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Second order state-dependent sweeping process with unbounded perturbation

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Abstract. We establish, in the setting of an infinite dimensional Hilbert space, results concerning the existence of solutions of second order "nonconvex sweeping process" for a class of uniformly prox-regular sets depending on time and state. The perturbation considered here is general and takes the form of a sum of a single-valued Carathéodory mapping and a set-valued unbounded mapping. We deal also with a delayed perturbation, that is the external forces applied on the system in presence of a finite delay. We extend a discretization approach known for the time-dependent case to the time and state-dependent sweeping process.

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1 Introduction

The second order perturbed state-dependent nonconvex sweeping process has been a particular attraction for many authors during the last years, it takes the following form: let H be a Hilbert space, T_0 and T be two non-negative real numbers with $0 \leq T_0 < T$, and $D(t,x)$ be a nonempty closed subset of H for each $t \in [T_0,T]$ and $x \in H$. Given $b \in H$ and $a \in D(T_0,b)$, we have to find two absolutely continuous mappings $u, v : [T_0,T]$ satisfying

$$(P_F)\begin{cases} -\dot{u}(t) \in N_{D(t,v(t))}(u(t)) + F(t,v(t),u(t)), \quad \text{a.e. } t \in [T_0,T] \\ v(t) = b + \displaystyle\int_{T_0}^{t} u(s)ds, \quad u(t) = a + \displaystyle\int_{T_0}^{t} \dot{u}(s)ds, \ \forall t \in [T_0,T], \\ \qquad\qquad u(t) \in D(t,v(t)), \ \forall t \in [T_0,T], \end{cases}$$

where $N_{D(t,v(t))}(u(t))$ denotes the normal cone to $D(t,v(t))$ at the point $u(t)$, $F : [T_0,T] \times H \times H \rightharpoonup H$ is a set-valued mapping. Such problem is an extension of the so-called Moreau's sweeping process for Lagrangian system to frictionless unilateral constraints. The differential inclusion (P_F) was studied for the first time when the sets $D(t,v(t))$ are convex and compact and $F \equiv 0$ by [9], then by [17] and [21]. The nonconvex case has been considered by [16], the authors proved the existence of solutions to (P_F) for uniformly prox-regular sets $D(t,v(t))$ with absolutely continuous variation in space and Lipschitz variation in time and with a single-valued perturbation. By means of a generalized version of the Shauder's theorem, [12] provided another approach to prove the existence for uniformly prox regular and ball-compact sets $D(t, v(t))$ with absolutely continuous variation in time, without perturbation and for the perturbed problem (even in presence of a delay). The existence of solution for such problem is established by proving the convergence of the Moreau's catching-up algorithm. For other approaches, we refer to [1–6, 11, 24, 25].

Our main purpose in this paper is to study, in an infinite dimensional Hilbert space, the second order sweeping process with two perturbations

$$(\mathcal{P})\begin{cases} -\dot{u}(t) \in N_{D(t,v(t))}(u(t)) + F(t, v(t), u(t)) + f(t, v(t), u(t)), \text{ a. e. } t \in [T_0, T]; \\ v(t) = b + \int_{T_0}^{t} u(s)ds; \ \ u(t) = a + \int_{T_0}^{t} \dot{u}(s)ds, \ \forall t \in [T_0, T]; \\ u(t) \in D(t, v(t)), \ \forall t \in [T_0, T], \end{cases}$$

where $F : [T_0, T] \times H \times H \rightharpoonup H$ is an upper semicontinuous set-valued map with nonempty closed convex values unnecessarily bounded and without any compactness condition and $f : [T_0, T] \times H \times H \to H$ is a Carathéodory mapping satisfying the linear growth condition. This work is motivated by the recent results obtained for the same problem by [20] and [22], where reduction approaches have been used. In [20], only a single-valued "Lipschitz" perturbation is considered, the authors reduced the problem for second order time and state-dependent sweeping process to a first order time-dependent one. They make use of the Shauder's fixed point argument in the line of the approach of [16]. Whereas the reduction approach of [22] is valid only in finite dimensional setting. Our aim in this paper is to generalize all the results obtained in the two cases, using a different approach, we weaken the hypotheses on the perturbation by taking a Carathéodory mapping satisfying a linear growth condition and an unbounded set-valued perturbation for which only the element of minimum norm satisfies a linear growth condition.

On the other hand, we extend another reduction approach, known for the time-dependent sweeping process in presence of delay; it consists to reduce a second order sweeping process with delayed perturbation to a problem without delay. We show that this approach is still valid in the case of time and state-dependent sweeping process. The paper is organized as follows. In Section 2, we recall some basic notations, definitions and useful results which are used throughout the paper. In Section 3, we provide the existence results for the problem $(\mathcal{P})$. The delayed problem is studied in the last section.

2 Notation and Preliminaries

We begin with some notations used in the paper. Let H be a real separable Hilbert space whose inner product is denoted by $\langle \cdot, \cdot \rangle$, and the associated norm by $\| \cdot \|$. We denote by $\overline{\mathbf{B}}_H$ the unit closed ball of H, $\mathcal{L}([T_0, T])$ the σ-algebra of Lebesgue measurable subsets of $[T_0, T]$ and by $\mathcal{B}(H)$ the Borel tribe on H. We denote also by $L^1_H([T_0, T])$ the space of all Lebesgue-Bochner integrable H-valued mappings defined on $[T_0, T]$, by $\mathcal{C}_H([T_0, T])$ the Banach space of all continuous mappings $u : [T_0, T] \to H$ endowed with the norm of uniform convergence.

For any nonempty closed subset S, S' of H, we denote by:
• $d(\cdot, S)$ the usual distance function associated with S;
• $\delta^*(x', S) = \sup\limits_{y\in S}\langle x', y\rangle$ the support function of S at $x' \in H$. If S is closed convex subset $d(x, S) = \sup\limits_{x'\in\overline{\mathbf{B}}_H}\big(\langle x', x\rangle - \delta^*(x', S)\big)$;
• $\mathrm{Proj}_S(u)$ the projection of u onto S defined by

$$Proj_S(u) = \{y \in S : \quad d(u, S) = \|u - y\|\},$$

is unique whenever S is closed convex;
• $\mathcal{H}$ the Hausdorff distance between S and S', defined by

$$\mathcal{H}(S, S') = \max\{\sup_{u\in S} d(u, S'), \sup_{v\in S'} d(v, S)\};$$

• $co(S)$ the convex hull of S and $\overline{co}(S)$ its closed convex hull, characterized by

$$\overline{co}(S) = \{x \in H : \ \forall x' \in H,\ \langle x', x\rangle \le \delta^*(x', S)\}.$$

Recall that $f : [T_0, T] \times H \to H$ is called a Carathéodory mapping if $f(\cdot, u)$ is measurable on $[T_0, T]$ for all $u \in H$ and $f(t, \cdot)$ is continuous on H for every $t \in [T_0, T]$.
A set-valued mapping $G : H \to H$ is called :
• upper semicontinuous if, for any open subset $\mathcal{V} \subset H$, the set $\{x \in H : \ G(x) \subset \mathcal{V}\}$ is open in H;
• scalarly upper semicontinuous on H if for every $h \in H$, $\delta^*(h, G(\cdot))$ is upper semicontinuous on H.

We need in the sequel to recall some definitions and results that will be used throughout the paper. Let A be an open subset of H and $\varphi : A \to (-\infty, +\infty]$ be a lower semicontinuous function, the proximal subdifferential $\partial^P\varphi(x)$, of φ at x (see [19]) is the set of all proximal subgradients of φ at x, any $\xi \in H$ is a proximal subgradient of φ at x if there exist positive numbers η and ς such that

$$\varphi(y) - \varphi(x) + \eta\|y - x\|^2 \ge \langle \xi, y - x\rangle,\ \forall y \in x + \varsigma\overline{\mathbf{B}}_H.$$

Let x be a point of $S \subset H$, we recall (see [19]) that the proximal normal cone to S at x is defined by $N^P_S(x) = \partial^P\Psi_S(x)$, where Ψ_S denotes the indicator function of S, i.e. $\Psi_S(x) = 0$ if $x \in S$ and $+\infty$ otherwise. Note that the proximal normal cone is also given by

$$N^P_S(x) = \{\xi \in H : \ \exists\varrho > 0 \ \text{ s.t. } \ x \in \mathrm{Proj}_S(x + \varrho\xi)\}.$$

When S is a closed set one has $\partial^P d(x, S) = N^P_S(x) \cap \overline{\mathbf{B}}_H$.
If φ is a real-valued locally-Lipschitz function defined on H, the Clarke subdifferential $\partial^C\varphi(x)$ of φ at x is the nonempty convex compact subset of H given by

$$\partial^C\varphi(x) = \{\xi \in H : \ \varphi^\circ(x; v) \ge \langle \xi, v\rangle, \forall v \in H\},$$

where

$$\varphi^\circ(x;v) = \limsup_{y\to x,\ t\downarrow 0} \frac{\varphi(y+tv)-\varphi(y)}{t}$$

is the generalized directional derivative of φ at x in the direction v (see [19]). The Clarke normal cone $N_S^C(x)$ to S at $x \in S$ is defined by polarity with T_S^C, that is,

$$N_S^C(x) = \{\xi \in H : \ \langle \xi, v\rangle \leq 0, \ \forall v \in T_S^C\},$$

where T_S^C denotes the clarke tangent cone, and is given by

$$T_S^C = \{v \in H : \ d^\circ(x, S; v) = 0\}.$$

Recall now, that for a given $r \in]0, +\infty]$ the subset S is uniformly r-prox-regular (see [19]) or equivalently r-proximally smooth ([23]) if and only if for all $\overline{x} \in S$ and all $0 \neq \xi \in N_S^P(\overline{x})$ one has

$$\langle \frac{\xi}{\|\xi\|}, x - \overline{x}\rangle \leq \frac{1}{2r}\|x-\overline{x}\|^2,$$

for all $x \in S$. We make the convention $\frac{1}{r} = 0$ for $r = +\infty$. Recall that for $r = +\infty$ the uniform r-prox-regularity of S is equivalent to the convexity of S. It's well known that the class of uniformly r-prox-regular sets is sufficiently large to include the class of convex sets, p-convex sets, $C^{1,1}$ submanifolds (possibly with boundary) of a Hilbert space and many other nonconvex sets (see [15, 20]). Furthermore, the following properties hold for a closed uniformly r-prox-regular set S:
- for any $N_S^P(x) = N_S^C(x) = N_S(x)$;
- the proximal subdifferential of $d(.,S)$ coincides with its Clarke subdifferential at all points $x \in H$ satisfying $d(x,S) < r$;
- for all $x \in H$ with $d(x,S) < r$, $\mathrm{Proj}_S(x)$ is a singleton of H.

The next proposition provides an upper semicontinuity property of the support function of the proximal subdifferential of the distance function to uniformly r-prox-regular sets.

Proposition 1. *Let $D : [T_0, T] \times H \rightharpoonup H$ be a uniformly r-prox regular closed valued mapping satisfying*

$$|d(u, D(t,x)) - d(v, D(s,y))| \leq ||u-v|| + v(t) - v(s) + L||x-y||$$

for all u, x, v, y in H and for all $s \leq t$ in $[T_0, T]$, where $v : [T_0, T] \to \mathbf{R}^+$ is a nondecreasing absolutely continuous function and L is a positive constant. Then the convex weakly compact valued mapping $(t, x, y) \to \partial^p d(y, D(t,x))$ satisfies the upper semicontinuity property: let (t_n, x_n) be a sequence in $[T_0, T] \times H$ converging to some $(t, x) \in [T_0, T] \times H$, and (y_n) be a sequence in H with $y_n \in D(t_n, x_n)$ for all n, converging to $y \in D(t, x)$, then, for any $z \in H$,

$$\limsup_{n\to\infty} \delta^*(z, \partial^p d(y_n, D(t_n, x_n))) \leq \delta^*(z, \partial^p d(y, D(t,x))).$$

3 Main results

The following assumption will be useful.
Assumption 1: Let $D : [T_0, T] \times H \to H$ be a set-valued mapping with nonempty closed and uniformly r-prox regular values such that:

($\mathcal{A}_1$) There is a positive constant L and a nondecreasing absolutely continuous function $\zeta : [T_0, T] \to \mathbf{R}_+$ such that, for all $s \leq t$ in $[T_0, T]$ and $x_i, y_i \in H (i = 1, 2)$,

$$|d\big(x_1, D(t, y_1)\big) - d\big(x_2, D(s, y_2)\big)| \leq \|x_1 - x_2\| + \zeta(t) - \zeta(s) + L\|y_1 - y_2\|;$$

($\mathcal{A}_2$) for all $(t, x) \in [T_0, T] \times H,\ D(t, x)$ is contained in a compact set Γ.

Let us start with an existence result for second order state-dependent sweeping process without perturbations, it will be used in the next theorem. The proof is a careful adaptation of Theorem 3.2 and 3.4 in [12]. Remark that, here the sets $D(t, u)$ are with absolutely continuous variation in time while in Theorem 3.2 of [12] the variation in time is Lipschitz.

Theorem 1. *Assume that Assumption 1 holds. Then, for every $b \in H$ and for every $a \in D(T_0, b)$, there exist two absolutely continuous mappings $u : [T_0, T] \to H$ and $v : [T_0, T] \to H$ satisfying*

$$\begin{cases} -\dot{u}(t) \in N_{D(t,v(t))}(u(t)), \quad a.e.\ \ t \in [T_0, T]; \\ v(t) = b + \displaystyle\int_{T_0}^{t} u(s)ds, \quad u(t) = a + \displaystyle\int_{T_0}^{t} \dot{u}(s)ds, \ \forall t \in [T_0, T]; \\ u(t) \in D(t, v(t)), \ \ \forall t \in [T_0, T], \end{cases}$$

with

$$\|\dot{u}(t)\| \leq \dot{\zeta}(t)(1 + L\alpha) \quad a.e. \quad t \in [T_0, T].$$

Proof. By assumption ($\mathcal{A}_2$), for some $\alpha > 0$ we have $D(t, x) \subset \Gamma \subset \alpha\overline{\mathbf{B}}_H$. Consider a partition of $[T_0, T]$ by the points $t_k^n = T_0 + ke_n,\ e_n = \frac{T-T_0}{n}, k \in \{0, 1, 2, ..., n\}$ and set

$$\sigma_k^n = \zeta(t_{k+1}^n) - \zeta(t_k^n)$$

and

$$\sigma^n = \max_{0 \leq k \leq n-1} \sigma_k^n.$$

As the sequences (σ^n) and (e_n) converge to 0, one can fix a positive integer n_0 such that for any $n \geq n_0$

$$(\sigma^n + e_n)(1 + L\alpha) < r.$$

Construction of approximate solutions: For each $t \in [t_0^n, t_1^n]$, we define

$$v_n(t) = b + (t - t_0^n)a$$

$$u_n(t) = x_0^n + \frac{\zeta(t) - \zeta(t_0^n)}{\sigma_0^n + e_n}(x_1^n - x_0^n),$$

where $x_0^n = a \in D(T_0, b)$ and $x_1^n = \mathrm{Proj}_{D(t_1^n, v_n(t_1^n))}(x_0^n)$. Despite the absence of the convexity of the images of D, the last equality is well defined. Indeed, we have

$$\begin{array}{rcl} d(x_0^n, D(t_1^n, v_n(t_1^n))) & = & |d(x_0^n, D(t_0^n, v_n(t_0^n))) - d(x_0^n, D(t_1^n, v_n(t_1^n)))| \\ & \leq & \zeta(t_1^n) - \zeta(t_0^n) + L\|v_n(t_1^n) - v_n(t_0^n)\| \\ & \leq & \sigma_0^n + Le_n\|x_0^n\| \leq (\sigma^n + e_n)(1 + L\alpha) \leq r. \end{array}$$

Hence $v_n(t_0^n) = b,\ u_n(t_0^n) = a$ and for $t \in]t_0^n, t_1^n[$, we have $\dot{v}_n(t) = a$ and

$$\dot{u}_n(t) = \dot{\zeta}(t)\frac{x_1^n - x_0^n}{\sigma_0^n + e_n} \in -N_{D(t_1^n, v_n(t_1^n))}(x_1^n),$$

with

$$\|\dot{u}_n(t)\| \leq \dot{\zeta}(t)(1 + L\alpha).$$

By induction, suppose that $(v_n), (u_n)$ are well defined on $]t_0^n, t_k^n]$ with $u_n(t_k^n) = x_k^n$ and $\|\dot{u}_n(t)\| \leq \dot{\zeta}(t)(1 + L\alpha)$. For each $t \in]t_k^n, t_{k+1}^n]$, we define

$$v_n(t) = v_n(t_k^n) + (t - t_k^n)u_n(t_k^n)$$

and

$$u_n(t) = x_k^n + \frac{\zeta(t) - \zeta(t_k^n)}{\sigma_k^n + e_n}(x_{k+1}^n - x_k^n),$$

where $x_{k+1}^n = \mathrm{Proj}_{D(t_{k+1}^n, v_n(t_{k+1}^n))}(x_k^n)$ and $d(x_k^n, D(t_{k+1}^n, v_n(t_{k+1}^n))) \leq r$.
Then for $t \in]t_k^n, t_{k+1}^n]$, we have $\dot{v}_n(t) = u_n(t_k^n)$ and

$$\dot{u}_n(t) = \dot{\zeta}(t)\frac{x_{k+1}^n - x_k^n}{\sigma_n^k + e_n} \in -N_{D(t_{k+1}^n, v_n(t_{k+1}^n))}(x_{k+1}^n),$$

with

$$\|\dot{u}_n(t)\| \leq \dot{\zeta}(t)(1 + L\alpha) \quad \text{and} \quad \|\dot{v}_n(t)\| \leq \alpha.$$

Defining for each $t \in [T_0, T]$ and each $n \geq n_0$,

$$p_n(t) = \begin{cases} t_k^n & \text{if} \quad t \in [t_k^n, t_{k+1}^n[\\ T & \text{if} \quad t = T; \end{cases}$$

$$q_n(t) = \begin{cases} T_0 & \text{if} \quad t = T_0 \\ t_{k+1}^n & \text{if} \quad t \in]t_k^n, t_{k+1}^n], \end{cases}$$

we get

$$\dot{u}_n(t) \in -N_{D(q_n(t), v_n(q_n(t)))}(u_n(q_n(t))) \ \ a.e.\ \ [T_0, T];$$

$$u_n(q_n(t)) \in D(q_n(t), v_n(q_n(t))),\ \forall [T_0, T];$$

$$v_n(t) = b + \int_{T_0}^{t} u_n(p_n(s))ds,\ \forall [T_0, T];$$

$$\lim_{n\to\infty} p_n(t) = \lim_{n\to\infty} q_n(t) = t,\ \ \forall [T_0, T];$$

$$\|\dot{v}_n(t)\| = \|u_n(p_n)(t))\| = ||x_k^n|| \leq \alpha, \ \ \forall\, k \leq n, \ \forall t \in [T_0, T]$$

and

$$\|\dot{u}_n(t)\| \leq \dot{\zeta}(t)(1 + L\alpha) = \rho(t). \tag{1}$$

Thus

$$\lim_{n\to\infty} ||u_n(p_n(t)) - u_n(t)|| = 0. \tag{2}$$

Convergence of approximate sequences:
We have $u_n(p_n(t)) \in D(p_n(t), v_n(p_n(t))) \subset \Gamma$, so that; $u_n(p_n(t))$ is relatively compact for every $t \in [T_0, T]$ in H, so is $(u_n(t))$ thanks to (2). By (1), $(u_n(\cdot))$ is equicontinuous. Thus (u_n) is relatively compact in $\mathcal{C}_H([T_0, T])$, consequently (u_n) converges in $\mathcal{C}_H([T_0, T])$ to the absolutely continuous mapping u. By (1) again, $(\dot{u}_n)$ weakly converges in $L^1_H[T_0, T]$ to a function z with $\|z(t) \leq \rho(t)$ a.e. in $[T_0, T]$ (see Proposition 6.2.3 in [10]) and (u_n) converges pointwise on $[T_0, T]$ with respect to the weak topology to an absolutely continuous function u and

$$u(t) = a + \int_{T_0}^{t} \dot{u}(s)ds, \ \forall [T_0, T]$$

with $\dot{u} = z$. From the convergence of (u_n) we deduce that of (v_n) to an absolutely continuous function v with

$$v(t) = b + \int_{0}^{t} u(s)ds, \ \forall [T_0, T].$$

For the rest of the demonstration we can consult the proof of Theorem 2 below. □

Now, we give the main result in this section.

Theorem 2. *Assume that Assumption 1 holds. Let $F : [T_0, T] \times H \times H \rightharpoonup H$ be a set-valued map with nonempty closed convex values such that:*

- $(\mathcal{A}_{F_1})$ *F is $\mathcal{L}([T_0, T]) \otimes \mathcal{B}(H) \otimes \mathcal{B}(H)$-measurable and for all $t \in [T_0, T]$, $F(t, \cdot, \cdot)$ is scalarly upper semicontinuous on $H \times H$;*
- $(\mathcal{A}_{F_2})$ *there exists a real $\beta > 0$, such that, for all $(t, u, v) \in [T_0, T] \times H \times H$,*

$$d(0, F(t, u, v)) \leq \beta(1 + \|u\| + \|v\|).$$

And let $f : [T_0, T] \times H \times H \to H$ be a Carathéodory mapping satisfies

- $(\mathcal{A}_f)$ *there exists a non-negative function $\gamma \in \mathbf{L}^1_{\mathbf{R}^+}([T_0, T])$ such that, for all $t \in [T_0, T]$ and for all $(u, v) \in H \times H$,*

$$\|f(t, u, v)\| \leq \gamma(t)(1 + \|u\| + \|v\|).$$

Then, for any $a, b \in H$ with $a \in D(T_0, b)$, there exist two absolutely continuous mappings $u, v : [T_0, T] \to H$ satisfying $(\mathcal{P})$.

Proof. **Step 1.** We begin by a single-valued integrable mapping $m \in L^1_H([T_0,T])$. Put for all $t \in [T_0,T]$,

$$m_1(t) = \int_{T_0}^{t} m(s)ds \ \ and \ \ m_2(t) = \int_{T_0}^{t} m_1(s)ds$$

and consider the set-valued map $C : [T_0,T] \times H \rightharpoonup H$ defined by

$$C(t,z) = D(t, z - m_2(t)) + m_1(t) \quad \forall \quad (t,z) \in [T_0,T] \times H.$$

Obviously, C satisfies $(\mathcal{A}_2)$, let verify $(\mathcal{A}_1)$. For any $w_1,\ w_2,\ z_1,\ z_2$ in H and any $s \le t$ in $[T_0,T]$, we have

$$\begin{aligned} &|d(w_1, C(t,z_1)) - d(w_2, C(s,z_2))| \\ &= |d(w_1 - m_1(t), D(t, z_1 - m_2(t))) - d(w_2 - m_1(s), D(s, z_2 - m_2(s)))| \\ &\le \|w_1 - w_2\| + \|m_1(t) - m_1(s)\| + L\|m_2(t) - m_2(s)\| + \zeta(t) - \zeta(s) + L\|z_1 - z_2\| \\ &\le \|w_1 - w_2\| + \zeta_1(t) - \zeta_1(s) + L\|z_1 - z_2\| \end{aligned}$$

where

$$\zeta_1(t) = \int_{T_0}^{t} \Big(\dot\zeta(\omega) + \|m(\omega)\| + L\int_{T_0}^{\omega} \|m(\tau)\|d\tau\Big)d\omega$$

is an absolutely continuous nondecreasing mapping. Hence, C satisfies $(\mathcal{A}_1)$, as $a \in C(T_0,b) = D(T_0,b)$, from Theorem 1, there exist two absolutely continuous mappings $x : [T_0,T] \to H$ and $y : [T_0,T] \to H$ such that

$$\begin{cases} -\dot y(t) \in N_{C(t,x(t))}(y(t)), \ \text{ a.e. } \ t \in [T_0,T]; \\ x(t) = b + \displaystyle\int_{T_0}^{t} y(s)ds, \ \ y(t) = a + \int_{T_0}^{t} \dot y(s)ds, \ \forall t \in [T_0,T]; \\ y(t) \in C(t,x(t)), \ \forall t \in [T_0,T]. \end{cases}$$

Let $u(t) = y(t) - m_1(t)$ and $v(t) = x(t) - m_2(t)$, the mappings $u(\cdot)$ and $v(\cdot)$ satisfy

$$\begin{cases} -\dot u(t) \in N_{D(t,v(t))}(u(t)) + m(t), \ \text{ a.e. } \ t \in [T_0,T]; \\ v(t) = b + \displaystyle\int_{T_0}^{t} u(s)ds, \ \ u(t) = a + \int_{T_0}^{t} \dot u(s)ds, \ \forall t \in [T_0,T]; \\ u(t) \in D(t,v(t)), \ \forall t \in [T_0,T]. \end{cases}$$

with

$$\|\dot u(t)\| \le \Big(1 + L\alpha\Big)\left(\dot\zeta(t) + 2\|m(t)\| + L\int_{T_0}^{s} \|m(\tau)\|d\tau\right)ds.$$

Step 2. For each $(t,u,v) \in [T_0,T] \times H \times H$, let $P(t,x,y)$ be the element of minimal norm of the closed convex set $F(t,x,y)$ of H, that is

$$P(t,x,y) = \mathrm{Proj}_{F(t,x,y)}(0), \ \ \forall\ (t,u,v) \in [T_0,T] \times H \times H.$$

Since F is $\mathcal{L}([T_0,T])\otimes\mathcal{B}(H)\otimes\mathcal{B}(H)$-measurable, so $P(\cdot,\cdot,\cdot)=d(0,F(\cdot,\cdot,\cdot))$, is measurable. In view of $(\mathcal{A}_{F_2})$

$$\|P(t,x,y)\|\leq\beta(1+\|x\|+\|y\|). \tag{3}$$

We put

$$g(t,x,y)=f(t,x,y)+P(t,x,y)$$

and

$$\Lambda(t)=\gamma(t)+\beta,$$

by (3) and $(\mathcal{A}_f)$, we get for all $(t,u,v)\in[T_0,T]\times H\times H$,

$$\|g(t,x,y)\|\leq\Lambda(t)(1+\|x\|+\|y\|). \tag{4}$$

Construction of sequences: Consider, for every $n\in\mathbf{N}$, a partition of $[T_0,T]$ defined by $t_i^n=T_0+i\frac{T-T_0}{n}$ $(0\leq i\leq n)$. We are going to construct a sequence of maps $(u_n(\cdot))$ and $(v_n(\cdot))$ via Step 1, by considering a perturbation g with fixed second and third variables in each subinterval $[t_i^n,t_{i+1}^n]$. So, for $a\in D(T_0,b)$, let us consider the following problem on the interval $[T_0,t_1^n]$:

$$(P_0)\begin{cases}-\dot{u}(t)\in N_{D(t,v(t))}(u(t))+g(t,b,a)\text{ a.e. }t\in[T_0,t_1^n]\\ v(T_0)=a,\ u(T_0)=a\in D(T_0,b)\end{cases}$$

where $g(\cdot,b,a)$ is a mapping depending only on t and is $L^1_H([T_0,t_1^n])$. By Step 1, there are two absolutely continuous mappings that we denote by $u_0^n(.),v_0^n(.):[T_0,t_1^n]\to H$ solutions of (P_0). Now, since $u_0^n(t_1^n)\in D(t_1^n,v_0^n(t_1^n))$ is well defined in the interval $[t_1^n,t_2^n]$ the problem

$$(P_1)\begin{cases}-\dot{u}_1^n(t)\in N_{D(t,v_1^n(t))}(u_1^n(t))+g(t,v_0^n(t_1^n),u_0^n(t_1^n))\text{ a.e. }t\in[t_1^n,t_2^n];\\ u_0^n(t_1^n)\in D(t_1^n,v_0^n(t_1^n)).\end{cases}$$

admits an absolutely continuous solution $(u_1^n(\cdot),v_1^n(\cdot))$ with $u_1^n(t_1^n)=u_0^n(t_1^n)$ and $v_1^n(t_1^n)=v_0^n(t_1^n)$. By induction, for each n, there exist two finite sequence of absolutely continuous mappings $u_i^n(\cdot),v_i^n(\cdot):[t_i^n,t_{i+1}^n]\to H$ with $u_i^n(t_i^n)=u_{i-1}^n(t_i^n)$ and $v_i^n(t_i^n)=v_{i-1}^n(t_i^n)$ such that, for each $i\in\{0,...,n-1\}$,

$$(P_i)\begin{cases}-\dot{u}_i^n(t)\in N_{D(t,v_i^n(t))}(u_i^n(t))+g(t,v_{i-1}^n(t_i^n),u_{i-1}^n(t_i^n))\text{ a.e. }t\in[t_i^n,t_{i+1}^n];\\ u_{i-1}^n(t_i^n)\in D(t_i^n,v_{i-1}^n(t_i^n)),\end{cases}$$

where $u_{-1}^n(T_0)=a,v_{-1}^n(T_0)=b$ and

$$\|\dot{u}(t)\|\leq\Big(1+L\alpha\Big)\Big(\dot{\zeta}(t)+2\|g(t,v_{i-1}^n(t_i^n),u_{i-1}^n(t_i^n))\|$$
$$+L\int_{t_i^n}^t\|g(\tau,v_{i-1}^n(t_i^n),u_{i-1}^n(t_i^n))\|d\tau\Big),$$

a.e. $t \in [t_i^n, t_{i+1}^n]$. We define the absolutely continuous mappings $u_n, v_n : [T_0, T] \to H$ by $u_n(t) = u_i^n(t)$ and $v_n(t) = v_i^n(t)$ for all $t \in [t_i^n, t_{i+1}^n],\ i \in \{0, \cdots, n\}$. One can write

$$\begin{cases} \dot{u}_n(t) \in -N_{D(t,v_n(t))}(u_n(t)) + g(t, v_n(p_n(t)), u_n(p_n(t))) \text{ a.e. } t \in [T_0, T]; \\ v_n(t) = b + \displaystyle\int_{T_0}^t u_n(s)ds,\ u_n(t) = a + \int_{T_0}^t \dot{u}_n(s)ds,\ \forall t \in [T_0, T]; \\ u_n(t) \in D(t, v_n(t)),\ \forall t \in [T_0, T],\ u_n(T_0) = a,\ v_n(T_0) = b, \end{cases}$$

with a.e. $t \in [T_0, T]$

$$\|\dot{u}_n(t)\| \le \Big(1 + L\alpha\Big)\Big(\dot{\zeta}(t) + 2\|g(t, v_n(p_n(t)), u_n(p_n(t)))\|$$

$$+L\int_{p_n(t)}^t \|g(\tau, v_n(p_n(\tau)), u_n(p_n(\tau)))\|d\tau\Big).$$

Since for all $t \in [T_0, T],\ u_n(p_n(t)) \in D(p_n(t), v_n(p_n(t)))$, then

$$\|u_n(p_n(t))\| \le \alpha \quad \text{and} \quad \|v_n(p_n(t))\| \le \|b\| + (T - T_0)\alpha.$$

By (4), we get for almost every $t \in [T_0, T]$

$$\|g(t, v_n(p_n(t)), u_n(p_n(t)))\| = \Big(1 + \|b\| + (T+1)\alpha\Big)\Lambda(t) = c_1(t). \tag{5}$$

Then

$$\|\dot{u}_n(t)\| \le \Big(1 + L\alpha\Big)\Big(\dot{\zeta}(t) + \big(2 + L\int_{T_0}^T \Lambda(\tau)d\tau)(1 + \|b\| + (T+1)\alpha\big)\Big) = c_2(t). \tag{6}$$

Convergence of sequences: Since for each $t,\ u_n(t) \in D(t, v_n(t)) \subset \Gamma$, for all $n \in \mathbf{N}$ such that $(u_n(t))$ is relatively compact in H for every $t \in [T_0, T]$. Using Ascoli-Arzelà theorem, (u_n) is relatively compact in $\mathcal{C}_H([T_0, T])$. Then there exists a subsequence again denoted by (u_n) which converges to a mapping u. According to (6), we may suppose that $(\dot{u}_n)$ weakly converges in $L_H^1([T_0, T])$ to a mapping z with $\|z(t)\| \le c_2(t)$ a.e. in $[T_0, T]$. Thus

$$\lim_{n\to\infty} u_n(t) = a + \lim_{n\to\infty}\int_{T_0}^t \dot{u}_n(s)ds = a + \int_{T_0}^t z(s)ds,$$

then, $u(t) = a + \displaystyle\int_{T_0}^t z(s)ds$. Consequently, $u(t)$ is absolutely continuous with $\dot{u} = z$. Furthermore,

$$|p_n(t) - t| \le |t_{k+1}^n - t_k^n| = \frac{T - T_0}{n},$$

so $\lim\limits_{n\to\infty} |p_n(t) - t| = 0$ and

$$||u_n(p_n(t)) - u_n(t)|| \le \int_{p_n(t)}^t \|\dot{u}_n(s)\|ds \le \int_{p_n(t)}^t c_2(s)ds,$$

since $c_2 \in L^1_{\mathbf{R}_+}([T_0,T])$, we get $\lim\limits_{n\to\infty} ||u_n(p_n(t)) - u_n(t)|| = 0$, so that

$$\lim_{n\to\infty} ||u_n(p_n(t)) - u(t)|| \le \lim_{n\to\infty} \Big(||u_n(p_n(t)) - u_n(t)|| + ||u_n(t) - u(t)|| \Big) = 0.$$

The convergence of the sequence $(u_n(p_n(\cdot))$ to $(u(\cdot))$ is obtained.
From the convergence of $(u_n(\cdot))$ we deduce that of $(v_n(\cdot))$ to an absolutely continuous function $v(\cdot)$ with

$$v(t) = b + \int_{T_0}^{t} u(s)ds, \;\; \forall t \in [T_0,T]$$

and

$$\lim_{n\to\infty} ||v_n(p_n(t)) - v_n(t)|| = 0.$$

Let us set for all $t \in [T_0,T]$,

$$f(t, v_n(p_n(t)), u_n(p_n(t))) = l_n(t)$$

and

$$P(t, v_n(p_n(t)), u_n(p_n(t))) = \eta_n(\cdot).$$

By the continuity of the mapping $f(t,\cdot,\cdot)$ we get $l_n(t)$ converges to $l(t) = f(t, u(t), v(t))$ and

$$\|l(t)\| \le \Big(1 + \|b\| + (T+1)\alpha\Big)\gamma(t).$$

On the other hand, for all $n \ge n_0$ and for all $t \in [T_0,T]$, we have

$$\|\eta_n(t)\| \le \| \Big(1 + \|b\| + (T+1)\alpha\Big)\beta,$$

so $(\eta_n(\cdot))$ is bounded, taking a subsequence if necessary, we may conclude that $(\eta_n(\cdot))$ weakly converges to some mapping $\eta \in L^1_H([T_0,T])$ with

$$\|\eta(t)\| \le \Big(1 + \|b\| + (T+1)\alpha\Big)\beta.$$

Now, we proceed to prove that

$$\dot u(t) \in -N_{D(t,v(t))}(u(t)) + F(t, v(t), u(t)) + f(t, v(t), u(t)) \; a.e. \; t \in [T_0,T].$$

First, we check that $u(t) \in D(t, v(t))$. For every $t \in [T_0,T]$ and for every n, we have

$$\begin{aligned} d(u_n(t), D(t,v(t))) &\le ||u_n(t) - u_n(p_n(t))|| + d(u_n(p_n(t)), D(t,v(t))) \\ &\le ||u_n(t) - u_n(p_n(t))|| + \mathcal{H}(D(p_n(t), v_n(p_n(t))), D(t,v(t))) \\ &\le ||u_n(t) - u_n(p_n(t))|| + |\zeta(t) - \zeta(p_n(t))| + L||v_n(p_n(t)) - v_n(t)||, \end{aligned}$$

Passing to the limit when $n \to \infty$, in the preceding inequality, we get $u(t) \in D(t, v(t))$. According to (5) and (6), we obtain

$$\| -\dot{u}_n(t) + l_n(t) + \eta_n(t)\| \le c_1(t) + c_2(t) := \lambda(t),$$

so

$$-\dot{u}_n(t) + l_n(t) + \eta_n(t) \in \lambda(t)\overline{\mathbf{B}}_H$$

since

$$-\dot{u}_n(t) + l_n(t) + \eta_n(t) \in N_{D(t,v_n(t))}(u_n(t)),$$

we get

$$-\dot{u}_n(t) + l_n(t) + \eta_n(t) \in \lambda(t)\partial d(u_n(t), D(t, v_n(t))).$$

Remark that $(-\dot{u}_n+l_n+\eta_n, \eta_n)$ weakly converges in $L^1_{H\times H}([T_0, T])$ to $(-\dot{u}+l+\eta, \eta)$. An application of the Mazur's Theorem to $(-\dot{u}_n + l_n + \eta_n, \eta_n)$ provides a sequence (w_n, ζ_n) with

$$w_n \in co\{-\dot{u}_m + l_m + \eta_m :\ m \ge n\} \quad \text{and} \quad \zeta_n \in co\{\eta_m :\ m \ge n\}$$

such that (w_n, ζ_n) converges strongly in $L^1_{H\times H}([0, T])$ to $(-\dot{u} + l + \eta, \eta)$. We can extract from (w_n, ζ_n) a subsequence which converges a.e. to $(-\dot{u} + l + \eta, \eta)$. Then, there is a Lebesgue negligible set $S \subset [0, T]$ such that for every $t \in [0, T] \setminus S$

$$-\dot{u}(t) + l(t) + \eta(t) \in \bigcap_{n\ge 0} \overline{\{w_m(t) :\ m \ge n\}}$$

$$\subset \bigcap_{n\ge 0} \overline{co}\{-\dot{u}_m(t) + l_m(t) + \eta_m(t) :\ m \ge n\}, \tag{7}$$

$$\eta(t) \in \bigcap_{n\ge 0} \overline{\{\zeta_m(t) :\ m \ge n\}} \subset \bigcap_{n\ge 0} \overline{co}\{\eta_m(t) :\ m \ge n\}. \tag{8}$$

Fix any $t \in [0, T] \setminus S, n \ge n_0$ and $\mu \in H$, then the relation (7) gives

$$\langle \mu, -\dot{u}(t) + l(t) + \eta(t)\rangle \le \limsup_{n\to\infty} \delta^*(\mu, \lambda(t)\partial d(u_n(t), D(t, v_n(t))))$$

$$\le \delta^*(\mu, \lambda(t)\partial d(u(t), D(t, v(t)))),$$

where the first inequality follows from the characterization of convex hull and the second one follows from Proposition 1. Taking the supremum over $\mu \in H$, we deduce that

$$\delta(-\dot{u}(t) + l(t) + \eta(t), \lambda(t)\partial d(u(t), D(t, v(t)))) =$$

$$\delta^{**}(-\dot{u}(t) + l(t) + \eta(t), \lambda(t)\partial d(u(t), D(t, v(t)))) \le 0$$

which entails

$$-\dot{u}(t) + l(t) + \eta(t) \in \lambda(t)\partial d(u(t), D(t, v(t))) \subset N_{D(t,v(t))}(u(t)).$$

Further, the relation (8) gives

$$\langle \mu, \eta(t) \rangle \leq \limsup_{n\to\infty} \delta^*(\mu, F(t, v_n(p_n(t)), u_n(p_n(t)))),$$

since $\delta^*(\mu, F(t, \cdot, \cdot))$ is upper semicontinuous on $H \times H$ then

$$\langle \mu, \eta(t) \rangle \leq \delta^*(\mu, F(t, v(t), u(t))),$$

so, we get $d(\eta(t), F(t, v(t), u(t))) \leq 0$, because F has closed convex values. Consequently $\eta(t) \in F(t, v(t), u(t))$ a.e $t \in [T_0, T]$. Then

$$\dot{u}(t) \in -N_{D(t,v(t))}(u(t)) + F(t, v(t), u(t)) + f(t, v(t), u(t)).$$

This completes the proof of the theorem. □

Remark 1. As in [22], the result remains valid if we replace the uniformly r-prox regular sets by a family of equi-uniformly subsmooth sets.

In the next theorem we prove the existence of solution on the whole interval $\mathbf{R}_+ = [0 + \infty[$.

Theorem 3. *Let $D : \mathbf{R}_+ \times H \to H$ be a set-valued mapping with nonempty closed and uniformly r-prox regular values such that:*

- (i) *There is a positive constant L and a nondecreasing absolutely continuous function $\zeta : \mathbf{R}_+ \to \mathbf{R}_+$ such that, for all $s \leq t$ in $\mathbf{R}_+$ and $x_i, y_i \in H (i = 1, 2)$,*

$$|d\big(x_1, D(t, y_1)\big) - d\big(x_2, D(s, y_2)\big)| \leq \|x_1 - x_2\| + \zeta(t) - \zeta(s) + L\|y_1 - y_2\|;$$

- (ii) *for all $(t, x) \in \mathbf{R}_+ \times H$, $D(t, x)$ is contained in a compact set Γ.*

Let $F : \mathbf{R}_+ \times H \times H \to H$ be a set-valued map with nonempty closed convex values such that:

- (iii) *F is $\mathcal{L}(\mathbf{R}_+) \otimes \mathcal{B}(H) \otimes \mathcal{B}(H)$-measurable and for all $t \in \mathbf{R}_+$, $F(t, \cdot, \cdot)$ is scalarly upper semicontinuous on $H \times H$;*
- (vi) *there exists a non-negative function $\beta(\cdot) \in L^\infty_{loc}(\mathbf{R}_+)$, such that, for all $(t, u, v) \in \mathbf{R}_+ \times H \times H$,*

$$d(0, F(t, u, v)) \leq \beta(t)(1 + \|u\| + \|v\|).$$

Then, for any $a, b \in H$ with $a \in D(T_0, b)$, there exist two absolutely continuous mappings $u, v : \mathbf{R}_+ \to H$ satisfying

$$(\mathcal{P}_{\mathbf{R}_+}) \begin{cases} -\dot{u}(t) \in N_{D(t,v(t))}(u(t)) + F(t, v(t), u(t)), \quad a.e.\ t \in \mathbf{R}_+; \\ v(t) = b + \displaystyle\int_{T_0}^t u(s)ds, \quad u(t) = a + \displaystyle\int_{T_0}^t \dot{u}(s)ds,\ \forall t \in \mathbf{R}_+; \\ \qquad u(t) \in D(t, v(t)),\ \forall t \in \mathbf{R}_+. \end{cases}$$

Proof. Since $\mathbf{R}_+ = \bigcup_{k\in\mathbf{N}} [k, k+1]$, for all $k \in \mathbf{N}$ applying Theorem 2 on each interval $[k, k+1]$, there exist two absolutely continuous mappings $u^k, v^k : [k, k+1] \to H$ satisfying

$$\begin{cases} -\dot{u}^k(t) \in N_{D(t,v^k(t))}(u^k(t)) + F(t, v^k(t), u^k(t)), \quad \text{a.e. } t \in [k, k+1]; \\ u^k(t) \in D(t, v^k(t)),\ \forall t \in [k, k+1], ;\ u^k(k) = u^{k-1}(k) \text{ and } v^k(k) = v^{k-1}(k). \end{cases}$$

Let $u : \mathbf{R}_+ \to H$ and $v : \mathbf{R}_+ \to H$ be defined by $u(t) = u^k(t)$ and $v(t) = v^k(t)$ for $t \in [k, k+1],\ k \in \mathbf{N}$, then it is easy to conclude that u, v are absolutely continuous solutions of the problem $(\mathcal{P}_{\mathbf{R}_+})$. This completes the proof of the theorem. $\square$

4 Delayed sweeping process

Now, we proceed, in the infinite dimensional setting, to an existence result for second order functional differential inclusion governed by the time and state-dependent nonconvex sweeping process, that is when the perturbation contains a finite delay. This problem was addressed by [22] using the discretization approach based on the Moreau's catching-up algorithm. Here, we provide another technique initiated in [10] for the first order time-dependent case, which consists to subdivide the interval $[0, T]$ in a sequence of subintervals and to reformulate the problem with delay to a sequence of problems without delay and apply the results known in this case. For second order functional problems regarding the time-dependent sweeping process, we refer to [7,8]. We will extend this approach for the case of time and state-dependent sweeping process with unbounded delayed perturbation. For a question of clarity and shortness, we will restrict ourselves to Theorem 2 for uniformly prox-regular sets and one set-valued perturbation, but it is clear that this remains valid for equi-uniformly subsmooth sets as well as for the sum of two perturbations.
Let $\tau > 0$ be a positive number and $\mathcal{C}_0 = \mathcal{C}_H([-\tau, 0])$ (resp. $\mathcal{C}_T = \mathcal{C}_H([-\tau, T])$ the Banach space of H-valued continuous functions defined on $[-\tau, 0]$ (resp. $[-\tau, T]$) equipped with the norm of uniform convergence. Let $u : [-\tau, T] \to H$, then for every $t \in [0, T]$ we define the function $u_t = \mathcal{T}(t)u$ on $[-\tau, 0]$ by $(\mathcal{T}(t)u)(s) = u(t+s),\ \forall s \in [-\tau, 0]$. Clearly, if $u \in \mathcal{C}_T$, then $u_t \in \mathcal{C}_0$ and the mapping $u \to u_t$ is continuous.
Consider the following problem

$$(\mathcal{P}_\tau) \quad \begin{cases} -\dot{u}(t) \in N_{D\big(t,v(t)\big)}(u(t)) + G\Big(t, \mathcal{T}(t)v, \mathcal{T}(t)u\Big) \ \text{ a.e. } t \in [0, T]; \\ u(t) = \psi(0) + \displaystyle\int_0^t \dot{v}(s)ds, \quad v(t) = \varphi(0) + \int_0^t u(s)ds, \quad \forall t \in [0, T]; \\ v(t) \in D\big(t, u(t)\big), \quad \forall t \in [0, T]; \\ u \equiv \psi \ \text{ and } \ v \equiv \varphi \ \text{ on } [-\tau, 0]. \end{cases}$$

Theorem 4. *Assume that $D : [0, T] \times H \rightharpoonup H$ satisfies Assumption 1 and let $G : [0, T] \times \mathcal{C}_0 \times \mathcal{C}_0 \rightharpoonup H$ be a set-valued mapping with nonempty closed convex values such that:*

$(\mathcal{A}_{G_1})$ *G is $\mathcal{L}([0,T])\otimes\mathcal{B}(\mathcal{C}_0)\otimes\mathcal{B}(\mathcal{C}_0)$-measurable and for all $t\in\mathbf{R}_+$, $G(t,\cdot,\cdot)$ is scalarly upper semicontinuous on $\mathcal{C}_0\times\mathcal{C}_0$;*

$(\mathcal{A}_{G_2})$ *there exists a real $\beta>0$, such that, for all $(t,\varphi,\psi)\in[T_0,T]\times\mathcal{C}_0\times\mathcal{C}_0$,*

$$d(0,G(t,\varphi,\psi))\leq\beta(1+\|\varphi(0)\|+\|\psi(0)\|).$$

Then for every $(\varphi,\psi)\in\mathcal{C}_0\times\mathcal{C}_0$ verifying $\psi(0)\in D(0,\varphi(0))$, there exist two absolutely continuous mappings $u:[0,T]\to H$ and $v:[0,T]\to H$ satisfying $(\mathcal{P}_\tau)$.

Proof. Let $a=\psi(0)$ and $b=\varphi(0)$, then $a\in D(0,b)$. We consider the same partition of $[0,T]$ by the points $t_k^n=ke_n$, $e_n=\frac{T}{n}, (k=0,1,...,n)$. For each $(t,u,v)\in[-\tau,t_1^n]\times H\times H$, we define $f_0^n:[-\tau,t_1^n]\times H\to H$, $g_0^n:[-\tau,t_1^n]\times H\to H$ by

$$f_0^n(t,v)=\begin{cases}\varphi(t) & \forall t\in[-\tau,0],\\ \varphi(0)+\frac{n}{T}t(v-\varphi(0)) & \forall t\in\,]0,t_1^n],\end{cases}$$

$$g_0^n(t,u)=\begin{cases}\psi(t) & \forall t\in[-\tau,0],\\ \psi(0)+\frac{n}{T}t(u-\psi(0)) & \forall t\in\,]0,t_1^n].\end{cases}$$

We have $f_0^n(t_1^n,v)=v$ and $g_0^n(t_1^n,v)=u$ for all $(u,v)\in H\times H$. Observe that the mapping $(u,v)\to\Big(\mathcal{T}(t_1^n)f_0^n(\cdot,v),\mathcal{T}(t_1^n)g_0^n(\cdot,u)\Big)$ from $H\times H$ to $\mathcal{C}_0\times\mathcal{C}_0$ is nonexpansive since for all $(v_1,v_2)\in H\times H$

$$\|\mathcal{T}(t_1^n)f_0^n(\cdot,v_1)-\mathcal{T}(t_1^n)f_0^n(\cdot,v_2)\|_{\mathcal{C}_0}=$$

$$\sup_{s\in[-\tau,0]}\|f_0^n(s+t_1^n,v_1)-f_0^n(s+t_1^n,v_2)\|=$$

$$\sup_{s\in\left[-\tau+\frac{T}{n},\frac{T}{n}\right]}\|f_0^n(s,v_1)-f_0^n(s,v_2)\|=$$

$$\sup_{0\leq s\leq\frac{T}{n}}\left\|\frac{n}{T}s(v_1-\varphi(0))-\frac{n}{T}s(v_2-\varphi(0))\right\|=$$

$$\sup_{0\leq s\leq\frac{T}{n}}\left\|\frac{n}{T}s(v_1-v_2)\right\|=\|v_1-v_2\|.$$

Similarly, for all $(u_1,u_2)\in H\times H$ we get

$$\|\mathcal{T}(t_1^n)g_0^n(\cdot,u_1)-\mathcal{T}(t_1^n)g_0^n(\cdot,u_2)\|_{\mathcal{C}_0}=\|u_1-u_2\|.$$

Hence the mapping $(u,v)\to\Big(\mathcal{T}(t_1^n)f_0^n(\cdot,v),\mathcal{T}(t_1^n)g_0^n(\cdot,v)\Big)$ from $H\times H$ to $\mathcal{C}_0\times\mathcal{C}_0$ is nonexpansive, so the set-valued mapping with nonempty closed convex values $G_0^n:[0,t_1^n]\times H\times H\rightharpoondown H$ defined by

$$G_0^n(t,u,v)=G(t,\mathcal{T}(t_1^n)f_0^n(\cdot,v),\mathcal{T}(t_1^n)g_0^n(\cdot,u))$$

is globally measurable and scalarly upper semicontinuous on $H \times H$, thanks to by $(\mathcal{A}_{G_1})$ and

$$d(0, G_0^n(t,v,u) = d(0, G(t, \mathcal{T}(t_1^n) f_0^n(\cdot, v), \mathcal{T}(t_1^n) g_0^n(\cdot, u))$$

$$\leq \beta\left(1 + \|v\| + \|u\|\right),$$

for all $(t,v,u) \in [0,t_1^n] \times H \times H$ since, $\mathcal{T}(t_1^n) f_0^n(0,v) = u, \mathcal{T}(t_1^n) g_0^n(0,u) = v$. Hence G_0^n verifies conditions of Theorem 2, then there exist two absolutely continuous mappings $u_0^n : [0,t_1^n] \to H$ and $v_0^n : [0,t_1^n] \to H$ such that

$$\begin{cases} -\dot{u}_0^n(t) \in N_{D(t,v_0^n(t))}(u_0^n(t)) + G_0^n(t, v_0^n, u_0^n) \quad \text{a.e on} \quad [0,t_1^n]; \\ v_0^n(t) = b + \displaystyle\int_0^t u_0^n(s)\, ds, \ \ u_0^n(t) = a + \int_0^t \dot{u}_0^n(s)\, ds \ \ \forall t \in [0,t_1^n]; \\ u_0^n(t) \in D(t, v_0^n(t)) \ \ \forall t \in [0,t_1^n]; \\ v_0^n(0) = b = \varphi(0), \ \ u_0^n(0) = a = \psi(0), \end{cases}$$

with

$$\|v_0^n(t)\| \leq \|b\| + T\alpha, \ \ \|u_0^n(t)\| \leq \alpha, \ \ \|\dot{u}_0^n(t)\| \leq c_2.$$

Set

$$v_n(t) = \begin{cases} \varphi(t) & \forall t \in [-\tau, 0], \\ v_0^n(t) & \forall t \in]0, t_1^n], \end{cases}$$

$$u_n(t) = \begin{cases} \psi(t) & \forall t \in [-\tau, 0], \\ u_0^n(t) & \forall t \in]0, t_1^n]. \end{cases}$$

Then, u_n and v_n are well defined on $[-\tau, t_1^n]$, with $v_n = \varphi$, $u_n = \psi$ on $[-\tau, 0]$, and

$$\begin{cases} -\dot{u}_n(t) \in N_{D(t,v_n(t))}(u_n(t)) + G_0(t, v_n(t), u_n(t)) \quad \text{a.e on} \quad [0,t_1^n]; \\ v_n(t) = b + \displaystyle\int_0^t u_n(s)\, ds, \\ u_n(t) = a + \displaystyle\int_0^t \dot{u}_n(s)\, ds, \ \ \forall t \in [0,t_1^n]; \\ u_n(t) \in D(t, v_n(t)), \ \ \forall t \in [0,t_1^n]; \\ v_n(0) = b = \varphi(0), \ \ u_n(0) = a = \psi(0), \end{cases}$$

By induction, suppose that u_n and v_n are defined on $[-\tau, t_k^n]$ $(k \geq 1)$ with $v_n = \varphi, u_n = \psi$ on $[-\tau, 0]$ and satisfy

$$v_n(t) = \begin{cases} v_0^n(t) = b + \displaystyle\int_0^t u_n(s) ds \ \ \forall t \in [0, t_1^n], \\ v_1^n(t) = v_n(t_1^n) + \displaystyle\int_{t_1^n}^t u_n(s)\, ds \ \ \forall t \in]t_1^n, t_2^n], \\ \cdots \\ v_{k-1}^n(t) = v_n\left(t_{k-1}^n\right) + \displaystyle\int_{t_{k-1}^n}^t u_n(s)\, ds \ \ \forall t \in \left]t_{k-1}^n, t_k^n\right], \end{cases}$$

$$u_n(t)=\begin{cases} u_0^n(t)=b+\int_0^t \dot{u}_n(s)\,ds \ \ \forall t\in[0,t_1^n]; \\ u_1^n(t)=u_n(t_1^n)+\int_{t_1^n}^t \dot{u}_n(s)\,ds \ \ \forall t\in\,]t_1^n,t_2^n]; \\ \cdots \\ u_{k-1}^n(t)=u_n\left(t_{k-1}^n\right)+\int_{t_{k-1}^n}^t \dot{u}_n(s)\,ds \ \ \forall t\in\left]t_{k-1}^n,t_k^n\right], \end{cases}$$

u_n and v_n are solutions of

$$\begin{cases} -\dot{u}_n(t)\in N_{D(t,v_n(t))}(u_n(t))+G\Big(t,\mathcal{T}(t_k^n)f_{k-1}^n(\cdot,v_n(t)),\mathcal{T}(t_k^n)g_{k-1}^n(\cdot,u_n(t))\Big); \\ v_n(t)=v_{k-1}^n(t)=v_n\left(t_{k-1}^n\right)+\int_{t_{k-1}^n}^t u_n(s)\,ds; \\ u_n(t)=u_{k-1}^n(t)=u_n\left(t_{k-1}^n\right)+\int_{t_{k-1}^n}^t \dot{u}_n(s)\,ds; \\ u_n(t)\in D(t,v_n(t)) \end{cases}$$

on $\left]t_{k-1}^n,t_k^n\right]$, where f_{k-1}^n and g_{k-1}^n are defined for any $(v,u)\in H\times H$ as follows

$$f_{k-1}^n(t,v)=\begin{cases} v_n(t) & \forall t\in\left[-\tau,t_{k-1}^n\right], \\ v_n\left(t_{k-1}^n\right)+\frac{n}{T}\left(t-t_{k-1}^n\right)\left(v-v_n\left(t_{k-1}^n\right)\right) & \forall t\in\left]t_{k-1}^n,t_k^n\right], \end{cases} \tag{9}$$

$$g_{k-1}^n(t,u)=\begin{cases} u_n(t) & \forall t\in\left[-\tau,t_{k-1}^n\right], \\ u_n\left(t_{k-1}^n\right)+\frac{n}{T}\left(t-t_{k-1}^n\right)\left(u-u_n\left(t_{k-1}^n\right)\right) & \forall t\in\left]t_{k-1}^n,t_k^n\right]. \end{cases} \tag{10}$$

Similarly we can define $f_k^n,g_k^n:\left[-\tau,t_{k+1}^n\right]\times H\to H$ as

$$f_k^n(t,v)=\begin{cases} v_n(t) & \forall t\in[-\tau,t_k^n], \\ v_n(t_k^n)+\frac{n}{T}(t-t_k^n)(v-v_n(t_k^n)), & \forall t\in\left]t_k^n,t_{k+1}^n\right], \end{cases}$$

$$g_k^n(t,u)=\begin{cases} u_n(t) & \forall t\in[-\tau,t_k^n], \\ u_n(t_k^n)+\frac{n}{T}(t-t_k^n)(u-u_n(t_k^n)) & \forall t\in\left]t_k^n,t_{k+1}^n\right], \end{cases}$$

for any $(u,v)\in H\times H$. Note that for all $(u,v)\in H\times H$,

$$\mathcal{T}(t_{k+1}^n)f_k^n(0,v)=f_k^n\left(t_{k+1}^n,v\right)=v,$$

$$\mathcal{T}(t_{k+1}^n)g_k^n(0,u)=g_k^n\left(t_{k+1}^n,u\right)=u.$$

Note also that, for all $(u_1,v_1),(u_2,v_2)\in H\times H$, we have

$$\left\|\mathcal{T}(t_{k+1}^n)f_k^n(\cdot,v_1)-\mathcal{T}(t_{k+1}^n)f_k^n(\cdot,v_2)\right\|_{\mathcal{C}_0}=$$

$$\sup_{s\in[-\tau,0]}\left\|f_k^n\left(s+t_{k+1}^n,v_1\right)-f_k^n\left(s+t_{k+1}^n,v_2\right)\right\|=$$

$$\sup_{s\in\left[-\tau+\frac{(k+1)T}{n},\frac{(k+1)T}{n}\right]} \left\| f_k^n\left(s,u_1\right) - f_k^n\left(s,u_2\right)\right\|,$$

and

$$\left\|\mathcal{T}(t_{k+1}^n)g_k^n\left(\cdot,u_1\right) - \mathcal{T}(t_{k+1}^n)g_k^n\left(\cdot,u_2\right)\right\|_{\mathcal{C}_0} =$$

$$\sup_{s\in[-\tau,0]} \left\|g_k^n\left(s+t_{k+1}^n,u_1\right) - g_k^n\left(s+t_{k+1}^n,u_2\right)\right\| =$$

$$\sup_{s\in\left[-\tau+\frac{(k+1)T}{n},\frac{(k+1)T}{n}\right]} \left\| g_k^n\left(s,u_1\right) - g_k^n\left(s,u_2\right)\right\|.$$

We distinguish two cases:

(1) if $-\tau+\dfrac{(k+1)\,T}{n} < \dfrac{kT}{n}$, we have

$$\sup_{s\in\left[-\tau+\frac{(k+1)T}{n},\frac{(k+1)T}{n}\right]} \left\| f_k^n\left(s,v_1\right) - f_k^n\left(s,v_2\right)\right\| =$$

$$\sup_{s\in\left[\frac{kT}{n},\frac{(k+1)T}{n}\right]} \left\| f_k^n\left(s,v_1\right) - f_k^n\left(s,v_2\right)\right\| =$$

$$\sup_{\frac{kT}{n}\leq s\leq\frac{(k+1)T}{n}} \left\|\frac{n}{T}\left(s-t_k^n\right)\left(v_1-v_2\right)\right\| = \left\|v_1-v_2\right\|$$

and

$$\sup_{s\in\left[-\tau+\frac{(k+1)T}{n},\frac{(k+1)T}{n}\right]} \left\| g_k^n\left(s,u_1\right) - g_k^n\left(s,u_2\right)\right\| =$$

$$\sup_{s\in\left[\frac{kT}{n},\frac{(k+1)T}{n}\right]} \left\| g_k^n\left(s,u_1\right) - g_k^n\left(s,u_2\right)\right\| =$$

$$\sup_{\frac{kT}{n}\leq s\leq\frac{(k+1)T}{n}} \left\|\frac{n}{T}\left(s-t_k^n\right)\left(u_1-u_2\right)\right\| = \left\|u_1-u_2\right\|;$$

(2) if $\frac{kT}{n} \leq -\tau+\frac{(k+1)T}{n} \leq \frac{(k+1)T}{n}$, we have

$$\sup_{s\in\left[-\tau+\frac{(k+1)T}{n},\frac{(k+1)T}{n}\right]} \left\| f_k^n\left(s,v_1\right) - f_k^n\left(s,v_2\right)\right\| =$$

$$\sup_{s\in\left[\frac{kT}{n},\frac{(k+1)T}{n}\right]} \left\| f_k^n\left(s,v_1\right) - f_k^n\left(s,v_2\right)\right\| =$$

$$\sup_{\frac{kT}{n}\leq s\leq\frac{(k+1)T}{n}} \left\|\frac{n}{T}(s-t_k^n)\left(v_1-v_2\right)\right\| = \left\|v_1-v_2\right\|$$

and

$$\sup_{s\in\left[-\tau+\frac{(k+1)T}{n},\frac{(k+1)T}{n}\right]} \left\| g_k^n\left(s,u_1\right) - g_k^n\left(s,u_2\right)\right\| =$$

$$\sup_{s\in\left[\frac{kT}{n},\frac{(k+1)T}{n}\right]} \|g_k^n(s,u_1) - g_k^n(s,u_2)\| =$$

$$\sup_{\frac{kT}{n}\le s\le\frac{(k+1)T}{n}} \left\|\frac{n}{T}(s-t_k^n)(u_1-u_2)\right\| = \|u_1-u_2\|.$$

So the mapping $(v,u) \to \Big(\mathcal{T}(t_{k+1}^n)f_k^n(\cdot,v), \mathcal{T}(t_{k+1})g_k^n(\cdot,u)\Big)$ from $H\times H$ to $\mathcal{C}_0\times\mathcal{C}_0$ is nonexpansive. Hence the set-valued mapping $G_k^n : \left[t_k^n, t_{k+1}^n\right]\times H\times H \rightharpoondown H$ defined by

$$G_k^n(t,u,v) = G\Big(t, \mathcal{T}(t_{k+1}^n)f_k^n(.,u), \mathcal{T}(t_{k+1}^n)g_k^n(.,v)\Big)$$

globally measurable and scalarly upper semicontinuous on $H\times H$, with nonempty closed convex values. As above we can easily check that

$$d(0, G_k^n(t,v,u) \le (1+\|u\|+\|v\|), \ \forall (t,u,v)\in\left[t_k^n,t_{k+1}^n\right]\times H\times H.$$

Applying Theorem 2, there exist two absolutely continuous mappings $u_k^n : \left[t_k^n,t_{k+1}^n\right]\to H$ and $v_k^n : \left[t_k^n,t_{k+1}^n\right]\to H$ such that

$$\begin{cases} -\dot{u}_k^n(t) \in N_{D(t,v_k^n(t))}(u_k^n(t)) + G_k^n(t, v_k^n(t), u_k^n(t)) \ \text{ a.e. on } \left[t_k^n,t_{k+1}^n\right]; \\ v_k^n(t) = v_n(t_k^n) + \displaystyle\int_{t_k^n}^t u_k^n(s)\,ds, \ \ \forall t\in\left[t_k^n,t_{k+1}^n\right]; \\ u_k^n(t) = u_n(t_k^n) + \displaystyle\int_{t_k^n}^t \dot{u}_k^n(s)\,ds, \ \ \forall t\in\left[t_k^n,t_{k+1}^n\right]; \\ u_k^n(t)\in D(t,u_k^n(t)) \ \ \forall t\in\left[t_k^n,t_{k+1}^n\right], \end{cases}$$

with

$$\|u_k^n(t)\|\le\alpha, \ \ \|v_k^n(t)\|\le\|b\|+T\alpha, \ \ \|\dot{u}_k^n(t)\|\le c_2(t).$$

Thus, by induction, we can construct two continuous mappings $u_n, v_n : [-\tau,T]\to H\times H$ with

$$v_n(t) = \begin{cases} \varphi(t) & \forall t\in[-\tau,0], \\ v_k^n(t) & \forall t\in\left]t_k^n,t_{k+1}^n\right], \ \forall k=0,\cdots,n-1; \end{cases}$$

$$u_n(t) = \begin{cases} \psi(t) & \forall t\in[-\tau,0], \\ u_k^n(t) & \forall t\in\left]t_k^n,t_{k+1}^n\right], \ \forall k=0,\cdots,n-1, \end{cases}$$

such that their restriction on each interval $\left[t_k^n,t_{k+1}^n\right]$ is a pair solution to

$$\begin{cases} -\dot{u}(t)\in N_{D(t,v(t))}(u(t)) + G(t, \mathcal{T}(t_{k+1}^n)f_k^n(.,v(t)), \mathcal{T}(t_{k+1}^n)g_k^n(.,u(t))); \\ v(t) = v_n(t_k^n) + \displaystyle\int_{t_k^n}^t u(s)\,ds, \ u(t) = u_n(t_k^n) + \displaystyle\int_{t_k^n}^t \dot{u}(s)\,ds \\ u(t)\in D(t,v(t)). \end{cases}$$

Let $h_k^n : [t_k^n, t_{k+1}^n] \times \mathcal{C}_0 \times \mathcal{C}_0$ be the element of minimal norm of G_k^n, then

$$\begin{cases} h_k^n\left(t, v_k^n\left(t\right), u_k^n\left(t\right)\right) \in G_k^n(t, v_k^n\left(t\right), u_k^n\left(t\right)) \text{ a.e. on } \left[t_k^n, t_{k+1}^n\right], \\ -\dot{u}_k^n\left(t\right) \in N_{D(t,v_k^n(t))}\left(u_k^n(t)\right) + h_k^n\left(t, v_k^n\left(t\right), u_k^n\left(t\right)\right) \text{ a.e. on } \left[t_k^n, t_{k+1}^n\right], \\ v_k^n\left(t_k^n\right) = v_n\left(t_k^n\right), u_k^n\left(t_k^n\right) = u_n\left(t_k^n\right) \\ u_k^n(t) \in D(t, v_k^n\left(t\right)), \ \forall t \in \left[t_k^n, t_{k+1}^n\right]. \end{cases}$$

Let set for notational convenience, $h_n(t,v,u) = h_k^n(t,v,u)$, $\theta_n\left(t\right) = t_{k+1}^n$ and $\delta_n\left(t\right) = t_k^n$, for all $t \in \left]t_k^n, t_{k+1}^n\right]$. Then we get for almost every $t \in [0,T]$

$$\begin{cases} h_n(t, v_n, u_n) \in G(t, \mathcal{T}(\theta_n\left(t\right)) f^n_{\frac{n}{T}\delta_n(t)}\left(., v_n\left(t\right)\right), \mathcal{T}(\theta_n\left(t\right)) g^n_{\frac{n}{T}\delta_n(t)}\left(., u_n\left(t\right)\right)); \\ -\dot{u}_n\left(t\right) \in N_{D(t, v_n(\theta_n(t)))}\left(u_n(\theta_n\left(t\right))\right) + h_n\left(t, v_n(t), u_n(t)\right); \\ v_n\left(0\right) = b = \varphi\left(0\right), \ u_n\left(0\right) = a = \psi\left(0\right) \in D\left(0, b\right), \\ u_n\left(t\right) \in D\left(t, v_n\left(\theta_n\left(t\right)\right)\right), \forall t \in [0,T] \end{cases}$$

with for all $t \in [0,T]$

$$d\Bigg(0, G(t, \mathcal{T}(\theta_n\left(t\right)) f^n_{\frac{n}{T}\delta_n(t)}\left(., v_n\left(t\right)\right), \mathcal{T}(\theta_n\left(t\right)) g^n_{\frac{n}{T}\delta_n(t)}\left(., u_n\left(t\right)\right)\Bigg)$$

$$\leq \beta\left(1 + \|u_n\left(t\right)\| + \|v_n\left(t\right)\|\right).$$

We claim that $\mathcal{T}(\theta_n\left(t\right)) f^n_{\frac{n}{T}\delta_n(t)}\left(., v_n\left(t\right)\right)$ and $\mathcal{T}(\theta_n\left(t\right)) g^n_{\frac{n}{T}\delta_n(t)}\left(., u_n\left(t\right)\right)$ pointwise converge on $[0,T]$ to $\mathcal{T}(t)v$ and $\mathcal{T}(t)u$ respectively in $\mathcal{C}_0$. The proof is similar to the one given in Theorem 2.1 in [14].

Further, as $\|v_n\left(t\right)\| \leq \|b\| + T\alpha$, $\|\dot{u}(t)\| \leq c_2(t)$ and

$$\|h_n\left(t, v_n(t), u_n(t)\right)\| \leq \beta\Bigg(1 + \|u_n(t)\| + \|v_n(t)\|\Bigg)$$

$$\leq \beta\big(1 + \|b\| + (1+T)\alpha\big).$$

We can proceed as in Theorem 2 to conclude the convergence of (u_n) and (v_n) to the solution of $(\mathcal{P}_\tau)$. □

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