda_3 + \alpha\}$. This yields that $F_{20}$ has at most $\lambda$ real positive roots. Hence, Theorem 1 is proved. Moreover, we can choose the coefficients $a_i$,$c_i$, $d_i$,$A_i$,$C_i$,$D_i$ in such a way that $F_{20}$ has exactly $\lambda$ real positive roots. This completes the proof of Theorem 1. □

## 4 Example

We consider the differential system 2 with $k = n = 1, m = 3, \alpha = 1$

$$\begin{cases} \dot{x} = y, \\ \dot{y} = -x - \varepsilon(\left(-\frac{118}{65} + x\right)y^2 + \left(\left(-\frac{13}{427}x + \frac{1}{61}x^3\right)\right)y^3 + (1+x)\,y^4) \\ -\varepsilon^2\left(\left(-1 - \frac{1}{4}x\right)y^2 + \left(\frac{1}{80} + \frac{967}{34\,160}x^2 + \frac{1}{8}x^3\right)y^3 - xy^4\right) \end{cases} \tag{9}$$

An easy computation shows that $F_{10}(r)$ is identically zero, so to look for the limit cycles, we must solve the equation $F_{20}(r) = 0$ which is equivalent to

$$-\frac{1}{1280}r^3\left(r^6 - 6r^4 + 11r^2 - 6\right) = 0$$

This equation has exactly three positive roots $r_1 = 1, r_2 = \sqrt{2}, r_3 = \sqrt{3}$. According with Theorem 1, that system (9) has exactly three limit cycles bifurcating from the periodic orbits of the linear center $\dot{x} = y, \dot{y} = -x$.

## 5 Appendix

In this appendix, we recall some formulas used during this article; for more details see [7]. For $i \geq 0$ and $j \geq 0$, we have

$$\begin{aligned} \int_0^\theta \cos^i s \sin^j s ds &= \frac{\cos^{i-1}\theta \sin^{j+1}\theta}{i+j} + \frac{i-1}{i+\alpha}\int_0^\theta \cos^{i-2} s \sin^j s ds \\ &= \frac{\cos^{i-1}\theta \sin^{j+1}\theta}{i+j} + \frac{\alpha-1}{i+\alpha}\int_0^\theta \cos^i s \sin^{j-2} s ds, \end{aligned} \tag{10}$$

$$\begin{aligned}\int_0^\theta \cos^{2i} s ds &= \frac{\sin\theta}{2i}\sum_{l=1}^{i-1}\frac{(2i-1)(2i-3)\dots.(2i-2l+1)}{2^l(i-1)(i-2).(i-l)}\cos^{2i-2l-1}\theta \\ &\quad+\frac{\sin\theta}{2i}\cos^{2i-1}\theta+\frac{(2i-1)(2i-3)\dots.1}{2^i i!}\theta \\ &= \frac{1}{2^{2i-1}}\sum_{l=0}^{i-1}\begin{pmatrix}2i\\ l\end{pmatrix}\frac{\sin 2(i-l)\theta}{2(i-l)}+\frac{1}{2^{2i}}\begin{pmatrix}2i\\ i\end{pmatrix}\theta,\end{aligned} \tag{11}$$

$$\begin{aligned}\int_0^\theta \cos^{2i+1} s ds &= \frac{\sin\theta}{2i+1}\sum_{l=1}^{i-1}\frac{2^{l+1}i(i-1)\dots..(i-l)}{(2i-1)(2i-3)\dots.(2i-2l-1)}\cos^{2i-2l-2}\theta \\ &\quad+\frac{\sin\theta}{2i+1}\cos^{2i}\theta \\ &= \frac{1}{2^{2i}}\sum_{l=0}^{i-1}\begin{pmatrix}2i+1\\ l\end{pmatrix}\frac{\sin(2i-2l+1)\theta}{(2i-2l+1)},\end{aligned} \tag{12}$$

where $\begin{pmatrix}2i\\ p\end{pmatrix}=\frac{2i!}{p!(2i-p)!}$

$$\begin{aligned}&\int_0^\theta \cos^i s \sin^{2j} s ds \\ &= -\frac{\cos^{i+1}\theta}{2j+1}\sum_{l=1}^{j-1}\frac{(2j-1)(2j-3)\dots(2j-2l+1)}{(2j+i-2)(2j+i-4)\dots(2j+i-2l)}\sin^{2j-2l-1}\theta \\ &\quad+\frac{(2j-1)(2j-3)\dots1}{(2j+i)(2j+i-2)\dots(i+2)}\int_0^\theta \cos^i s ds,\end{aligned} \tag{13}$$

$$\begin{aligned}&\int_0^\theta \cos^i s \sin^{2j+1} s ds \\ &= -\frac{\cos^{i+1}\theta}{2j+i+1}\sum_{l=1}^{j-1}\frac{2^l j(j-1)\dots..(j-l+1)}{(2j+i-1)(2j+i-3)\dots(2j+i-2l+1)}\sin^{2j-2l}\theta \\ &\quad-\frac{\cos^{i+1}\theta}{2j+i+1}\sin^{2\alpha}\theta.\end{aligned} \tag{14}$$

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Sabah Benadouane
Laboratory of Fundamental and Numerical Mathematics
Department of Mathematics, University Ferhat Abbas Sétif
E-mail: *sabah.benadouane@univ-setif.dz*

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Aziza Berbache
Laboratory of Applied Mathematics, University Ferhat Abbas Sétif, Department of Mathematics, University of Bordj Bou Arréridj
E-mail: *azizaberbache@hotmail.fr*

Ahmed Bendjeddou
Laboratory of Applied Mathematics, Department of Mathematics, University Ferhat Abbas Sétif
E-mail: *bendjeddou@univ-setif.dz*