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Strong stability for multiobjective investment problem with perturbed minimax risks of different types and parameterized optimality

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Abstract. A multicriteria investment Boolean problem of minimizing lost profits with parameterized efficiency and different types of risks is formulated. The lower and upper bounds on the radius of the strong stability of efficient portfolios are obtained. Several earlier known results regarding strong stability of Pareto efficient and extreme portfolios are confirmed.

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1 Introduction

Many problems of making multi-purpose decisions (individual or group) in management, planning and design can be formulated as multicriteria discrete optimization problems. A characteristic feature of such problems is the inaccuracy of the initial parameters. This inaccuracy is due to the influence of various factors of uncertainty and randomness: the inadequacy of the mathematical models used real processes, measurement or rounding errors and other factors. To manage financial investments, G. Markovitz [1] developed an optimization model that demonstrates how an investor, choosing a portfolio of assets, can minimize the degree of risk for a given expected income level. This formulation involves the use of statistical and expert assessments of risks (financial, environmental, etc.) as input data. It is well known that complex calculations of such quantities are accompanied by large number of errors, which leads to a high degree of uncertainty of the initial information. Under these conditions, the question naturally arises about the plausibility of results obtained in solving such problems, which makes necessary to conduct a post-optimal analysis of the stability of solutions to perturbations of parameters.

Modern research on the stability of multicriteria discrete optimization problems is carried out in two directions: qualitative and quantitative. Within the framework of the first direction, the authors concentrate their attention on the definition and study of various types of stability (see monograph [2], and surveys [3,4]), establishing a connection between different types of stability as well as on the search and description of the region of stability of the problem [5, 6]. The second direction is focused on obtaining estimates of permissible changes in the initial data of the problem, at which a certain predetermined property of optimal solutions is preserved [7–12], and on the development of algorithms for calculating these estimates [13–15].

Our current work continues research towards a similar direction, with focus on a different optimality principle, namely, the so-called parameterized efficient solutions and their strong stability properties are investigated. The paper is organized as follows. In Section 2, we introduce basic concepts and formulate the problem. Section 3 contains auxiliary technical statements required for the proof of the main result. As a result of the parametric analysis, in Section 4 the lower and upper bounds on strong stability radius are obtained in the case with arbitrary Hölder's norms specified in the three spaces of the problem's initial data. Some previously known facts are confirmed in Section 5.

2 Problem formulation and basic definitions

Consider a multicriteria discrete variant of the investment optimization problem with the following parameters specified below: let

$N_n=\{1,2,\dots,n\}$ be a variety of alternatives (investment assets);

N_m be a set of possible financial market states (market situations, scenarios);

N_s be a set of possible risks;

r_{ijk} be a numerical measure of economic risk of type $k \in N_s$ if investor chooses project $j \in N_n$ given the market is in state $i \in N_m$;

$R= [r_{ijk}] \in \mathbf{R}^{m\times n\times s}$ be a matrix specifying risks;

$x= (x_1,x_2,\dots,x_n)^T \in \mathbf{E}^n$ be an investment portfolio, where $\mathbf{E}=\{0,1\}$, and

$$x_j=\begin{cases} 1 & \text{if investor chooses project } j, \\ 0 & \text{otherwise;} \end{cases}$$

$X \subset \mathbf{E}^n$ be a set of all admissible investment portfolios, i.e. those whose realization provides the investor with the expected income and does not exceed his/her initial capital;

$\mathbf{R}^m$ be a financial market state space; $\mathbf{R}^n$ be a portfolio space; $\mathbf{R}^s$ be a risk space.

In our model, we assume that the risk measure is addictive, i.e. the total risk of one portfolio is a sum of risks of the projects included in the portfolio. The risk of each project can be measured, for instance, by means of the associated implementation cost.

Efficiency of a chosen portfolio (Boolean vector) $x \in X$, $|X| \geq 2$, is evaluated by a vector objective function

$$f(x,\mathrm{R}) = (f(x,R_1),f(x,R_2),\dots,f(x,R_s))^T,$$

with each partial objective representing minimax Savage's risk criterion [17]:

$$f(x,R_k) = \max_{i\in N_m} r_{ik}x = \max_{i\in N_m} \sum_{j\in N_n} r_{ijk}x_j \to \min_{x\in X}, \quad k \in N_s,$$

$$r_{ik}=(r_{i1k},\ r_{i2k},\dots,r_{ink})\in\mathbf{R}^n,\quad i\in N_m,\quad k\in N_s.$$

In the formula above, $R_k\in\mathbf{R}^{m\times n}$ represents the k-th cut of the risk matrix $R=[r_{ijk}]\in\mathbf{R}^{m\times n\times s}$ with rows r_{ik}.

Certainly, the problem has practical interest due to its multicriteria nature and the criteria that could be interpreted as maximum risk minimizing attitude of an investor to market instability and uncertainty.

For arbitrary $v\in\mathrm{N}$ (dimension of a space), we define the Pareto dominance [16] between two vectors as the following binary relation in the real vector valued space $\mathbf{R}^v$: $y\succ y'\iff y\geq y'\ \&\ y\neq y'$,where $y=(y_1,y_2,\dots,y_v)^T\in\mathrm{R}^v$, and $y'=(y'_1,y'_2,\dots,y'_v)^T\in\mathrm{R}^v$.

Let $\emptyset\neq I\subseteq N_s$. Denote R_I a submatrix of the risk matrix $R=[r_{ijk}]\in\mathbf{R}^{m\times n\times s}$ consisting of $h=|I|$ cuts with numbers of the set I, i.e.

$$R_I=(R_{k_1},R_{k_2},\dots,R_{k_h})^T\in\mathbf{R}^{m\times n\times h},$$
$$I=\{k_1,k_2,\dots,k_h\},\ 1\leq k_1<k_2<\dots<k_h\leq s.$$

Thus for a fixed non-empty I and chosen $x\in X$, we have a vector function

$$f(x,R_I)=(f(x,R_{k_1}),f(x,R_{k_2}),\dots,f(x,R_{k_h}))^T,$$

with components being type of Savage's minimax risk criterion [17]:

$$f\,(x,R_k)=\max_{i\in N_m} r_{ik}x\ \to\min_{x\in X},\quad k\in I.$$

An investor in the conditions of economic instability and uncertainty of the market state is extremely cautious, optimizing the total risk of the portfolio in the most unfavorable situation, namely when the risk is maximum. Such caution is appropriate because any investment is the exchange of a certain current value for a possibly uncertain future income. Obviously, this approach is dictated by the safest and most protective rule prescribing to assume the worst.

Let $u\in N_s$ and $N_s=\bigcup_{v\in N_u}I_v$ be a partition of the set N_s in u non-empty subsets (types of risks), i.e. $I_v\neq\emptyset$, $v\in N_u$, and $i\neq j\implies I_i\bigcap I_j=\emptyset$.

Such partition may naturally arise in the situation when risks can be classified to the different groups, e.g. financial, industrial, ecological etc. Another situation with different types of risks may appear if risk measurement scales are different, e.g. some risks are measured on a monetary scale whereas the others are measured on various subjective preference scales.

As following definition shows, inside a group of a certain type, Pareto dominance binary relation is used while comparing portfolios. For the given partition, we introduce a set of $(I_1,I_2,\dots,I_u)$-efficient portfolios according to the following formula:

$$G^s_m(R,I_1,I_2,\dots,I_u)=\{x\in X\colon \exists v\in N_u\ \ (X(x,R_{I_v})=\emptyset)\}\,,\tag{1}$$

where

$$X\,(x,R_{I_v})=\left\{x'\in X\colon\, f\,(x,R_{I_v})\succ f\left(x',R_{I_v}\right)\right\}.\tag{2}$$

For brevity, we sometimes refer to the set of $(I_1, I_2, \ldots, I_u)$ – efficient portfolios as $G_m^{su}(R)$ and name them *efficient*. It is easy to see that the set of efficient portfolios is non-empty.

In one particular case, if $u = 1$, i.e. $I{=}N_s$, any N_s – efficient portfolio $x \in G_m^s(R, N_s)$ is also Pareto efficient (optimal). Therefore, the set $G_m^s(R, N_s)$ is identical to the Pareto set [18] defined as follows:

$$P_m^s(R) = \{x \in X\colon X(x, R) = \emptyset\},$$

where

$$X(x, R) = \left\{x' \in X\colon f(x, R) \geq f\left(x', R\right) \ \& \ f(x, R) \neq f\left(\mathrm{x}', R\right)\right\}.$$

In another particular case, if $u{=}$ s, i.e. $I_v = \{v\}$ for $v \in N_u = N_s$, the set $G_m^s(R, \{1\}, \{2\}, \ldots, \{s\})$ is a set of all the so-called extreme portfolios (see e.g. [19]). The set of extreme portfolios is defined as

$$E_m^s(R) = \{x \in X\colon \exists k \in N_s \ (X(x, R_k) = \emptyset\},$$

where

$$X(x, R_k) = \left\{x' \in X\colon f(x, R_k) > f\left(x', R_k\right)\right\}.$$

The choice of extreme portfolios can be interpreted as finding best solutions for each of s criteria, and then combining them into one set. The vector composed of optimal objective values constitutes the ideal vector that is of great importance in theory and methodology of multiobjective optimization [19].

The problem of finding the set of efficient portfolios

$$G_m^s(R, I_1, I_2, \ldots, I_u) = G_m^{su}(R)$$

is referred to as multicriteria investment Boolean problem with Savage's risk criteria of different types and denoted by $Z_m^s(R, I_1, I_2, \ldots, I_u)$, or shortly, $Z_m^{su}(R)$.

For the fixed non-empty $I \subseteq N_s$, we introduce the following sets:

$$P(R_I) = \{x \in X\colon X(x, R_I) = \emptyset\},$$

$$E(R_I) = \{x \in X\colon \exists k \in I \ (X(x, R_k) = \emptyset\},$$

where

$$X(x, R_I) = \left\{x' \in X\colon f(x, R_I) \succ f\left(x', R_I\right)\right\}.$$

In particular, for fixed $k \in N_s$ and $I = \{k\}$, $|I| = 1$, the two sets $P(R_k)$ and $E(R_k)$ are identical. Both sets represent a set of optimal portfolios for the scalar problem with respect to the k-th risk:

$$f(x, R_k) = \max_{i \in N_m} r_{ik} x \ \to \min_{x \in X}.$$

Due to (1), we have the following equality:

$$G_m^s(R, I_1, I_2, \ldots, I_u) = \{x \in X\colon \exists v \in N_u \ (x \in P(R_{I_v}))\}. \tag{3}$$

Therefore, we have

$$G_m^s\left(R, I_1, I_2, \ldots, I_u\right)=\bigcup_{v \in N_u} P\left(R_{I_v}\right), \quad \bigcup_{v \in N_u} I_v=N_s.$$

Obviously, all the sets specified above are non-empty for any risk matrix $R \in \mathbf{R}^{m \times n \times s}$.

We will perturb the elements of the three–dimensional risk matrix $R \in \mathbf{R}^{m \times n \times s}$ by adding elements of the risk perturbing matrix $R' \in \mathbf{R}^{m \times n \times s}$. Thus the problem $Z_m^{su}(R+R')$ with perturbed risks has the following form:

$$f(x, R+R') \rightarrow \min_{x \in X}.$$

The set of $(I_1, I_2, \ldots, I_u)$ – efficient portfolios in the perturbed problem is denoted by $G_m^s(R+R', I_1, I_2, \ldots, I_u)$, or shortly $G_m^{su}(R+R')$.

Recall that Hölder's norm l_p (also known as p-norm) in vector space $\mathbf{R}^n$ is the number

$$\|a\|_p=\begin{cases} \left(\sum_{j \in N_n}|a_j|^p\right)^{\frac{1}{p}} & \text{if } \ 1 \leq p<\infty, \\ \max\left\{|a_j| \ : j \in N_n\right\} & \text{if } \ p=\infty, \end{cases}$$

where $a=(a_1, a_2, \ldots, a_n)^T \in \mathbf{R}^n$.

In the spaces $\mathbf{R}^n, \mathbf{R}^m$ and $\mathbf{R}^s$ we define three Hölder's norms l_p, l_q and l_t, where $p, q, t \in [1, \infty]$. So, the norm of matrix $R \in \mathbf{R}^{m \times n \times s}$ is the following number:

$$\|R\|_{pqt}=\|(\|R_1\|_{pq}, \|R_2\|_{pq}, \ldots, \|R_s\|_{pq})\|_t,$$

with cuts

$$\|R_k\|_{pq}=\|(\|r_{1k}\|_p, \|r_{2k}\|_p, \ldots, \|r_{mk}\|_p)\|_q, \quad k \in N_s.$$

For any numbers $p, q, t \in [1, \infty]$ the following inequalities are valid:

$$\|r_{ik}\|_p \leq \|R_k\|_{pq} \leq \|R\|_{pqt}, \quad i \in N_m, \quad k \in N_s. \tag{4}$$

While solving investment problems, it is necessary to take into account the inaccuracy of the input information (statistical and expert risks evaluation errors) that are very common in real life. Under these conditions, it is highly recommended to get numerical bounds of possible changes to the input data that for any small perturbation the efficiency of at least one originally extreme portfolio is preserved.

Following [3], the strong stability (in terminology of [4], T_1-stability) radius of $Z_m^s\left(R, I_1, I_2, \ldots, I_u\right), \ s, m \in \mathbf{N}$, with Hölder's norms l_p, l_q and l_t in spaces $\mathbf{R}^n, \mathbf{R}^m$ and $\mathbf{R}^s$, respectively, is defined as:

$$\rho=\rho_m^{su}(p, q, t)=\begin{cases} \sup \Xi_{pqt} & \text{if } \ \Xi_{pqt} \neq \emptyset, \\ 0 & \text{if } \ \Xi_{pqt}=\emptyset. \end{cases}$$

where

$$\Xi_{pqt}=\{\varepsilon>0 \ : \ \forall R' \in \Omega_{pqt}(\varepsilon) \quad (G_m^{su}(R+R') \cap G_m^{su}(R) \neq \emptyset)\};$$

$\Omega_{pqt}(\varepsilon) = \{R' \in \mathbf{R}^{m\times n\times s} : \|R'\|_{pqt} < \varepsilon\}$ is the set of perturbing matrices R' with cuts $R'_k \in \mathbf{R}^{m\times n}$, $k \in N_s$;

$G^{su}_m(R + R')$ is the set of $(I_1, I_2, \dots, I_u)$-solutions of the perturbed problem $Z^{su}_m(R + R')$;

$\|R'\|_{pqt}$ is the norm of matrix $R' = [r'_{ijk}]$.

Thus the strong stability radius of the problem $Z^{su}_m(R)$ is an extreme level of independent perturbations of elements of matrix $R \in \mathbf{R}^{m\times n\times s}$ such that the sets $G^{su}_m(R)$ and $G^{su}_m(R + R')$ are never disjoint.

Obviously, if $G^{su}_m(R) = X$, then the strong stability radius is not bounded. For this reason, the problem with $X\backslash E^m_s(R) \neq \emptyset$ is called *non-trivial*.

3 Auxiliary statements and lemmas

Let v be any of the above-numbers p, q, t. For the number v, let v^* be the number conjugate to v and defined as:

$$1/v + 1/v^* = 1, \quad 1 < v < \infty.$$

We also set $v^* = 1$ if $v = \infty$, and $v^* = \infty$ otherwise. We assume that v and v^* be taken from $[1,\infty]$, and conjugate. In addition to the above, we assume that $1/v = 0$ if $v = \infty$.

Further we will use the well-known Hölder's inequality

$$|a^T b| \leq \|a\|_v \|b\|_{v^*} \tag{5}$$

that is true for any two vectors a and b of the same dimension.

It is also well-known that Hölder's inequality becomes an equality for $1 < v < \infty$ if and only if

a) one of a or b is the zero vector;

b) the two vectors obtained from non-zero vectors a and b by raising their components' absolute values to the powers of v and v^*, respectively, are linearly dependent (proportional), and sign $(a_i b_i)$ is independent of i.

When $v = 1$, (3) transforms into the following inequality:

$$|\sum_{i\in N_n} a_i b_i| \leq \max_{i\in N_n} |b_i| \sum_{i\in N_n} |a_i|.$$

The last holds as equality if, for example, b is the zero vector or if $a_j \neq 0$ for some j such that $|b_j| = \|b\|_\infty \neq 0$, and $a_i = 0$ for all $i \in N_n\backslash\{j\}$.

When $v = \infty$, (3) transforms into the following inequality:

$$|\sum_{i\in N_n} a_i b_i| \leq \max_{i\in N_n} |a_i| \sum_{i\in N_n} |b_i|.$$

The last holds as equality if, for example, b is the zero vector or if $a_i = \sigma sign(b_i)$ for all $i \in N_n$ and $\sigma \geq 0$.

It is easy to see that for any $a=(a_1,a_2,...,a_n)^T\in\mathbf{R}^n$ with

$$|a_j|=\alpha,\quad j\in N_n,$$

the following equality holds

$$\|a\|_v=\alpha n^{1/v} \tag{6}$$

for any $v\in[1,\infty]$.

The following two lemmas can easily be proven.

Lemma 1. *Given two portfolios $x,x^0\in X$, two market states $i,i'\in N_m$ and a fixed risk $k\in N_s$, the following statement is true for any $p,q\in[1,\infty]$:*

$$r_{ik}x-r_{i'k}x^0\geq-\|R_k\|_{pq}\|(\|x\|_{p^*},\|x^0\|_{p^*})\|_\nu,$$

where $R_k\in\mathbf{R}^{m\times n}$ is the k-th cut of matrix $R\in\mathbf{R}^{m\times n\times s}$ with rows $r_{1k},r_{2k},...,r_{mk}$, $\nu=\min\{p^*,q^*\}$.

Proof. Let $i\neq i'$. Then, using Hölder's inequality (5), we get

$$r_{ik}x-r_{i'k}x^0\geq-(\|r_{ik}\|_p\|x\|_{p^*}+\|r_{i'k}\|_p\|x\|_{p^*})\geq$$
$$\geq\|(\|r_{ik}\|_p,\|r_{i'k}\|_p)\|_q\ \|(\|x\|_{p^*},\|x^0\|_{p^*})\|_{q^*}\geq$$
$$\geq-\|R_k\|_{pq}\ \|(\|x\|_{p^*},\|x^0\|_{p^*})\|_{q^*}\geq-\|R_k\|_{pq}\ \|(\|x\|_{p^*},\|x^0\|_{p^*})\|_\nu.$$

For $i=i'$, using inequalities (4), and Hölder's inequality (5) we deduce

$$r_{ik}x-r_{i'k}x^0\geq-\|r_{ik}\|_p\ \|x-x^0\|_{p^*}\geq-\|R_k\|_{pq}\ \|x-x^0\|_{p^*}\geq$$
$$\geq-\|R_k\|_{pq}\ \|(\|x\|_{p^*},\|x^0\|_{p^*})\|_{q^*}\geq-\|R_k\|_{pq}\ \|(\|x\|_{p^*},\|x^0\|_{p^*})\|_\nu.$$

□

From the definition of $G_m^s(R,I_1,I_2,\ldots,I_u)$, the following claim holds straightforward.

Lemma 2. *A portfolio $x\notin G_m^s(R,I_1,I_2,\ldots,I_u)$ if and only if $x\notin P(R_{I_v})$ for any index $v\in N_u$.*

4 Main result

For non-trivial problem $Z_m^{su}(R)=Z_m^{su}(R,I_1,I_2,\ldots,I_u)$, we introduce the following notation

$$\varphi=\varphi_m^{su}(p,q)=\min_{x\notin G_m^{su}(R)}\ \min_{v\in N_u}\ \max_{x'\in P(x,R_{I_v})}\ \min_{k\in I_v}\frac{g(x.x',R_k)}{\|(\|x\|_{p^*},\|x'\|_{p^*})\|_\gamma},$$

$$\psi=\psi_m^{su}(p,q,t)=\max_{x'\in G_m^{su}(R)}\ \max_{v\in N_u}\ \min_{x\notin G_m^{su}(R)}\frac{\|[g(x,x',R_{I_v})]^+\|_t}{\|(\|x\|_{p^*},\|x'\|_{p^*})\|_\gamma},$$

$$\chi = \chi_m^{su}(p,q,t) = n^{1/p} m^{1/q} s^{1/t} \min_{x \notin G_m^{su}(R)} \max_{v \in N_u} \max_{x' \in G_m^{su}(R)} \max_{k \in I_v} \frac{g(x,x',R_k)}{\|x-x'\|_1},$$

where

$$g(x,x',R_k) = f(x,R_k) - f(x',R_k), \quad k \in I_v,$$

$$g(x,x',R_{I_v}) = f(x,R_{I_v}) - f(x',R_{I_v}),$$

$$P(x,R_{I_v}) = P(R_{I_v}) \cap X(x,R_{I_v}),$$

$$\gamma = \min\{p^*, q^*\}.$$

Here $[y]^+ = (y_1^+, y_2^+, \dots, y_h^+)$ is a positive projection of vector $y = (y_1, y_2, \dots, y_h) \in \mathbf{R}^h$, i.e. $y_k^+ = \max\{0, y_k\}$, $k \in N_h$. It is easy to see that $\varphi,\ \psi,\ \chi \geq 0$.

Theorem 1. *Given $s, m \in \mathbf{N}$, $u \in N_s$ and $p, q, t \in [1, \infty]$, for the strong stability radius $\rho = \rho_m^{su}(p,q,t)$ of s-criteria non-trivial problem $Z_m^{su}(R)$, the following bounds are valid:*

$$0 < \max\{\varphi_m^{su}(p,q),\ \psi_m^{su}(p,q,t)\} \leq \rho_s^m(p,q,t) \leq \min\{\chi_m^{su}(p,q,t),\ \|R\|_{pqt}\}.$$

Proof. Since

$$\forall x' \in G_m^{su}(R) \quad \forall x \notin G_m^{su}(R) \quad \exists v \in N_u \quad (f(x,R_{I_v}) \succ f(x',R_{I_v})),$$

the inequalities $\psi, \chi > 0$ are evident.

Now we show that

$$\rho = \rho_m^{su}(p,q,t) \geq \varphi_m^{su}(p,q) = \varphi.$$

If $\varphi = 0$, the inequality above is evident, so we assume $\varphi > 0$.

Let the perturbing matrix $R' = [r'_{ijk}] \in \mathbf{R}^{m \times n \times s}$ with cuts R'_k, $k \in N_s$, be taken from the set $\Omega_{pqt}(\varphi)$. According to the definition of the number φ, and due to inequality (4), we obtain

$$\forall v \in N_u \quad \forall x \notin G_m^{su}(R) \quad \exists x^0 \in P(x,R_{I_v}) \quad \forall k \in I_v$$

$$\left(\frac{g(x,x^0,R_k)}{\|(\|x\|_{p^*}, \|x^0\|_{p^*})\|_\gamma} \geq \varphi > \|R'\|_{pqt} \geq \|R'_k\|_{pq} \right).$$

Thus, due to Lemma 1, for any criterion $v \in N_u$ there exists a portfolio $x^0 \neq x$ such that

$$g(x,x^0,R_k+R'_k) = f(x,R_k+R'_k) - f(x^0,R_k+R'_k) =$$

$$= \max_{i \in N_m}(r_{ik}+r'_{ik})x - \max_{i \in N_m}(r_{ik}+r'_{ik})x^0 =$$

$$= \min_{i \in N_m} \max_{i' \in N_m} (r_{ik}x + r'_{ik}x - r_{i'k}x^0 - r'_{i'k}x^0) \geq$$

$$\geq f(x,R_k) - f(x^0,R_k) - \|R'_k\|_{pq}\ \|(\|x\|_{p^*}, \|x^0\|_{p^*})\|_\gamma =$$

$$= g(x, x^0, R_k) - \|R'_k\|_{pq} \ \|(\|x\|_{p^*}, \|x^0\|_{p^*})\|_\gamma > 0, \ k \in I_v,$$

where r'_{ik} is the i-th row of the k-th cut R'_k of the matrix R'. This implies

$$x \notin P(R_{I_v} + R'_{I_v}), \ v \in N_u.$$

Therefore according to Lemma 2, we obtain that

$$x \notin G_m^{su}(R + R').$$

Summarizing and taking into account that $x \notin G_m^{su}(R)$, we conclude that for any perturbing matrix $R' \in \Omega_{pqt}(\varphi)$, any portfolio $x \in G_m^{su}(R+R')$ is also an element of $G_m^{su}(R)$, i.e. inequality $\rho \geq \varphi$ is true.

Further, we prove the lower bound

$$\rho = \rho_m^{su}(p, q, t) \geq \psi_m^{su}(p, q, t) = \psi.$$

We already know that $\psi > 0$. Therefore in order to prove $\rho \geq \psi$, it suffices to show that there exists a portfolio x^* belonging to $G_m^{su}(R) \cap G_m^{su}(R+R')$ for any perturbing matrix $R' = [r'_{ijk}] \in \Omega_{pqt}(\psi)$.

Since the problem $Z_m^{su}(R)$ is non-trivial, according to the definition of ψ, we have

$$\exists x^0 \in G_m^{su}(R) \quad \exists w \in N_u \quad \forall x \notin G_m^{su}(R)$$

$$\left(\|[g(x, x^0, R_{I_w})]^+\|_t \geq \psi \|(\|x\|_{p*}, \|x^0\|_{p*})\|_\gamma > 0\right). \tag{7}$$

Further we show that the formula

$$\forall x \notin G_m^{su}(R) \quad \forall R' \in \Omega_{pqt}(\psi) \quad (x \notin X(x^0, R_{I_w} + R'_{I_w})) \tag{8}$$

holds.

We prove this by contradiction. Assume the opposite, i.e. that formula

$$\exists \tilde{x} \notin G_m^{su}(R) \quad \exists \tilde{R} \in \Omega_{pqt}(\psi) \quad (\tilde{x} \in X(x^0, R_{I_w} + \tilde{R}_{I_w}))$$

holds. Then we get

$$f(\tilde{x}, R_{I_w} + \tilde{R}_{I_w}) \prec f(x^0, R_{I_w} + \tilde{R}_{I_w}).$$

Using Lemma 1 for any index $k \in I_w$, we obtain

$$0 \geq g(\tilde{x}, x^0, R_k + \tilde{R}_k) = f(\tilde{x}, R_k + \tilde{R}_k) - f(x^0, R_k + \tilde{R}_k) =$$

$$= \max_{i \in N_m} (r_{ik} + \tilde{r}_{ik})\tilde{x} - \max_{i \in N_m} (r_{ik} + \tilde{r}_{ik})x^0 =$$

$$= \min_{i \in N_m} \ \max_{i' \in N_m} \ (r_{ik}\tilde{x} - r_{i'k}x^0 + \tilde{r}_{ik}\tilde{x} - \tilde{r}_{i'k}x^0) \geq$$

$$\geq g(\tilde{x}, x^0, R_k) - \|\tilde{R}_k\|_{pq} \ \|(\|\tilde{x}\|_{p^*}, \|x^0\|_{p^*})\|_\gamma.$$

Therefore, we get

$$g(\tilde{x}, x^0, R_k) \le \|\tilde{R}_k\|_{pq}\ \|(\|\tilde{x}\|_{p^*}, \|x^0\|_{p^*})\|_\gamma,\ \ k \in I_w.$$

Then we continue

$$[g(\tilde{x}, x^0, R_k)]^+ \le \|\tilde{R}_k\|_{pq}\ \|(\|\tilde{x}\|_{p^*}, \|x^0\|_{p^*})\|_\gamma,\ \ k \in I_w.$$

As a result we get a formula contradicting (7)

$$\|[g(\tilde{x}, x^0, R_{I_w})]^+\|_t \le \|\tilde{R}_{I_w}\|_{pqt}\ \|(\|\tilde{x}\|_{p^*}, \|x^0\|_{p^*})\|_\gamma \le$$

$$\le \|\tilde{R}\|_{pqt}\ \|(\|\tilde{x}\|_{p^*}, \|x^0\|_{p^*})\|_\gamma < \psi\|(\|\tilde{x}\|_{p^*}, \|x^0\|_{p^*})\|_\gamma.$$

This confirms the validity of (8).

Further we show a way of selecting a portfolio $x^* \in G_m^{su}(R) \cap G_m^{su}(R+R')$ where $R' \in \Omega_{pqt}(\psi)$. If $x^0 \in G_m^{su}(R+R')$, then we get $x^* = x^0$. If $x^0 \notin G_m^{su}(R+R')$, then due to Lemma 2 we obtain $x^0 \notin P(R_{I_v}+R'_{I_v})$ for any $v \in N_u$, and in particular for a fixed $w \in N_u$ we have $x^0 \notin P(R_{J_w}+R'_{I_w})$. Then due to external stability (see [16]) of the Pareto set $P(R_{J_w}+R'_{I_w})$, one can chose a portfolio $x^* \in P(R_{J_w}+R'_{I_w})$ (and hence $x^* \in G_m^{su}(R+R')$) such that $x^* \in X(x^0, R_{I_w}+R'_{I_w})$. Taking into account (8), it is easy to see that $x^* \in G_m^{su}(R)$. Thus, we just have $\rho \ge \psi$ proven.

Further, we prove the upper bound

$$\rho = \rho_m^{su}(p,q,t) \le \chi_m^{su}(p,q,t) = \chi.$$

According to the definition of χ and due to assumption about problem's non-triviality, we have

$$\exists x^0 = (x_1^0, x_2^0, ..., x_n^0)^T \notin G_m^{su}(R) \quad \forall v \in N_u \quad \forall x \in G_m^{su}(R) \quad \forall k \in I_v$$
$$\left(\chi\|x^0 - x\|_1 \ge n^{1/p} m^{1/q} s^{1/t} g(x^0, x, R_k)\right). \tag{9}$$

Let $\varepsilon > \chi$, and let the elements of perturbing matrix $R^0 = [r_{ijk}^0] \in \mathbf{R}^{m\times n\times s}$ be defined as:

$$r_{ijk}^0 = \begin{cases} -\delta & \text{if } \ i \in N_m, \ \ x_j^0 = 1, \ \ k \in N_s, \\ \delta & \text{if } \ i \in N_m, \ \ x_j^0 = 0, \ \ k \in N_s, \end{cases}$$

where δ satisfies

$$\chi < \delta n^{1/p} m^{1/q} s^{1/t} < \varepsilon. \tag{10}$$

From the above according to (6), we get

$$\|r_{ik}^0\|_p = \delta n^{1/p}, \quad i \in N_m, \quad k \in N_s,$$

$$\|R_k^0\|_{pq} = \delta n^{1/p} m^{1/q}, \quad k \in N_s,$$

$$\|R^0\|_{pqt} = \delta n^{1/p} m^{1/q} s^{1/t},$$

$$R^0 \in \Omega_{pqt}(\varepsilon).$$

In addition, all the rows $r^0_{ik},\ i \in N_m$, of any k-th cut $R^0_k,\ k \in N_s$, are constructed identically and composed of δ and $-\delta$. So, setting $c = r^0_{ik},\ i \in N_m,\ k \in N_s$, we deduce

$$c(x^0 - x) = -\delta\|x^0 - x\|_1 < 0$$

that is true for any portfolio $x \neq x^0$. Using (9) and (10), we conclude that for any portfolio $x \in G^{su}_m(R)$ and any $v \in N_u$, the following statements are true:

$$g(x^0, x, R_k + R^0_k) = f(x^0, R_k + R^0_k) - f(x, R_k + R^0_k) =$$

$$= \max_{i \in N_m}(r_{ik} + c)x^0 - \max_{i \in N_m}(r_{ik} + c)x = \max_{i \in N_m} r_{ik}x^0 - \max_{i \in N_m} r_{ik}x + c(x^0 - x) =$$

$$= g(x^0, x, R_k) + c(x^0 - x) \leq \left(\chi(n^{1/p}m^{1/q}s^{1/t})^{-1} - \delta\right) \|x^0 - x\|_1 < 0,\ k \in I_v.$$

This implies $x \notin P(R_{I_v} + R'_{I_v})$ for any $v \in N_u$. Then due to Lemma 2 we have $x \notin G^{su}_m(R + R^0)$. Thus, for any $\varepsilon > \chi$ there exists a perturbing matrix $R^0 \in \Omega_{pqt}(\varepsilon)$ such that $G^{su}_m(R) \cap G^{su}_m(R + R^0) = \emptyset$, i.e. $\rho < \varepsilon$ for any $\varepsilon > \chi$. Hence, $\rho \leq \chi$.

Finally, we show

$$\rho = \rho^{su}_m(p, q, t) \leq \|R\|_{pqt}.$$

Let $x^0 = (x^0_1, x^0_2, ..., x^0_n)^T \notin G^{su}_m(R)$ and $\varepsilon > \|R\|_{pqt}$, and let us fix δ satisfying condition

$$0 < \delta n^{1/p}m^{1/q}s^{1/t} < \varepsilon - \|R\|_{pqt}. \tag{11}$$

We introduce an auxiliary matrix $V = [v_{ijk}] \in \mathbf{R}^{m \times n \times s}$ with cuts $V_k,\ k \in N_s$, defined as follows:

$$v_{ijk} = \begin{cases} -\delta & \text{if} \quad i \in N_m, \quad x^0_j = 1, \quad k \in N_s, \\ \delta & \text{if} \quad i \in N_m, \quad x^0_j = 0, \quad k \in N_s. \end{cases}$$

Using (6), we obtain

$$\|V_k\|_{pq} = \delta n^{1/p}m^{1/q},\ k \in N_s,$$

$$\|V\|_{pqt} = \delta n^{1/p}m^{1/q}s^{1/t}. \tag{12}$$

It is easy to see that all rows of $V_k,\ k \in N_s$, are identical and composed of δ and $-\delta$. So, we get that for any $v \in N_u$ the following formula

$$f(x^0, V_k) - f(x, V_k) = -\delta\|x^0 - x\|_1 < 0,\ k \in I_v, \tag{13}$$

is true for any $x \neq x^0$, and in particular for $x \in G^{su}_m(R)$.

Further, let $R^0 \in \mathbf{R}^{m \times n \times s}$ be a perturbing matrix with cuts $R^0_k,\ k \in N_s$, defined as:

$$R^0_k = V_k - R_k,\ k \in N_s, \tag{14}$$

i.e. $R^0 = V - R$. Using (11) and (12), we deduce

$$\|R^0\|_{pqt} \leq \|V\|_{pqt} + \|R\|_{pqt} = \delta n^{1/p}m^{1/q}s^{1/t} + \|R\|_{pqt} < \varepsilon,$$

i.e. $R^0 \in \Omega_{pqt}(\varepsilon)$.

Additionally, using (13) and (14) for any index $v \in N_u$, we have

$$g(x^0, x, R_k + R_k^0) = f(x^0, R_k + R_k^0) - f(x, R_k + R_k^0) =$$

$$= f(x^0, V_k) - f(x, V_k) = -\delta\|x^0 - x\|_1 < 0, \ k \in I_v,$$

i.e. $x \notin P(R_{I_v} + R^0_{I_v})$ for any $v \in N_u$. Therefore, due to Lemma 2 $x \notin G^{su}_m(R + R^0)$. Summarizing, we get

$$\forall \varepsilon > \|R\|_{pqt} \quad \exists R^0 \in \Omega_{pqt}(\varepsilon) \quad \left(G^{su}_m(R) \cap G^{su}_m(R + R^0) = \emptyset\right).$$

The last implies $\rho \leq \|R\|_{pqt}$. □

5 Corollaries

From theorem 1 we obtain a series of known results. For the completeness of description we list most interesting of them below. The first corollary describes strong stability bounds for an extreme case $u = 1$ where the set of efficient portfolios $G^s_m(R, N_s)$ transforms into the set of Pareto efficient portfolios $P^s_m(R)$.

Corollary 1. *[8] For $s, m \in \mathbf{N}$ and $p, q, t \in [1, \infty]$, the strong stability radius $\rho^{s1}_m(p, q, t)$ of s-criteria non-trivial problem $Z^s_m(R, N_s)$ of finding the set of Pareto efficient portfolios $P^s_m(R)$ has the following valid lower and upper bounds:*

$$0 < \max\{\varphi^{s1}_m(p, q), \psi^{s1}_m(p, q, t)\} \leq \rho^{s1}_m(p, q, t) \leq \min\{\chi^{s1}_m(p, q, t), \|R\|_{pqt}\},$$

where

$$\varphi^{s1}_m(p, q) = \min_{x \notin P^s_m(R)} \max_{x' \in P(x, R)} \min_{k \in N_s} \frac{g(x, x', R_k)}{\|(\|x\|_{p^*}, \|x'\|_{p^*})\|_\gamma},$$

$$\psi^{s1}_m(p, q, t) = \max_{x' \in P^s_m(R)} \min_{x \notin P^s_m(R)} \frac{\|[g(x, x', R_k)]^+\|_t}{\|(\|x\|_{p^*}, \|x'\|_{p^*})\|_\gamma},$$

$$\chi^{s1}_m(p, q, t) = n^{1/p} m^{1/q} s^{1/t} \min_{x \notin P^s_m(R)} \max_{x' \in P^s_m(R)} \max_{k \in N_s} \frac{g(x, x', R_k)}{\|x - x'\|_1}.$$

Therefore, in particular case where $p = q = t = \infty$, we have

$$0 < \max_{x' \in P^s_m(R)} \min_{x \notin P^s_m(R)} \max_{k \in N_s} \frac{g(x, x', R_k)}{\|x + x'\|_1} \leq \rho^{s1}_m(\infty, \infty, \infty) \leq$$

$$\leq \min_{x \notin P^s_m(R)} \max_{x' \in P^s_m(R)} \max_{k \in N_s} \frac{g(x, x', R_k)}{\|x - x'\|_1}.$$

The second corollary describes strong stability bounds for another extreme case $u = s$ where the set of efficient portfolios $G^s_m(R, \{1\}, \{2\}, \ldots, \{s\})$ transforms into the set of extreme portfolios $E^s_m(R)$.

Corollary 2. *[20] For $s, m \in \mathbf{N}$ and $p, q, t \in [1, \infty]$, the strong stability radius $\rho_m^{ss}(p, q, t)$ of s-criteria non-trivial problem $Z_m^s(R, \{1\}, \{2\}, \dots, \{s\})$ of finding the set of extreme portfolios $E_m^s(R)$ has the following valid lower and upper bounds:*

$$0 < \max\{\varphi_m^{ss}(p, q), \psi_m^{ss}(p, q)\} \leq \rho_m^{ss}(p, q, t) \leq \min\{\chi_m^{ss}(p, q, t), \|R\|_{pqt}\},$$

where

$$\varphi_m^{ss}(p, q) = \min_{x \notin E_m^s(R)} \min_{k \in N_s} \max_{x' \in E(R_k)} \frac{g(x, x', R_k)}{\|(\|x\|_{p^*}, \|x'\|_{p^*})\|_\gamma},$$

$$\psi_m^{ss}(p, q) = \max_{x' \in E_m^s(R)} \max_{k \in N_s} \min_{x \notin E_m^s(R)} \frac{g(x, x', R_k)}{\|(\|x\|_{p^*}, \|x'\|_{p^*})\|_\gamma},$$

$$\chi_m^{ss}(p, q, t) = n^{1/p} m^{1/q} s^{1/t} \min_{x \notin E_m^s(R)} \max_{k \in N_s} \max_{x' \in E_m^s(R)} \frac{g(x, x', R_k)}{\|x - x'\|_1}.$$

Therefore, in particular case where $p = q = t = \infty$, we have

$$0 < \min_{x \notin E_m^s(R)} \min_{k \in N_s} \max_{x' \in E(R_k)} \frac{g(x, x', R_k)}{\|x + x'\|_1} \leq \rho_m^{ss}(\infty, \infty, \infty) \leq$$

$$\leq \min_{x \notin E_m^s(R)} \max_{k \in N_s} \max_{x' \in E_m^s(R)} \frac{g(x, x', R_k)}{\|x - x'\|_1}.$$

6 Conclusion

As a summary, it is worth mentioning that the bounds proven in Theorem 1 and corollaries, are mostly theoretical due to their analytical and enumerative structures. Even for a single objective, the difficulty of stability radius exact value calculation is a long-standing challenge pointed out in [13, 14]. In practical applications, one can try to get reasonable approximation of the bounds using some meta-heuristics, e.g. evolutionary algorithms or Monte-Carlo simulation. Another possibility to continue research in this direction is to specify some particular classes of problems where computational burden can be drastically reduced due to a unique structure of the set of efficient portfolios.

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